On some properties of Archimedean tiling graphs

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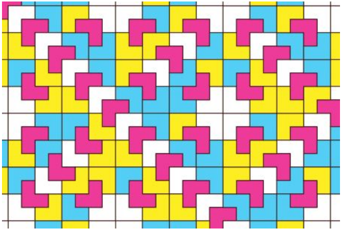
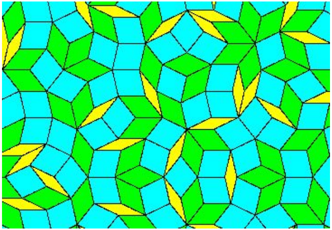
Plane tiling

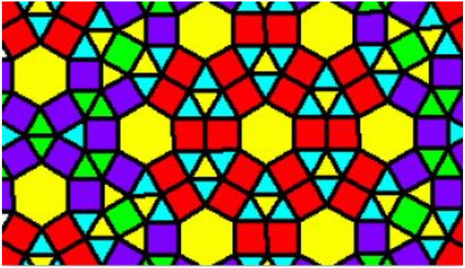
- A plane tiling $\mathcal{T}$ is a countable family of closed sets $\mathcal{T} = \{T_1, T_2, \cdots\}$ which cover the plane without gaps or overlaps.
- Every closed set $T_i \in \mathcal{T}$ is called a tile of $\mathcal{T}$.
- The intersection of any finite set of tiles of $\mathcal{T}$ (containing at least two distinct tiles) may be empty or may consist of a set of isolated points and arcs. In these cases, the points will be called vertices of the tiling and the arcs will be called edges.
- In a tiling with each tile is a polygon, if the corners and sides of a polygon coincide with the vertices and edges of the tiling, we say the tiling is edge-to-edge.





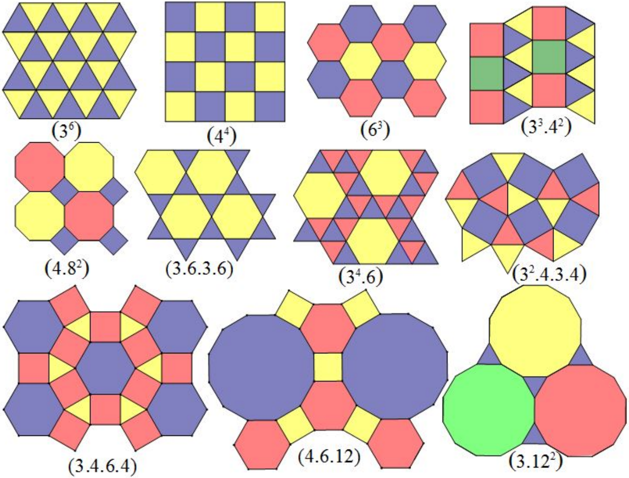
Plane tiling

- A so-called type of vertex describes its neighbourhood. If, for example, in some cyclic order around a vertex there are a triangle, then a square, next a hexagon, and last another square, then its type is $(3, 4, 6, 4)$.



Archimedean tilings

- Archimedean tilings are plane edge-to-edge tilings by regular polygons such that all vertices are of the same type. Thus, the vertex type will be defining our tiling. There exist precisely 11 distinct such tilings.



Archimedean tiling graphs

- The graph formed by an Archimedean tiling, which means that its vertex set and edge set are consisted of all vertices and edges of responding Archimedean tiling respectively, is called an Archimedean tiling graph.
- For the sake of convenience, we still use the notation for an Archimedean tiling, such as $(3^2.4.3.4)$, to denote the corresponding Archimedean tiling graph.
- Clearly, the lattice graph, the regular triangular lattice graph and the regular hexagonal lattice graph are all Archimedean tiling graphs.

Part I. Gallai's property of Archimedean tiling graphs

Gallai's property about longest paths

- In 1966, Gallai [1] raised the question whether (connected) graphs do exist such that each vertex is missed by some longest path. This property will be called Gallai's property.
- In 1969, Walther [2] firstly constructed such a planar graph with 25 vertices, which has connectivity 1.
- In 1975, Schmitz [3] found a planar graph with 17 vertices satisfying Gallai's property, which is the smallest planar graph with connectivity 1 up to now.

[1] T. Gallai, Problem 4, in: Theory of Graphs, Proc. Tihany 1966 (ed: P. Erdős and G. Katona), Academic Press, New York, 1968, 362.

[2] H. Walther, Über die Nichtexistenz eines Knotenpunktes, durch den alle längsten Wege eines Graphen gehen, J. Comb. Theory **6** (1969) 1-6.

[3] W. Schmitz, Über längste Wege und Kreise in Graphen, Rend. Sem. Mat. Univ. Padova **53** (1975) 97-103.

2-connected graphs with Gallai's property

- In 1972, Zamfirescu [4] asked about examples with higher connectivity, and presented the first 2-connected planar graph with 82 vertices satisfying Gallai's property.
- Soon a smaller example with 32 vertices was given [5].
- In 1996, Skupień [6] found a 2-connected graph with 26 vertices satisfying Gallai's property, which is the smallest 2-connected graph so far.

[4] T. Zamfirescu, A two-connected planar graph without concurrent longest paths, J. Combin. Theory B **13** (1972) 116-121.

[5] T. Zamfirescu, On longest paths and circuits in graphs, Math. Scand. **38** (1976) 211-239.

[6] Z. Skupień, Smallest sets of longest paths with empty intersection, Combin. Probab. Comput. **5** (1996), 429 – 436.

3-connected graphs with Gallai's property

- In 1974, Grünbaum [7] presented the first 3-connected graph with 484 vertices satisfying Gallai's property.
- In 2017, one 3-connected planar graph with 156 vertices satisfying Gallai's property was given. [8].

[7] B. Grünbaum, Vertices missed by longest paths or circuits, J. Combin. Theory A, **17** (1974) 31-38.

[8] M. Jooyandeh, B. D. McKay, P. R. J. Östergård, V. H. Pettersson, C. T. Zamfirescu, J. Graph Theory **84** (2017) 121 – 33.

Lattice graphs with Gallai's property

- Impulses coming from fault-tolerant designs in computer networks motivated studying Gallai's property with respect to finite graphs embeddable in lattices.
- Nadeem, Shabbir and Zamfirescu [9] considered the family of all graphs embeddable in planar lattice or regular hexagonal lattice graphs, and found that Gallai's question again receives a positive answer.
- And the embeddings in cubic lattice [10] and regular triangular lattice [11] have also been studied.

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[9] F. Nadeem, A. Shabbir, and T. Zamfirescu, Planar lattice graphs with Gallai's property, Graphs Combin. **29** (2013) 1523-1529.

[10] Y. Bashir, T. Zamfirescu, Lattice graphs with Gallai's property, Bull. Math. Soc. Sci. Math. Roumanie **56** (2013) 65-71.

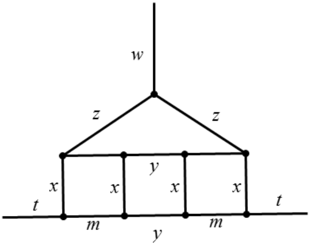
[11] A. D. Jumani and T. Zamfirescu, On longest paths in triangular lattice graphs, Util. Math. **89** (2012) 269-273.

Results

Archimedean tiling graphs	Connectivity=1	Connectivity=2
$(3^4.6)$	62	152
$(3^3.4^2)$	46	110
$(3^2.4.3.4)$	48	110
$(3.6.3.6)$	92	270
$(3.4.6.4)$	100	220
(4.8^2)	166	511
$(4.6.12)$	207	541
(3.12^2)	191	499

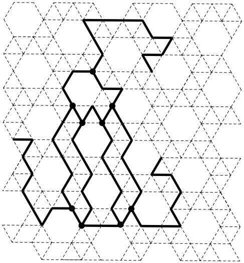
Embeddings of graphs with connectivity 1

Let G be a graph homeomorphic to the graph $G^{'}$ in the following figure. For each edge of G' the corresponding path of G has a number of vertices of degree 1 and 2, denoted by x, y, z, t, w, m respectively.

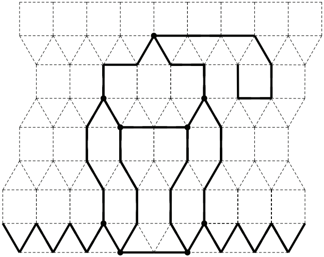


Lemma The longest paths of G have empty intersection if $0 \leq m \leq \min\{y, z\}$, $2x \geq y + 2z + 1$, $t \geq y + 2z - m + 1$, $t \geq x + z + 1$, $t \geq y + m + 1$, and $w = x + t - z$.

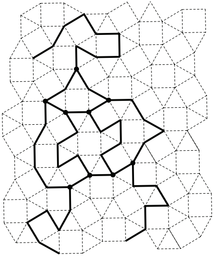
Embeddings of graphs with connectivity 1



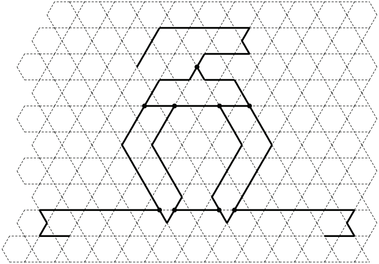
$(3^4.6)$ 62



$(3^3.4^2)$ 46

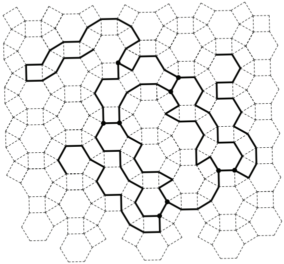


$(3^2.4.3.4)$ 48

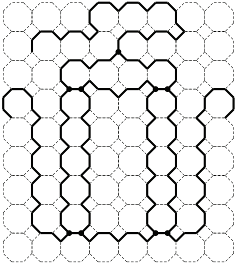


$(3.6.3.6)$ 92

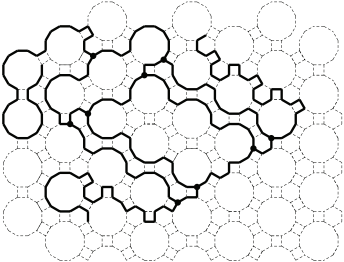
Embeddings of graphs with connectivity 1



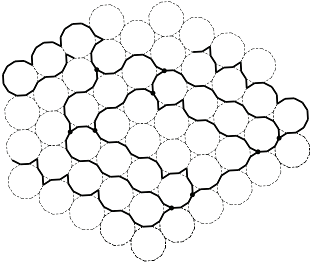
(3.4.6.4) 100



(4.8^2) 166



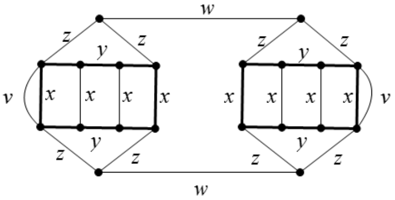
(4.6.12) 207



(3.12^2) 191

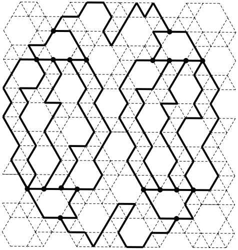
Embeddings of graphs with connectivity 2

Let H be a graph homeomorphic to H', depicted in the following figure, where the letters indicate the numbers of consecutive vertices of degree 2.

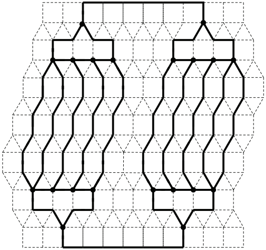


Lemma Let $x \geq v$. The longest paths of H have empty intersection if the following conditions are fulfilled.
(i) $v \geq y + 2z + 1$,
$(ii) x + v = 2z + w + 1$.

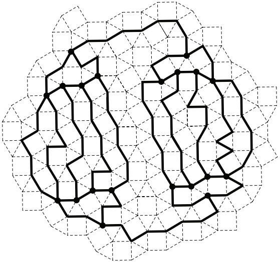
Embeddings of graphs with connectivity 2



$(3^4.6)$ 152

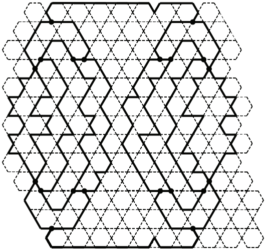


$(3^3.4^2)$ 110

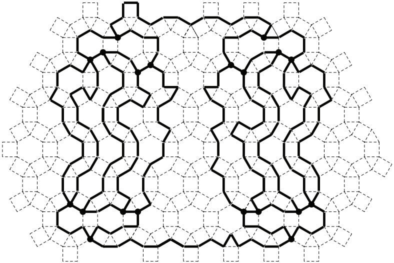


$(3^2.4.3.4)$ 110

Embeddings of graphs with connectivity 2



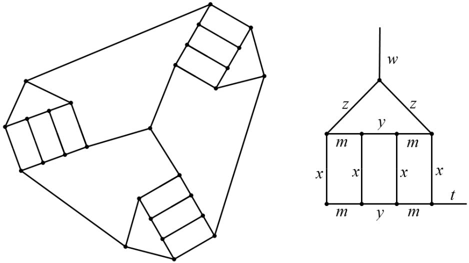
(3.6.3.6) 270



(3.4.6.4) 220

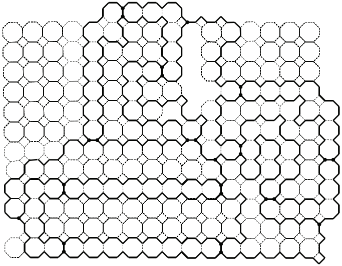
Embeddings of graphs with connectivity 2

Now consider the graph K' shown in following figure (left side), and the graph K which is homeomorphic to K', where x, y, z, t, w and m are numbers of vertices of degree 2, as before.

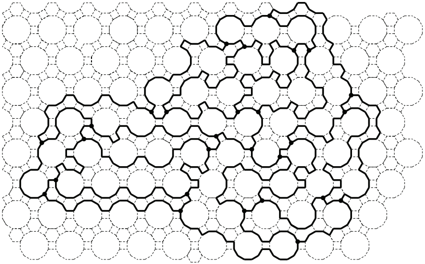


Lemma Let $x \geq v$. The longest paths of K have empty intersection if $y \geq 1$, $m \geq 1$ and $x = y + z - m \geq w = y + 2t - m + 1$.

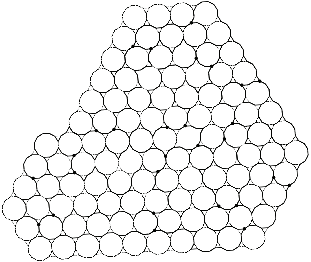
Embeddings of graphs with connectivity 2



(4.8^2) 511



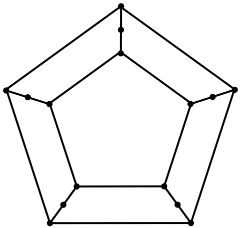
$(4.6.12)$ 541



(3.12^2) 499

Gallai's property about longest cycles

- For cycles instead of paths, Gallai's property means that all longest cycles have empty intersection.
- The first planar example, having 105 vertices and connectivity 2, was found by Walther [12] in 1969.
- Later on, Thomassen found an example with 15 vertices, denoted by G', as shown in the following figure.



[12] H. Walther, Über die Nichtexistenz eines Knotenpunktes, durch den alle längsten Wege eines Graphen gehen, J. Comb. Theory **6** (1969) 1-6.

Gallai's property about longest cycles

- In 1974, Grünbaum [13] presented a 3-connected planar graph with 124 vertices satisfying Gallai's property.
- 105 (Thomassen, 1976);
  57 (Hatzel, 1979);
  48 (Zamfirescu et al., 2007);
  42 (Araya et al., 2011)
- In 2017, some 3-connected plane graphs with 40 vertices satisfying Gallai's property were given. [14].

[13] B. Grünbaum, Vertices missed by longest paths or circuits, J. Comb. Theory A **17**, (1974) 31-38.

[14] M. Jooyandeh, B. D. McKay, P. R. J. Östergård, V. H. Pettersson, C. T. Zamfirescu, J. Graph Theory **84** (2017) 121 – 33.

Lattice graphs with Gallai's property

- The first example satisfying Gallai's property in a lattice is due to Menke [15], who found a graph in the square lattice of order 95.
- In 2013, Nadeem, Shabbir and Zamfirescu [16] found a subgraph of the regular hexagonal lattice satisfying Gallai's property, of order 170.
- An example with 60 vertices in the triangular lattice was presented by Shabbir and Zamfirescu [17] in 2016.

[15] B. Menke, On longest cycles in grid graphs, Studia Sci. Math. Hung. **36** (2000), 201-230.

[16] F. Nadeem, A. Shabbir, T. Zamfirescu, Planar lattice graphs with Gallai's property, Graphs Combin. **29** (2013) 1523-1529.

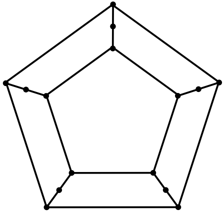
[17] A. Shabbir, T. Zamfirescu, Fault-tolerant designs in triangular lattice networks, Appl. Anal. **10** (2016), 447-456.

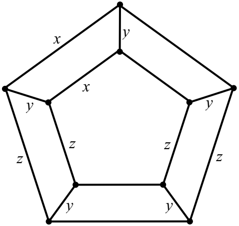
Results

Archimedean tiling graphs	connectivity=2
$(3^4.6)$	52
$(3^3.4^2)$	35
$(3^2.4.3.4)$	53
$(3.6.3.6)$	65
$(3.4.6.4)$	58
(4.8^2)	98
$(4.6.12)$	130
(3.12^2)	188

Embeddings of graphs with connectivity 2

Let G be a graph homeomorphic to the graph G' in the left figure. For each edge of G', the corresponding path of G has a number of vertices of degree 2, denoted by x, y, z, respectively, see the right figure.

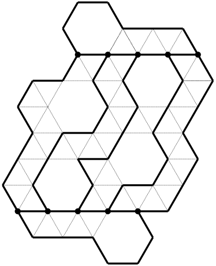




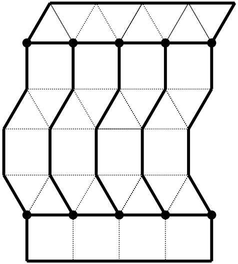
Lemma [18] The longest cycles of G have empty intersection if and only if $2y \geqslant x + 2z + 1$.

[18] A. Dino, C. T. Zamfirescu, T. I. Zamfirescu, Lattice graphs with non-concurrent longest cycles, Rend. Semin. Mat. Univ. Padova, **132** (2014), 75-82.

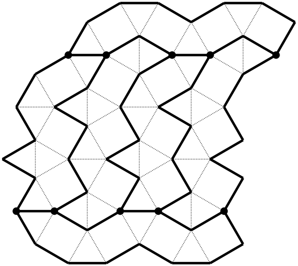
Embeddings of graphs with connectivity 2



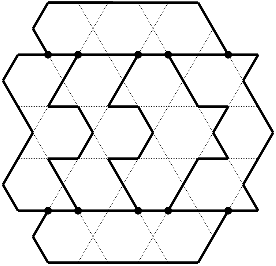
$(3^4.6)$ 52



$(3^3.4^2)$ 35

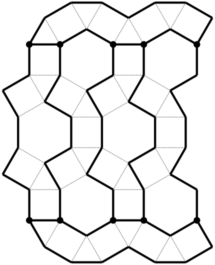


$(3^2.4.3.4)$ 53

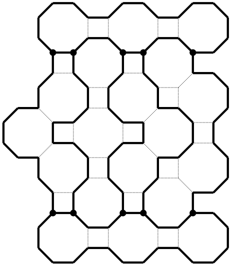


$(3.6.3.6)$ 65

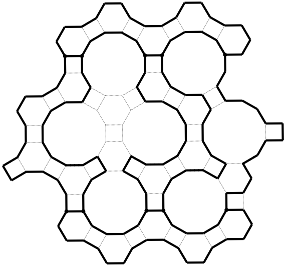
Embeddings of graphs with connectivity 2



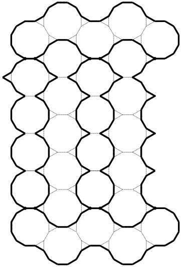
(3.4.6.4) 58



(4.8^2) 98



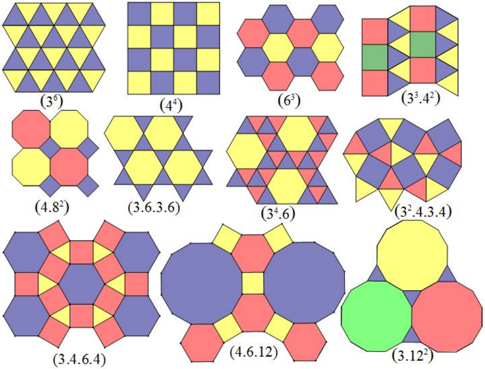
(4.6.12) 130



(3.12^2) 188

What's next?

- How about graphs with connectivity 3?



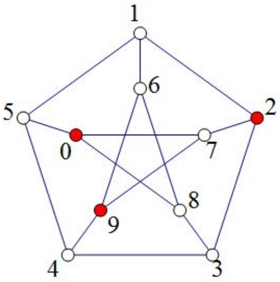
Part II. Dominating sets in Archimedean tiling graphs

Notation

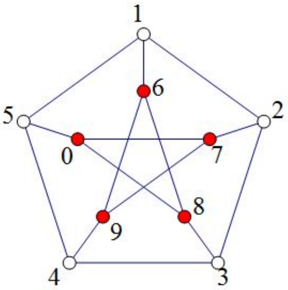
- Let $G = (V, E)$ be a graph, where V and E are vertex set and edge set of G, respectively.
- $N(v)$ denotes the open neighborhood of v in G, which is defined as $N_G(v) = \{x \in V : [vx] \in E\}$.
- $N_G[v] = N_G(v) \cup \{v\}$ is the closed neighborhood of v in G.
- The k-neighborhood of u in G is defined as $N_G^k[u] = \{x \in V : d(u,x) \leqslant k\}$, the set of vertices at distance at most k from u.
- We regard each vertex s in a graph G as a possible location for a monitoring that can monitor each vertex in its closed neighborhood $N[s]$ (or open neighborhood $N(s)$).

Dominating sets and open-dominating sets

A subset S of V is a dominating set (respectively, open-dominating set), abbreviated DS (respectively, ODS), of G if every vertex in the graph can be contained in the closed neighbor (respectively, the open neighbor) of a vertex in S, which means $\bigcup_{s \in S} N[s] = V$ (respectively, $\bigcup_{s \in S} N(s) = V$).



DS



ODS

Concepts involving domination

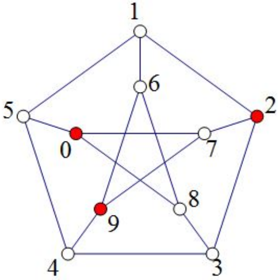
- Books written by Haynes, Hedetniemi, Slater [19] [20] studied the domination in graphs extensively.
- There are scores of graph-theoretic concepts involving domination, such as domination, independent domination, connected domination, total domination, locating-domination, paired-domination, and so forth.

[19] T. W. Haynes, S. T. Hedetniemi, P. J. Slater. *Fundamentals of Domination in Graphs*, Marcel Dekker, New York (1998).

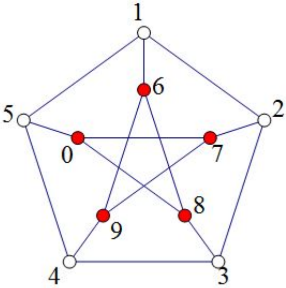
[20] T. W. Haynes, S. T. Hedetniemi, P. J. Slater. *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York (1998)

locating-dominating sets

A dominating set S of a graph G is called locating, if for any two distinct vertices $u, v \in V \setminus S$, $N(u) \cap S \neq N(v) \cap S$, which means that u, v do not have the same set of dominating vertices.



not locating



locating

locating-dominating sets

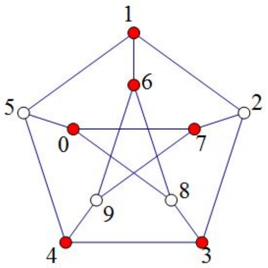
- Locating-dominating sets were introduced by Colbourn, Slater and Stewart [21].
- Charon, Hudry, Lobstein [22] proved that, given a graph G, the decision problem of the existence of a locating-dominating set of size at most k in G is NP-complete.
- On the other hand, many special graphs such as paths, cycles, trees etc. and (3^6), (4^4) have been investigated.

[21] C. J. Colbourn, P. J. Slater, L. K. Stewart. Locating-dominating sets in series-parallel networks. *Congr. Numer.*, 56 (1987) 135-162.

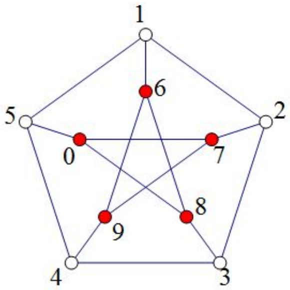
[22] I. Charon, O. Hudry, A. Lobstein. Minimizing the size of an identifying or locating dominating code in a graph is NP-hard. *Theoret. Comput. Sci.*, 290 (2003) 2109-2120.

Paired-dominating sets

A dominating set S of G is a paired-dominating set, denoted as PDS, if its induced subgraph $G[S]$ contains at least one perfect matching.



PDS



not PDS

Paired-dominating sets

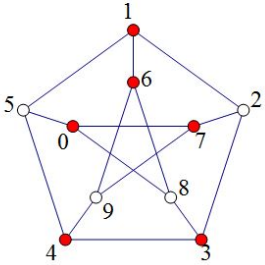
- The concept of paired-domination in graphs was introduced by Haynes and Slater [23] [24].
- Paired-domination is the model from the following actual problem: place monitoring devices in a system such that every site in the system (including the monitoring devices themselves) is adjacent to a monitor and every monitor is paired with a backup monitor.

[23] T. W. Haynes, P. J. Slater. Paired-domination and the paired-domatic number. *Congr. Numer.*, 109 (1995) 65-72.

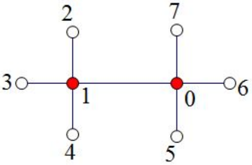
[24] T. W. Haynes, P. J. Slater. Paired-domination in graphs. *Networks*, 32 (1998) 199-206.

Locating-paired-dominating sets

A set $S \subset V$ is a locating-paired-dominating set, shorted for LPDS, of G if S is a PDS and for any two distinct vertices $u, v \in V \setminus S$, $N(u) \cap S \neq N(v) \cap S$.



LPDS



not LPDS

Locating-paired-dominating sets

- The concept of locating-paired-dominating sets was introduced by McCoy and Henning [25] as an extension of paired-dominating sets.
- The location of monitoring devices in a system when every monitor is paired with a backup monitor serves as the motivation for this concept.
- Niepel [26] studied the locating-paired-dominating sets in (4^4).

[25] J. McCoy, M. A. Henning. Locating and paired-dominating sets in graphs [J]. *Discrete Appl. Math.*, 157 (2009) 3268-3280.

[26] Ľ. Niepel. Locating-paired-dominating sets in square grids. *Discrete Math.*, 338 (2015) 1699-1705.

Density

- Let S be a DS of a graph G. The density of S, denoted by $D(S)$, in graph G is defined as

$$D(S) = \frac{|S|}{|V|}.$$

- It is possible to generalize the concept of density of a set to infinite local finite graphs. The density of $S \subset V$ in V is defined to be

$$D(S) = \limsup_{k\to\infty} \frac{|S \cap N_G^k[u]|}{|N_G^k[u]|}.$$

Density

- In 2002, Slater [27] proved the density of the optimal LDS in (4^4) is $3/10$.
- Honkala [28] [29] studied the optimal LDS in (3^6) and (6^3).
- Niepel [30] studied the LPDS in (4^4), and proved the optimal density is $1/3$.

[27] P. J. Slater. Fault-tolerant locating-dominating sets. it Discrete Math., 2002, 249 (1): 179-189.

[28] I. Honkala Ȧn optimal locating-dominating set in the infinite triangular grid. *Discrete Math.*, 2006, 306 (21): 2670 – 2681.

[29] I. Honkala , T. Laihonen. On locating-dominating sets in infinite grids. *European J. Combin.*, 2006, 27 (2): 218 – 227.

[30] Ľ. Niepel. Locating-paired-dominating sets in square grids. *Discrete Math.*, 338 (2015) 1699-1705.

Density of LPDS

Lemma [31] Let S be a LPDS in a graph G with maximum degree Δ, then each vertex in S has share at most $\frac{\Delta+2}{2}$ and $D(S) \geqslant \frac{2}{\Delta+2}$.

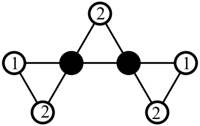
[31] Ľ. Niepel. Locating-paired-dominating sets in square grids. *Discrete Math.*, 338 (2015) 1699-1705.

Density of the optimal LPDS in Archimedean tiling graphs

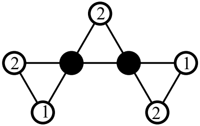
Archimedean tiling graphs	maximum degrees	densities of LPDS
$(3.6.3.6)$	4	$1/3$
(4.8^2)	3	$2/5$
$(3.4.6.4)$	4	$1/3$
$(4.6.12)$	3	$[2/5, 5/12]$
(3.12^2)	3	$[2/5, 4/9]$
$(3^3.4^2)$	5	$[2/7, 1/3]$
$(3^2.4.3.4)$	5	$[2/7, 1/3]$
$(3^4.6)$	5	$[2/7, 1/3]$

The optimal LPDS in $(3.6.3.6)$

Lemma Let S be an optimal LPDS of $(3.6.3.6)$ with perfect matching M, then every edge of M induces a pattern of type A or type B.



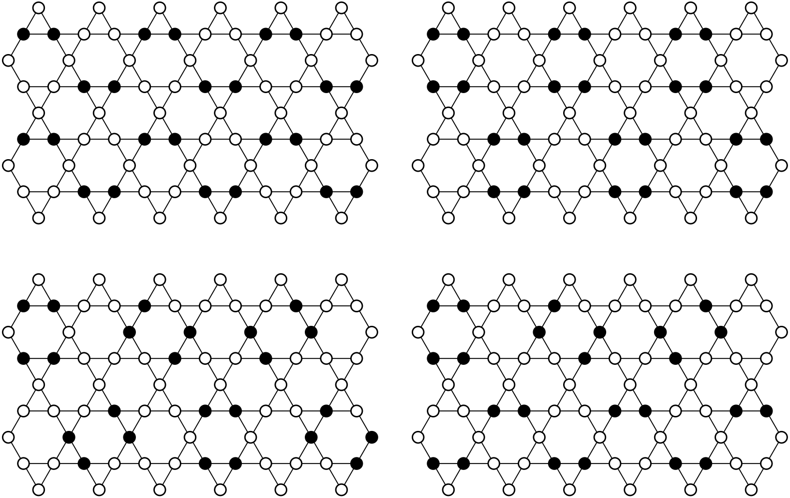
Type A



Type B

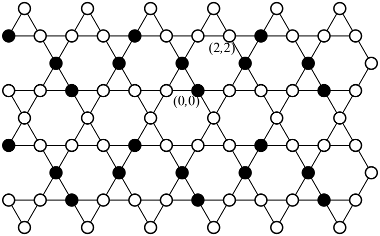
The optimal LPDS in $(3.6.3.6)$

Some examples of the optimal LPDS in $(3.6.3.6)$ with all patterns of type A.



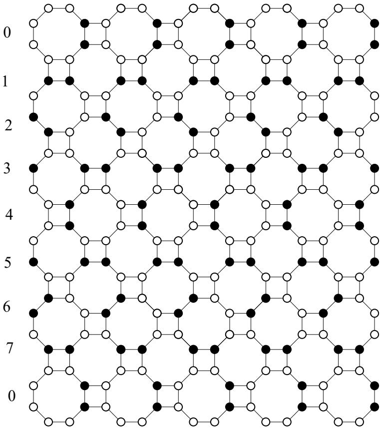
The optimal LPDS in $(3.6.3.6)$

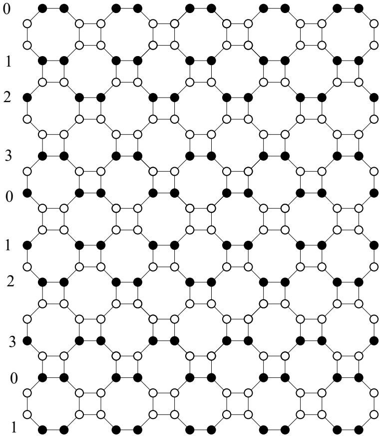
Some examples of the optimal LPDS in $(3.6.3.6)$ with all patterns of type B.



The optimal LPDS in (4.8^2)

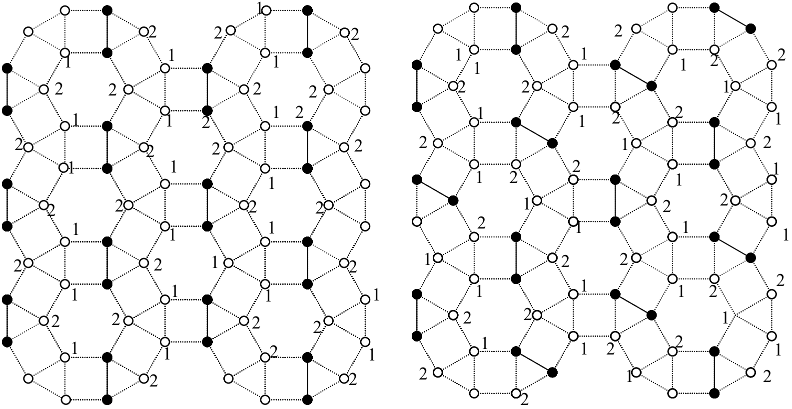
Some examples of the optimal LPDS in (4.8^2).



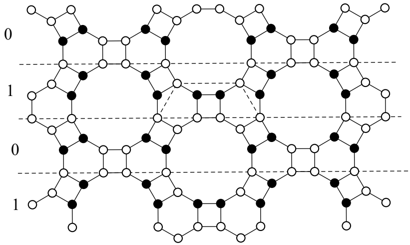


The optimal LPDS in $(3.4.6.4)$

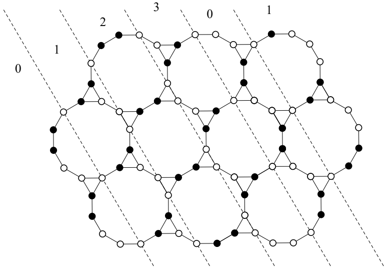
Some examples of optimal LPDS in $(3.4.6.4)$.



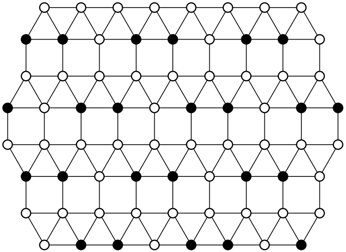
Examples of LPDS in the other graphs and their densitiesy



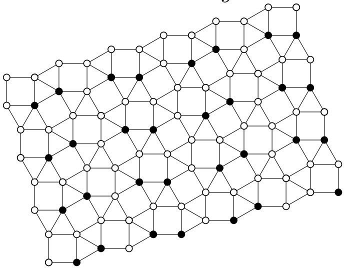
$(4.6.12)\quad \frac{5}{12}$



$(3.12^2)\quad \frac{4}{9}$

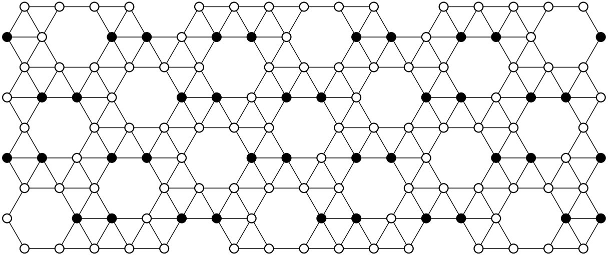


$(3^3.4^2)\quad \frac{1}{3}$



$(3^2.4.3.4)\quad \frac{1}{3}$

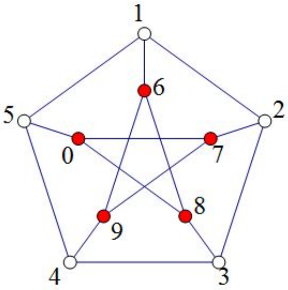
Examples of LPDS in the other graphs and their densities



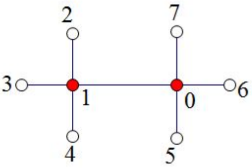
$(3^4.6)$ $\frac{1}{3}$

Open-locating-dominating sets

A set $S \subset V$ is an open-locating-dominating set, shorted for OLDS, of graph G if S is an ODS, and for any distinct vertices $u, v \in V$, $N(v) \cap S \neq N(u) \cap S$.



OLDS



not OLDS

Open locating dominating sets

- The concept of $OLDS$ was introduced by Seo and Slater [32] as a method by which one could identify the location of an event at a vertex where a vertex in the set can detect events at adjacent vertex, but cannot detect an event at itself.

[32] S. J. Seo, P. J. Slater, Open neighborhood locating-dominating sets. *Australas. J. Combin.*, 2010, 46: 109 – 120.

Density of OLDS

Lemma [33] Let G be a regular graph with degree Δ, and S be an OLDS in G, then $D(S) \geqslant \frac{2}{\Delta+1}$.

[33] S. J. Seo, P. J. Slater. Open neighborhood locating-dominating sets. *Australas. J. Combin.*, 2010, 46: 109 – 120.

Density of the optimal OLDS in Archimedean tiling graphs

Archimedean tiling graphs	degrees	densities of OLDS
(4^4) [34]	4	$2/5$
(6^3)[33]	3	$1/2$
(3^6) [35]	6	$4/13$
$(3.6.3.6)$ [36]	4	$[2/5, 5/12]$

[34] S. J. Seo, P. J. Slater. Open neighborhood locating-dominating sets. *Australas. J. Combin.*, 2010, 46: 109 – 120.

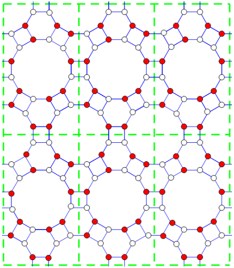
[35] R. Kincaid, A. Oldham, G. Yu. Optimal open-locating-dominating sets in infinite triangular grids. *Discrete Appl. Math.* 2015, 193: 139 – 144.

[36] D.B Sweigart, J. Presnell, R. Kincaid. An integer program for Open Locating Dominating sets and its results on the hexagon-triangle infinite grid and other graphs. *Systems and Information Engineering Design Symposium*, IEEE, 2014: 29-32.

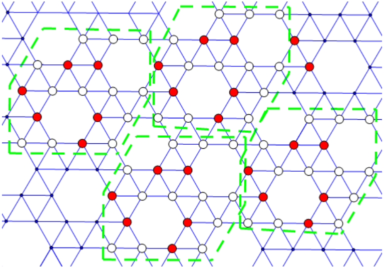
Densities of the optimal OLDS in other Archimedean tiling graphs

Archimedean tiling graphs	degrees	densities of OLDS
$(4.6.12)$	3	$1/2$
(4.8^2)	3	$1/2$
(3.12^2)	3	$1/2$
$(3^4.6)$	5	$1/3$
$(3^3.4^2)$	5	$1/3$
$(3^2.4.3.4)$	5	$1/3$
$(3.4.6.4)$	4	$[2/5, 5/12]$

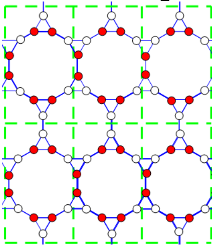
The optimal OLDS in other Archimedean tiling graphs



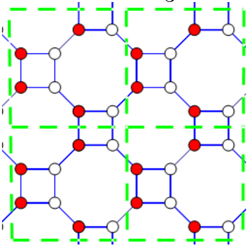
$(4.6.12)$ $\frac{1}{2}$



$(3^4.6)$ $\frac{1}{3}$

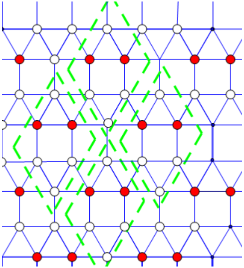


(3.12^2) $\frac{1}{2}$

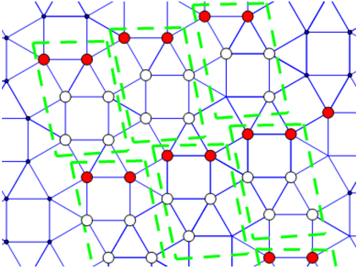


(4.8^2) $\frac{1}{2}$

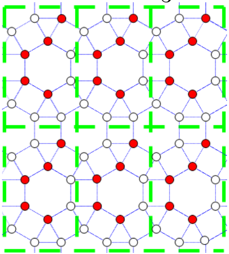
The optimal OLDS in other Archimedean tiling graphs



$(3^3.4^2)$ $\frac{1}{3}$



$(3^2.4.3.4)$ $\frac{1}{3}$



$(3.4.6.4)$ $[\frac{2}{5}, \frac{5}{12}]$

Thanks for your attention!