Induced nets and hamiltonicity of claw-free graphs

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Joint work with
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Introduction

Theorem (Dirac, 1952)

A graph of order $n \geq 3$ is hamiltonian if its minimum degree is at least $n/2$.

↓

Theorem (Matthews and Sumner, 1985)

Let G be a 2-connected claw-free graph of order n. If minimum degree of G is at least $(n-2)/3$, then G is hamiltonian.

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As this type of research progressed, some of these degree conditions became extremely complicated.

Theorem (F. and Yamashita, 2007)

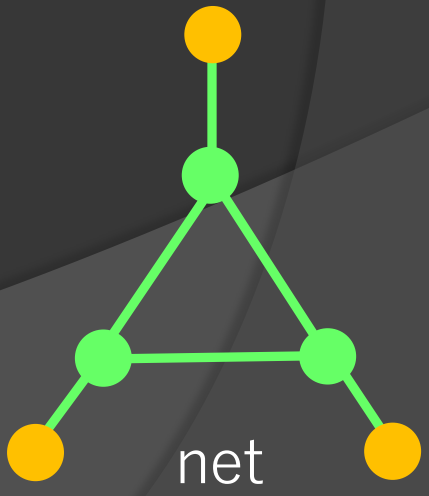
Let G be a 2-connected claw-free graph of order n. If each induced modified claw M has an end vertex x of M satisfying $d_G(x) \geq (n-2)/3$ or a pair of vertices y and z satisfying $d_G(y) + d_G(z) \geq n$, then G is hamiltonian.

However, there remains several unsolved problems that can only be stated in short, easily understandable form, and these problems still engage our interest.

Conjecture (Broersma, 1993)

Let G be a 2-connected claw-free graph of order n.
If every endvertex of each induced net of G has degree at least $(n-2)/3$, then G is hamiltonian.

: endvertices of the net



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Let G be a 2-connected claw-free graph of order n.
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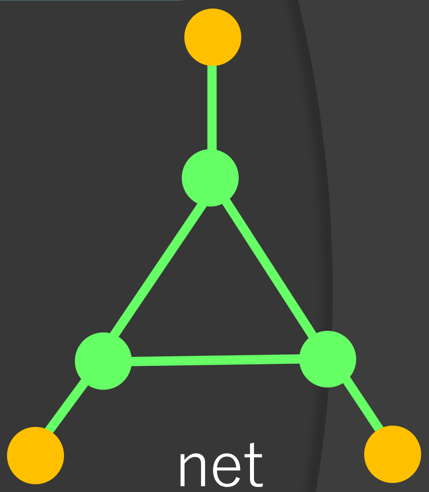
Theorem (Duffus, Gould and Jacobson, 1981)

Every 2-connected claw-free net-free
graph is hamiltonian.



net

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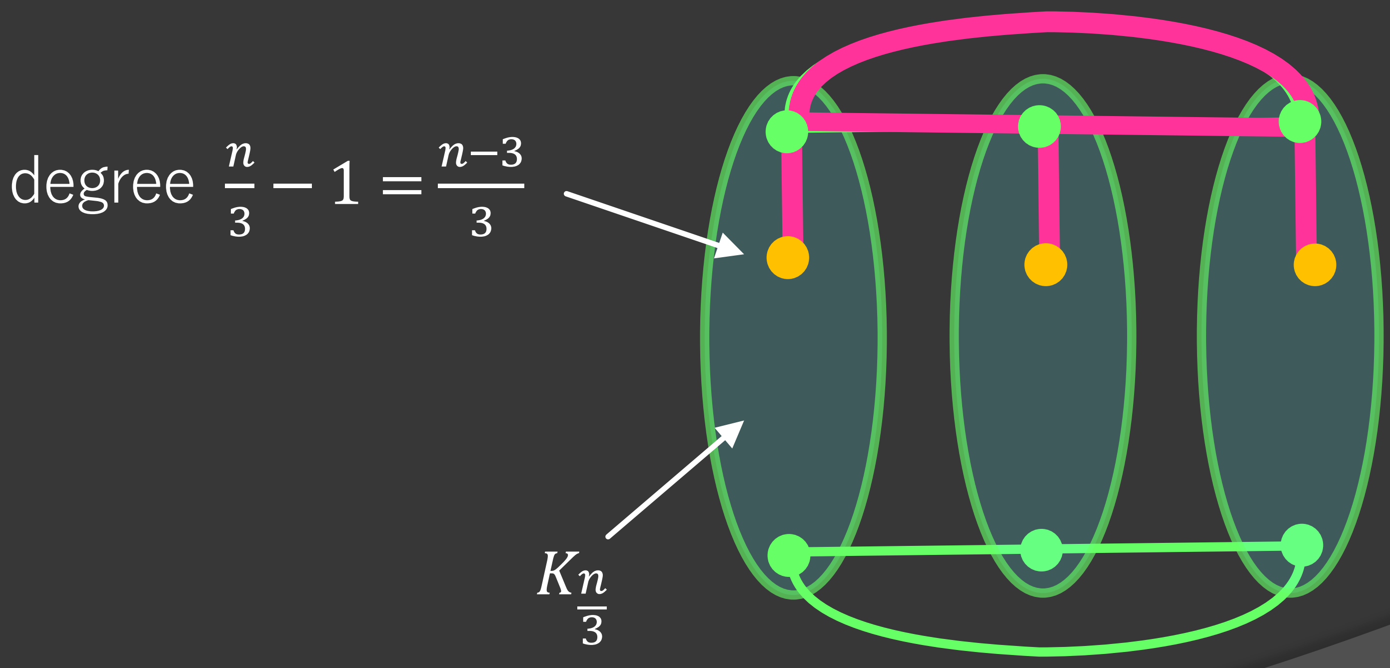

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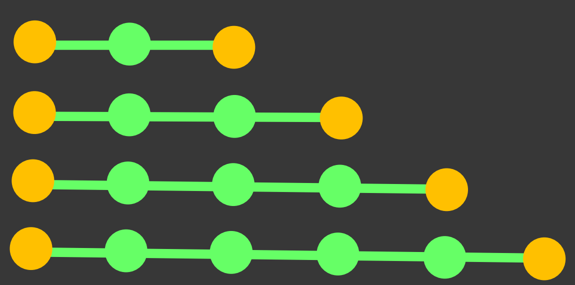
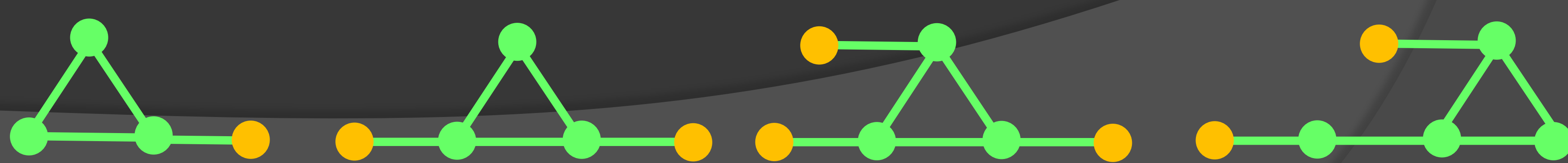
Theorem (Čada, Li, Ning and Zhang, 2016)

Let G be a 2-connected claw-free graph of order $n \geq 3$.
If every endvertex of each induced net of G has degree at least $(n+3)/3$, then G is hamiltonian.



Characterization of the graphs which yield the similar result is obtained as well.



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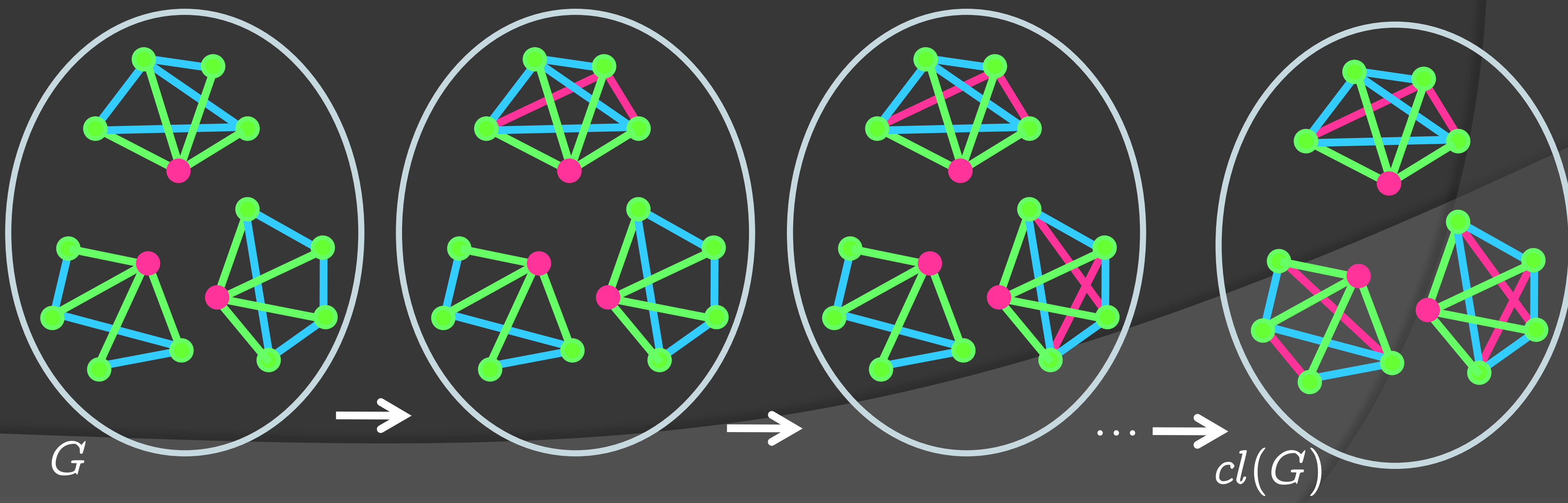
Theorem (Chiba, F.)

Broersma's conjecture is true.

Our proof relies heavily on claw-free closure.

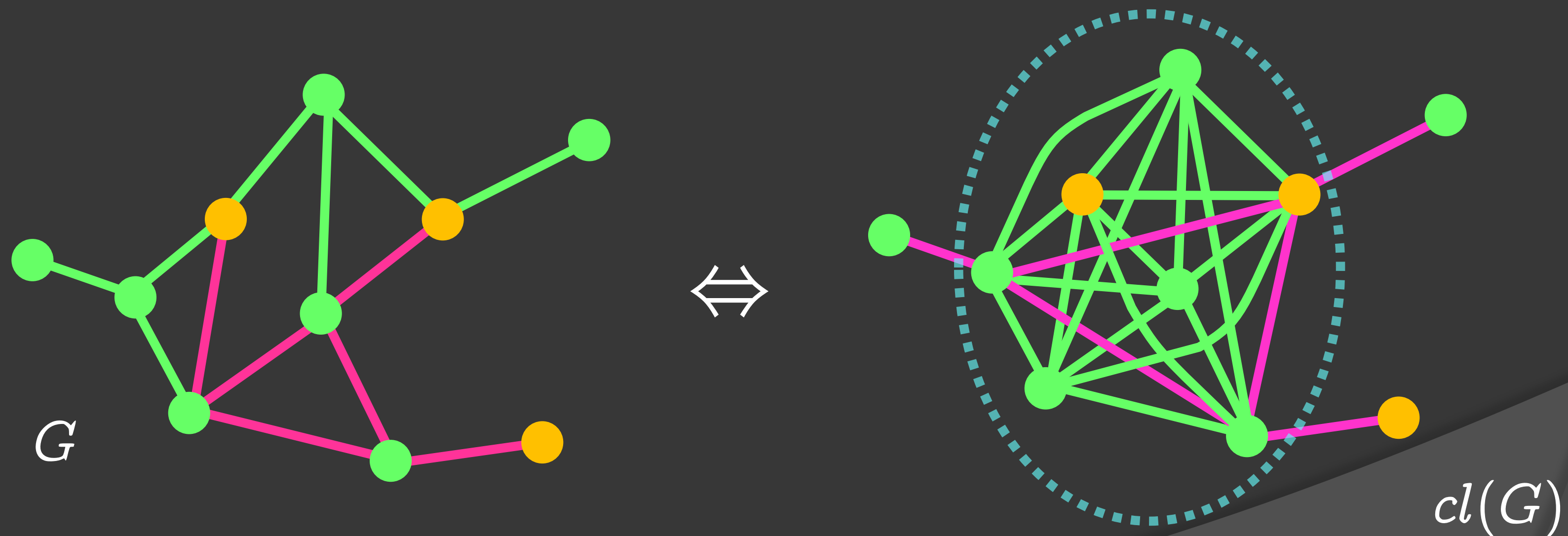


Obstacles

Theorem (Chiba, F.)

Let G be a 2-connected claw-free graph of order n.
If every endvertex of each induced net of G has degree at least $(n-2)/3$, then G is hamiltonian.

An induced net N in $cl(G)$ is not necessarily an induced net in G.

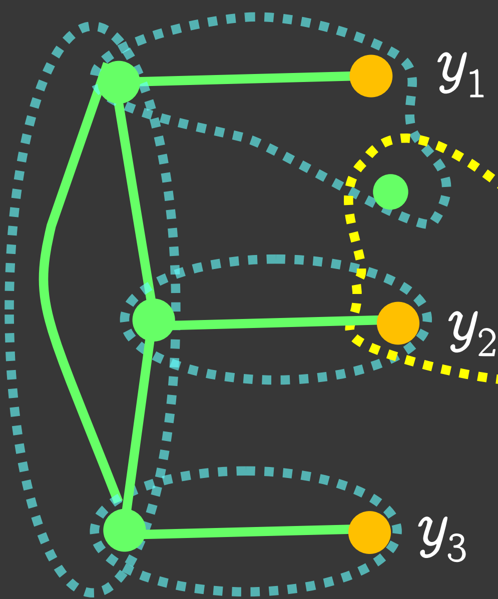


Heavy vertices may hide in a clique of $cl(G)$, so we cannot find out heavy vertices by the structure of $cl(G)$.

Theorem (Chiba, F.)

Let G be a 2-connected claw-free graph of order n.
If every endvertex of each induced net of G has degree at least $(n-2)/3$, then G is hamiltonian.

If we restrict our target to line graphs of a triangle-free graphs, we can easily find independent heavy vertices y_1, y_2 and y_3 in $cl(G)$.
But...

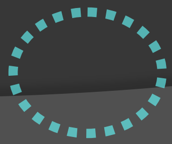


y_i and y_j may have two common neighbors.

What we obtain is

$$|N(y_1) \cup N(y_2) \cup N(y_3)| \geq 3 \cdot \frac{n-2}{3} + 3 - 2 \cdot 3 = n - 5,$$

which does not yield a contradiction immediately.

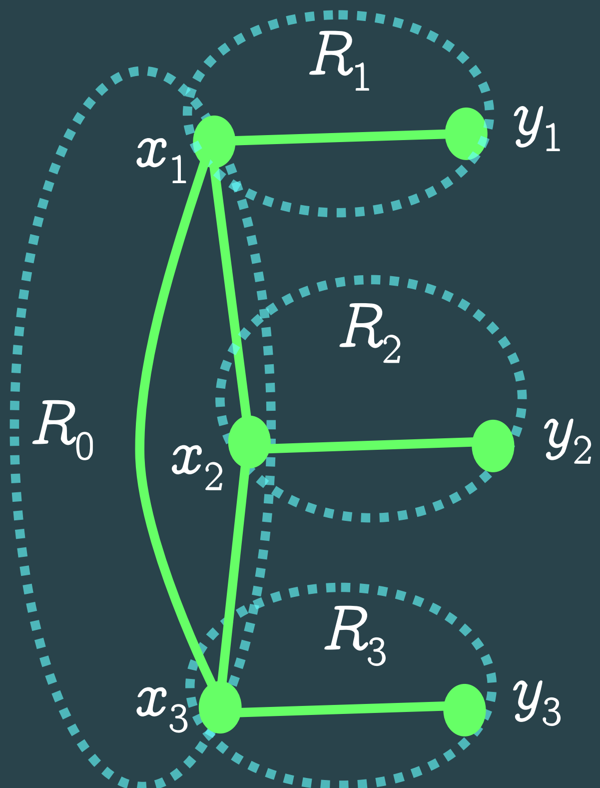


: a clique

Key Lemmas

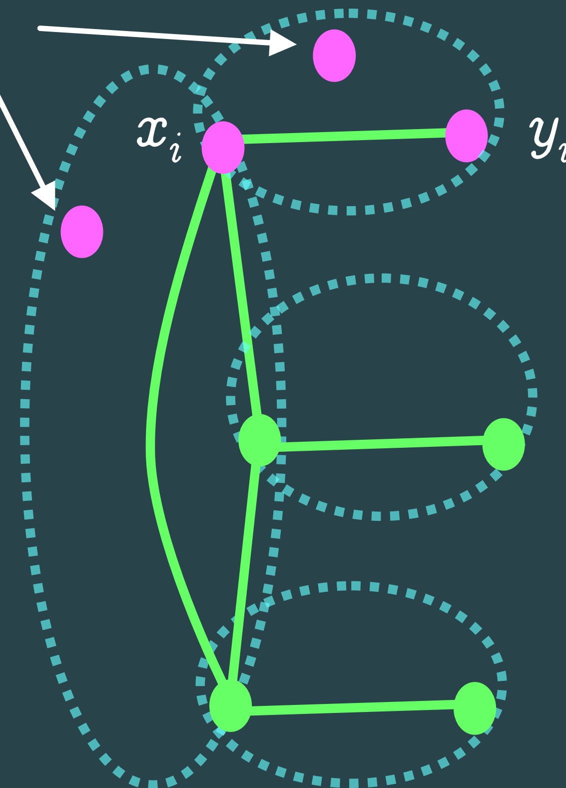
Lemma A.

(● :candidate for y_i')



$\Rightarrow$

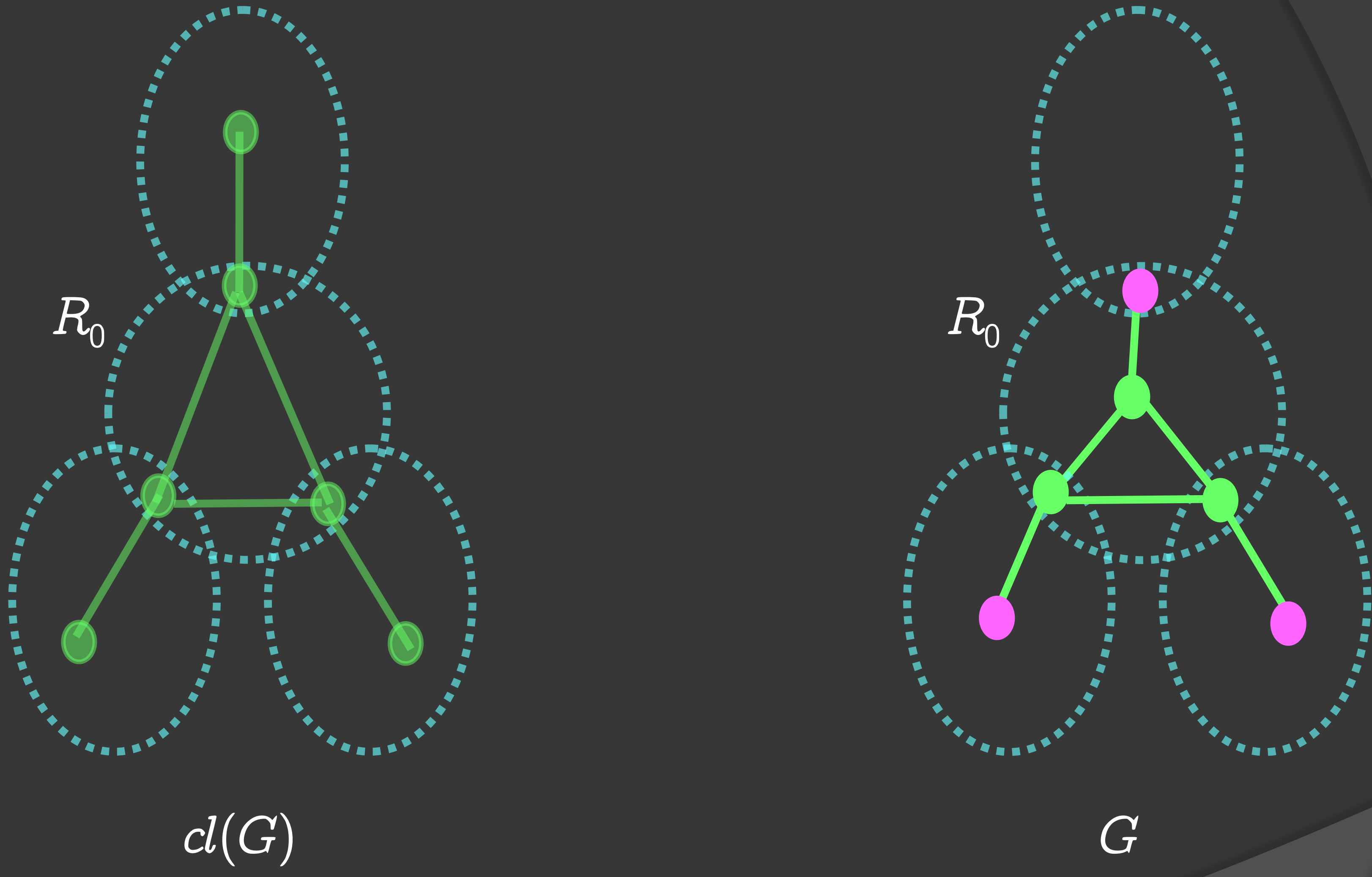
Locally connected in $cl(G)$



If there exists an induced net in $cl(G)$,

we can find an induced net of G with endvertices y_i' $(1 \leq i \leq 3)$

<observation>

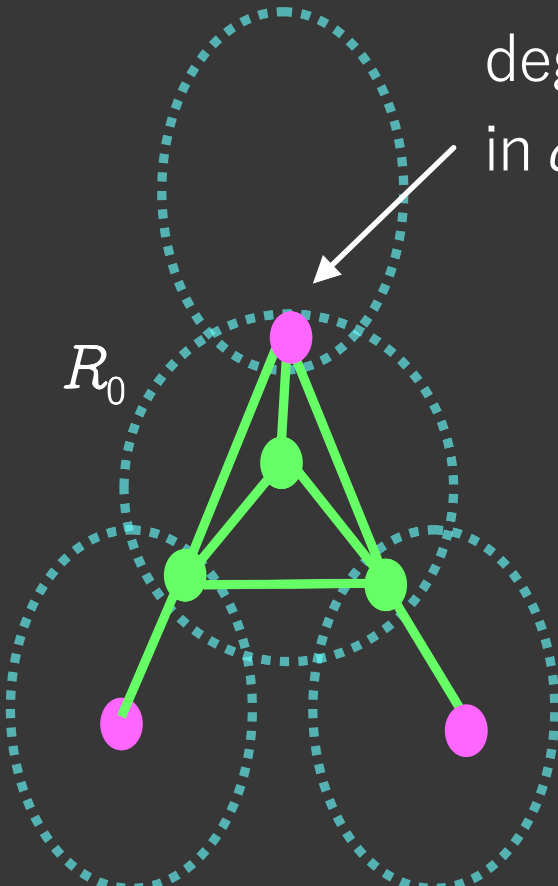


$cl(G)$ G

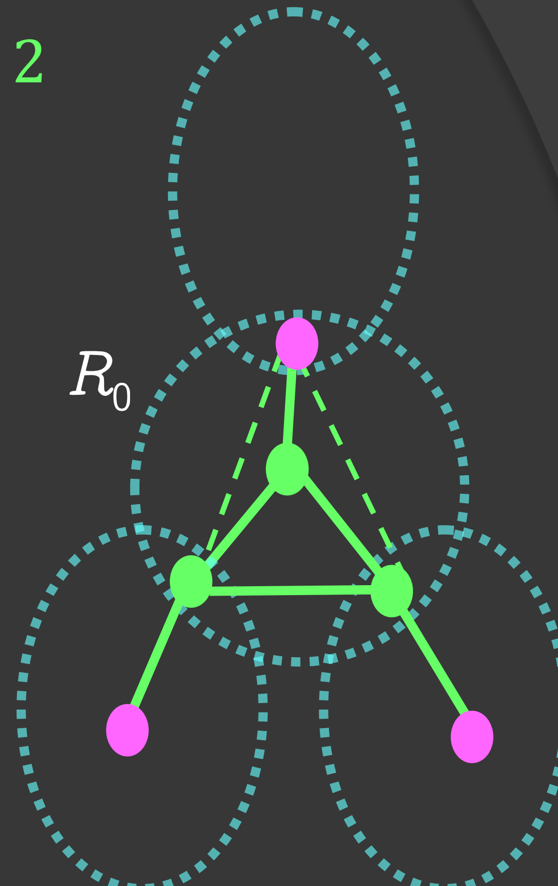
: degree at least $\frac{n-2}{3}$ in G

<observation>

degree at least $\frac{n-2}{3}+2$ in $cl(G)$



$cl(G)$



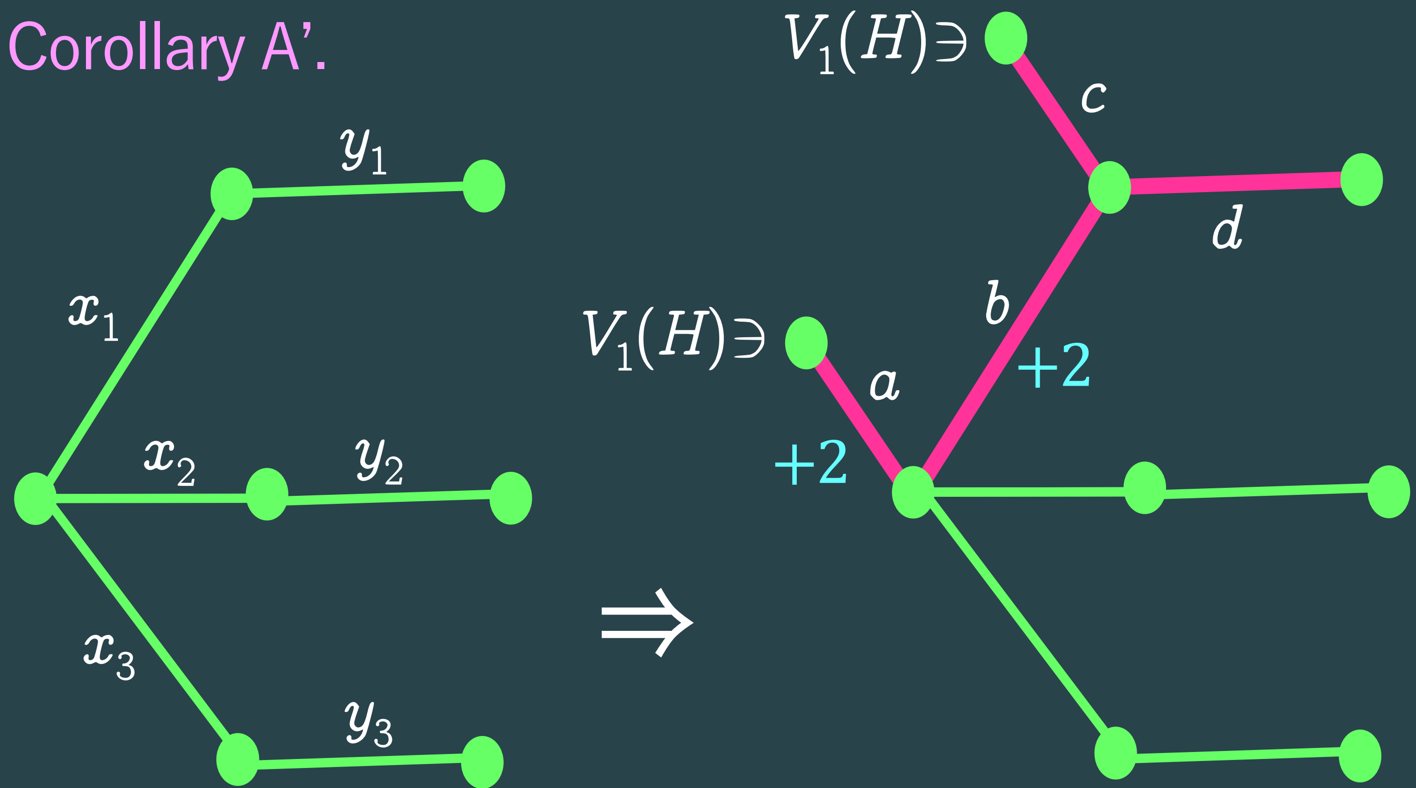
G

If an endvertex is found in R_0, then it is "extra-heavy" in $cl(G)$.

●: degree at least $\frac{n-2}{3}$ in G

In the words of the preimage H (that is, $L(H) = cl(G)$),

Corollary A'.



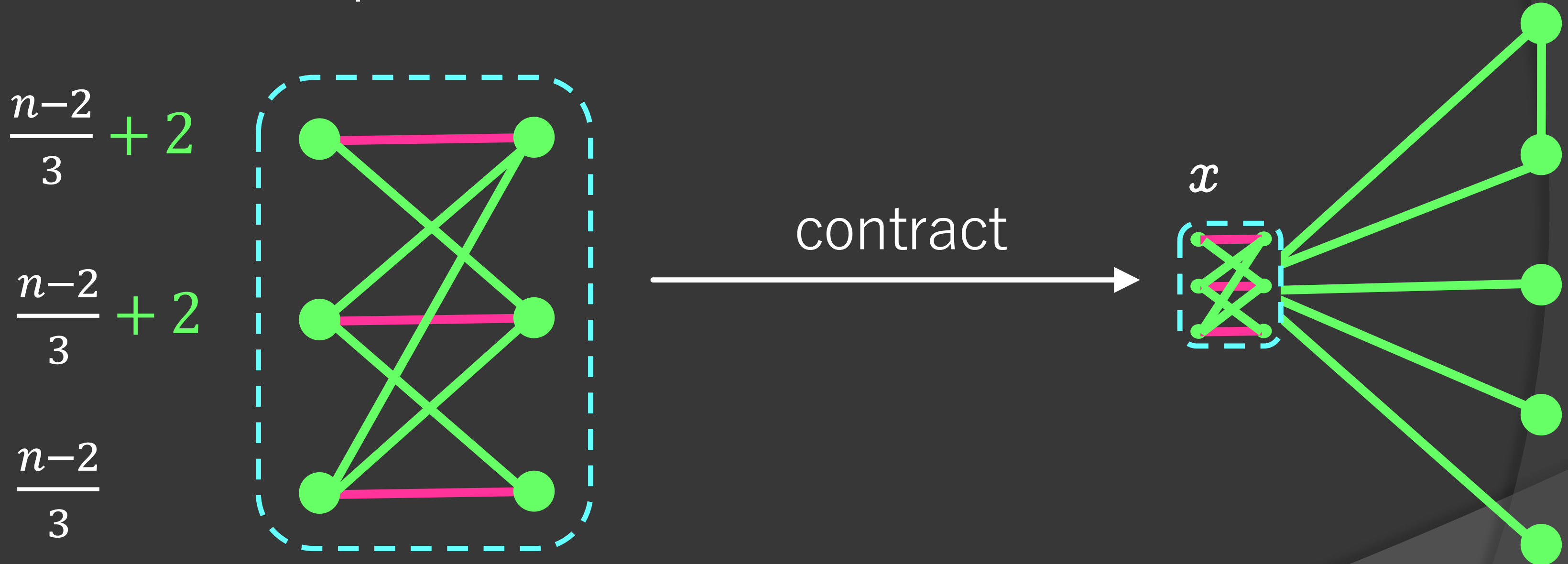
If there exists "a subdivided claw" in H,

there exists an edge y_i' with edge-degree at least $\frac{n-2}{3}$ for each i and, if $y_i' = a$ or b, then the edge-degree of y_i' is at least $\frac{n-2}{3} + 2$.

Lemma B.

Let H be an essentially 2-edge-connected triangle-free graph. If $M=\{e_1, e_2, e_3\}$ is a matching of H with edge-degree sum at least $n+2$, then H has a dominating closed trail.

<sketch of the proof>



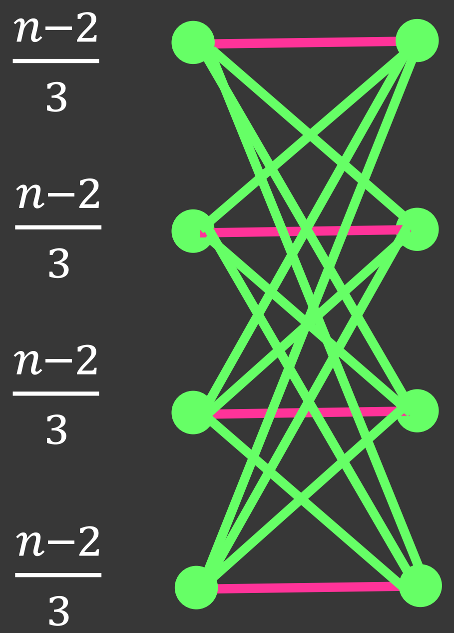
$H[V(M)]$ induces $K_{3,3}$ or $K^{-}_{3,3}$
⇒ collapsible

$H'-x$ has at most one edge
⇒ we can find a DCT.

A matching M of H is called heavy if every edge in M is heavy.

Lemma C.

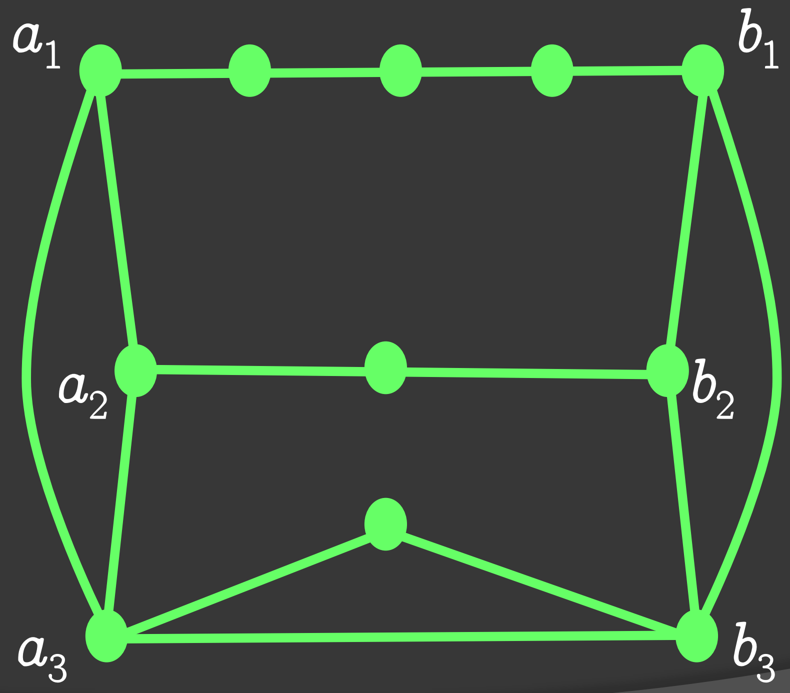
Let H be an essentially 2-edge-connected triangle-free graph with at least 33 edges. If $M=\{e_1, e_2, e_3, e_4\}$ is a matching of H, then M is not heavy.



$$n = |E(H)| \geq \frac{n-2}{3} \times 4 + 4 - 12$$
$$= n + \frac{n}{3} - \frac{32}{3} > n \quad (\text{if } n \geq 33)$$

Theorem (Brousek, 1998)

Every non-hamiltonian 2-connected claw-free graph contains an induced subgraph F such that F is obtained by taking two vertex-disjoint triangles $a_1 a_2 a_3$ and $b_1 b_2 b_3$ and by joining every pair of vertices $\{a_i, b_i\}$ by a path of length at least two or by a triangle.



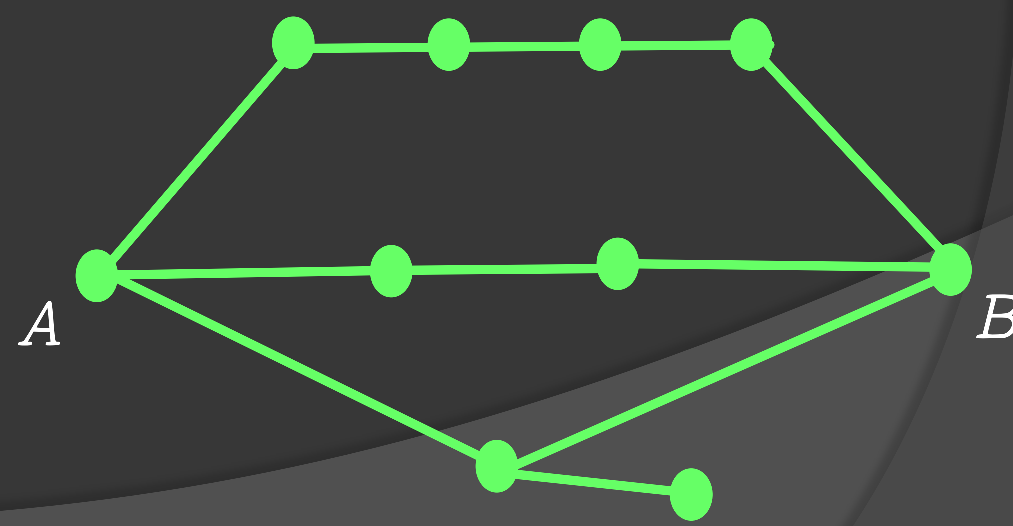
< Sketch of the proof of the main theorem in the case $n \geq 33$ >

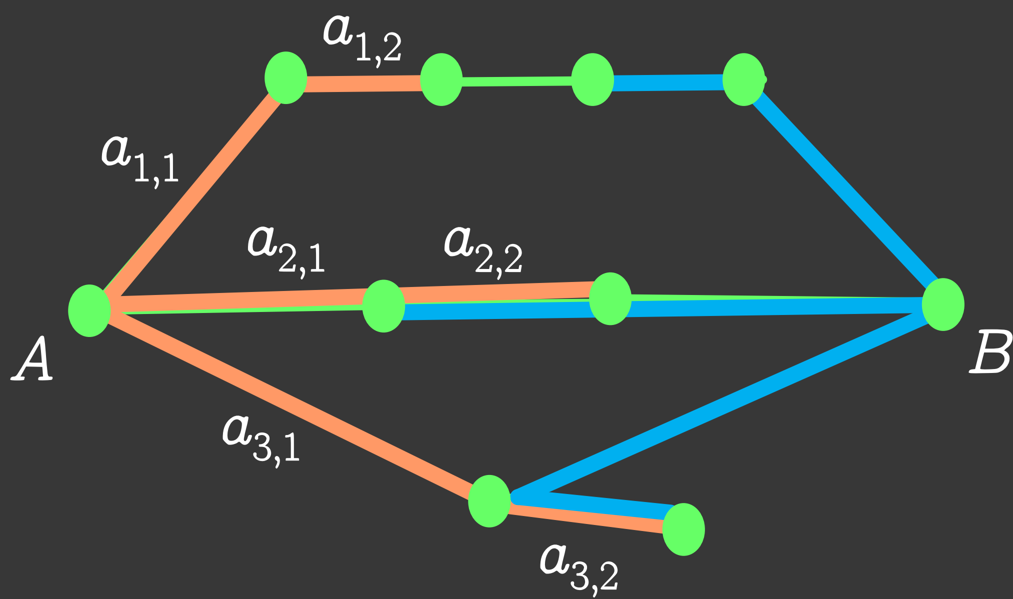
Let G be a non-hamiltonian claw-free graph satisfying the assumptions and let H be a triangle-free graph such that $L(H) = cl(G)$.

By Lemmas B and C, it suffices to prove the existence of

- a heavy matching of size 4 or
- a matching of size 3 with edge-degree sum $\geq n+2$.

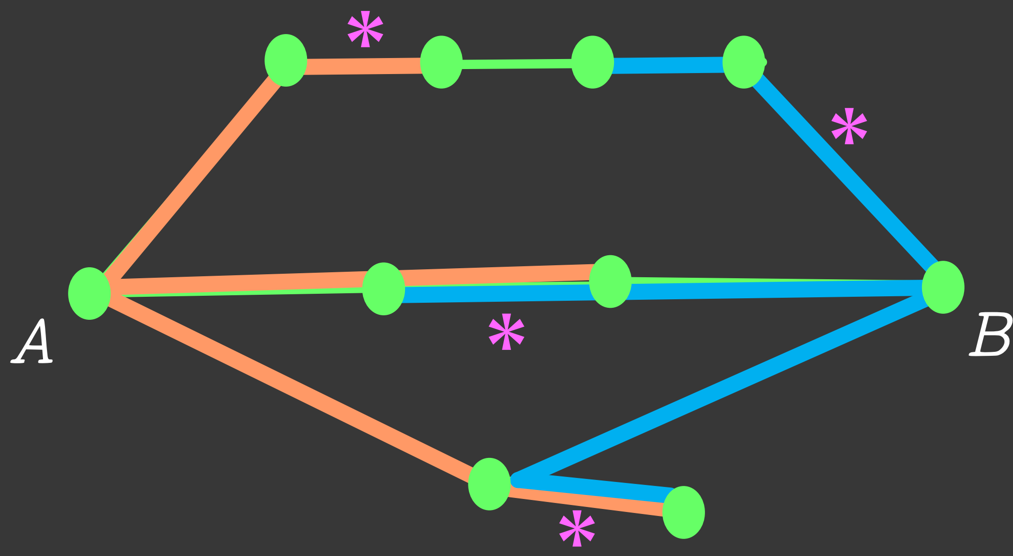
By Brousek's Theorem, H contains the following structure as a subgraph.





By Corollary A', either $a_{i,1}$ or $a_{i,2}$ is heavy for each i (and similar things happen in the B-side).

We need case analysis.



Heavy matching of size 4

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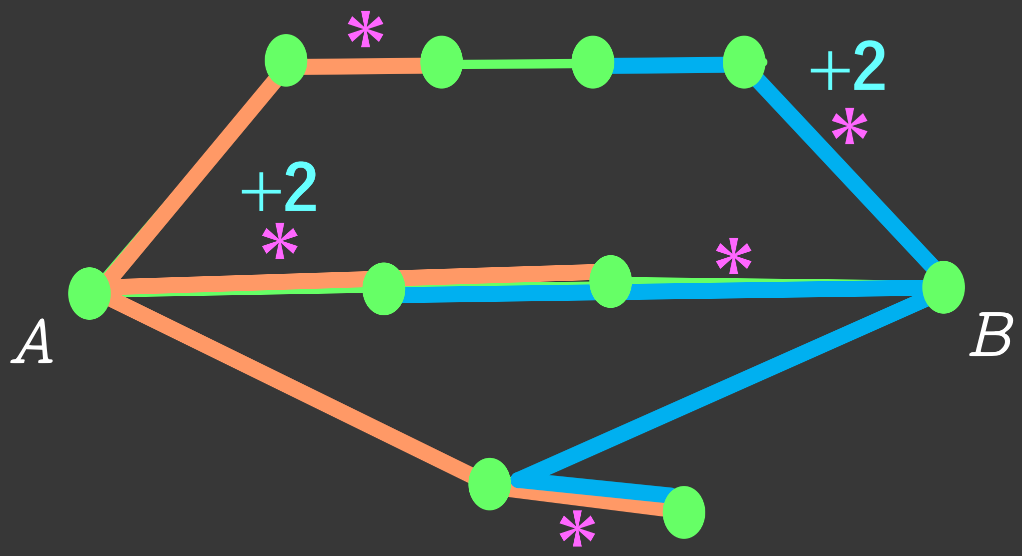

A matching of size 3 with edge-degree sum $\geq n+2$.

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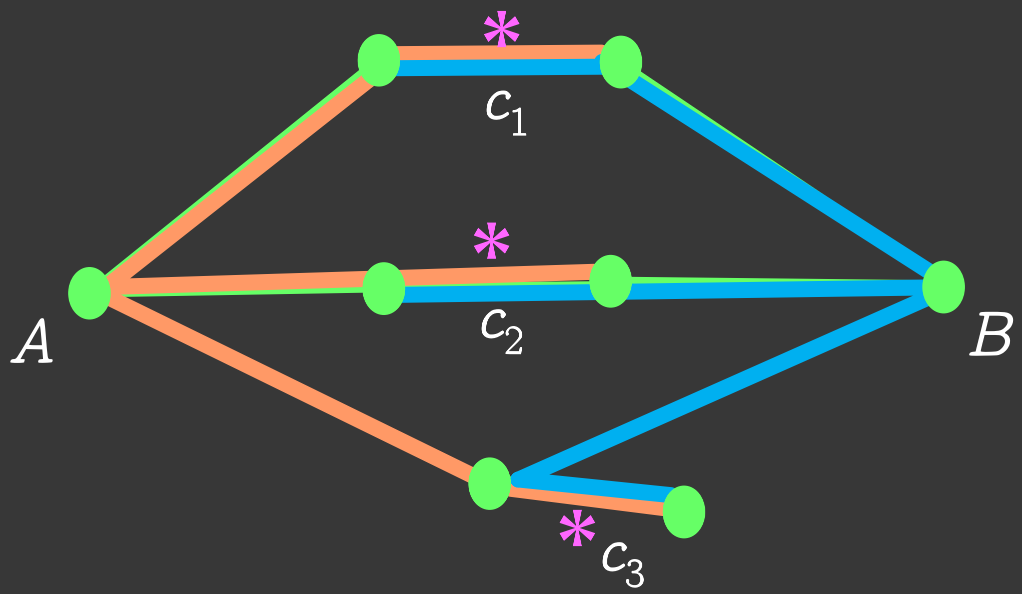
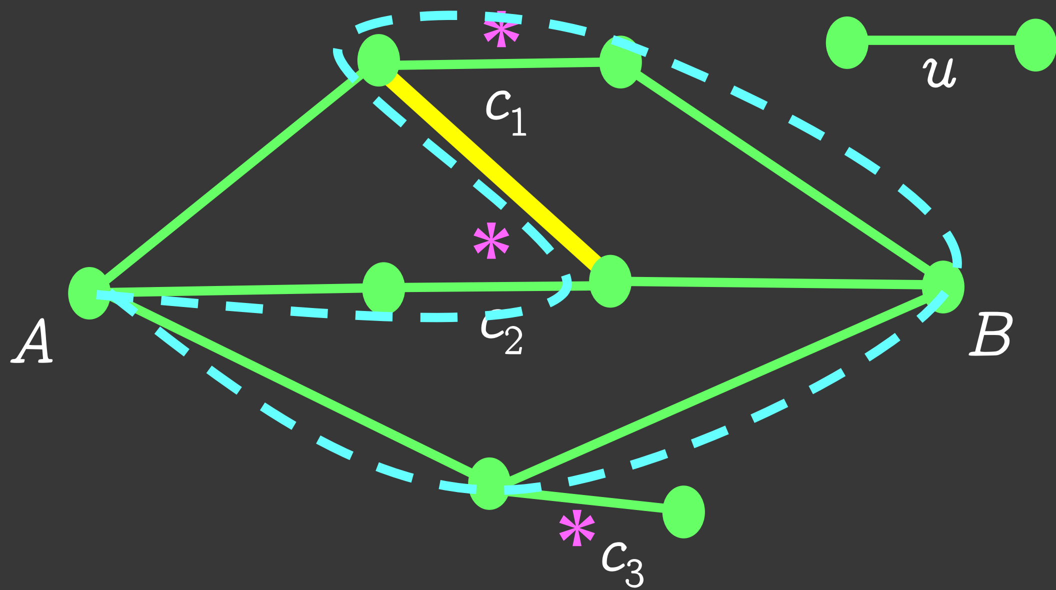


The worst case is that each (A,B)-path is short and heavy edges we can find is not around the center of the subdivided claw.

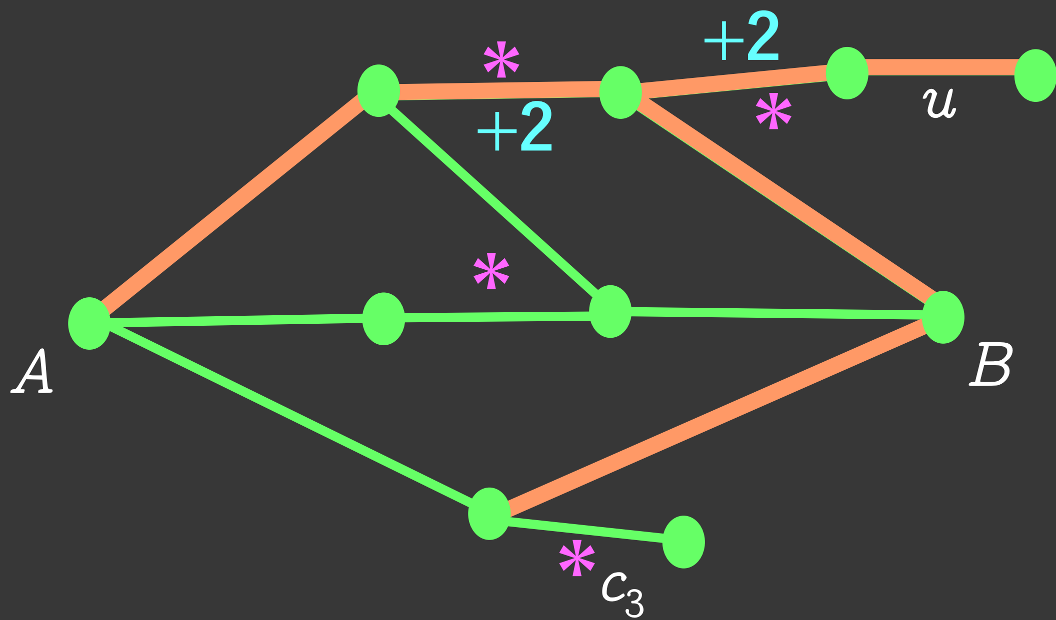


The worst case is that each (A,B)-path is short and heavy edges we can find is not around the center of the subdivided claw.

Since each c_i is heavy, there must be an edge between them.

We can find a closed trail T as above.

There exists an edge u which is not dominated by T.



The worst case is that each (A,B)-path is short and heavy edges we can find is not around the center of the subdivided claw.

Since each c_i is heavy, there must be an edge between them.

We can find a closed trail T as above.

There exists an edge u which is not dominated by T.

We can find another subdivided claw.

⋮

In the case $n \leq 32$, we need boring case analysis.

Thank you.