Reconfiguration graphs induced by rainbow spanning trees

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Joint work with
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Graphternoon @ Ghent University

Combinatorial reconfiguration

Combinatorial reconfiguration is an algorithmic concept that provide mathematical models and analysis for “transformations over state spaces”.

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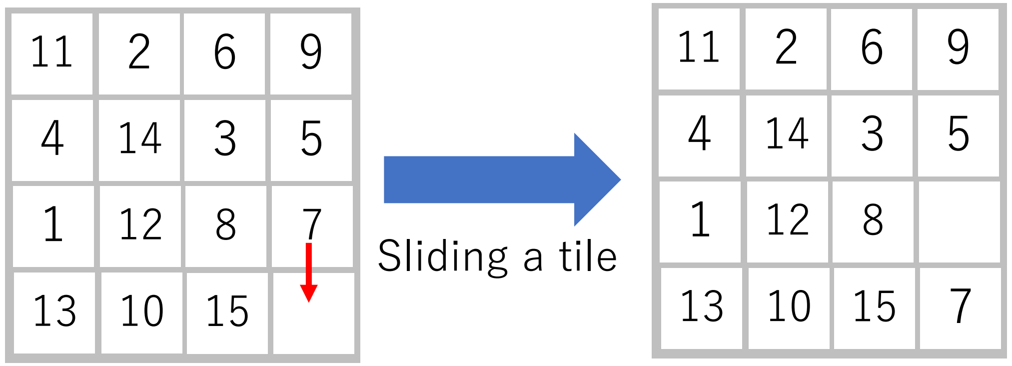
Object : 15-puzzle

Operation : Sliding tiles



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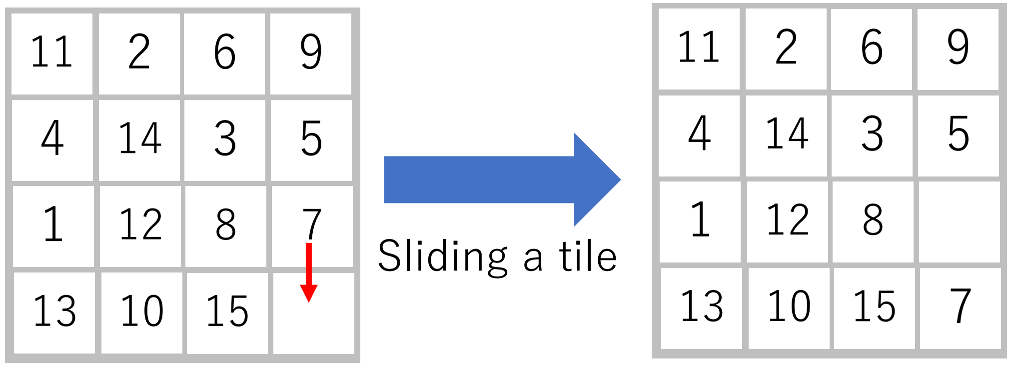
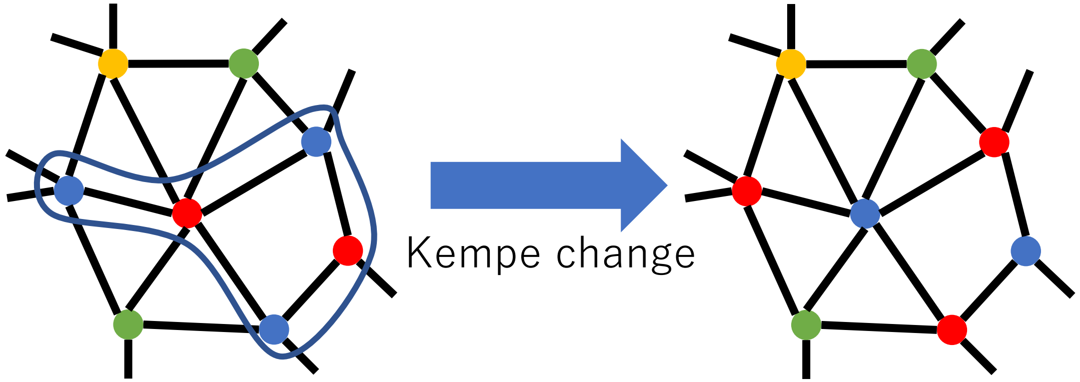


Object : Graph Coloring
Operation : Kempe change



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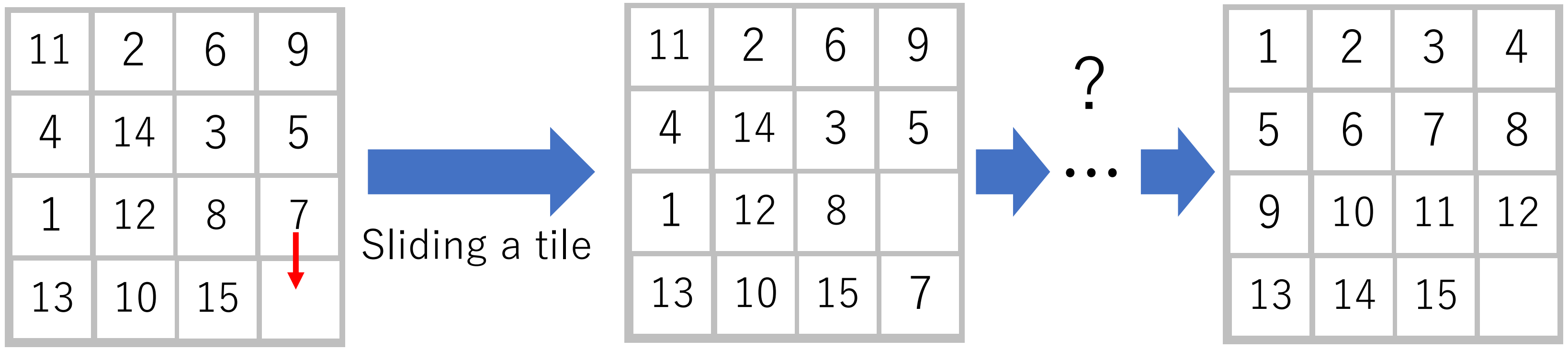

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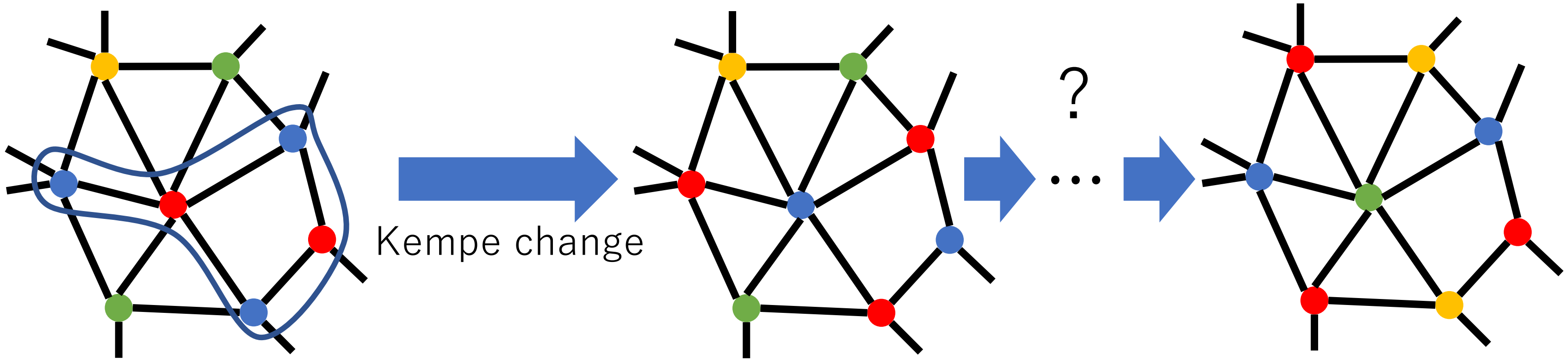
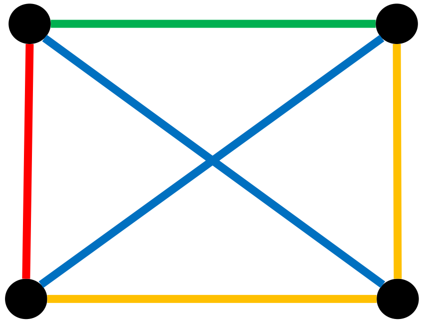


Reachability problem: For two inputs, we are asked to determine whether or not we can transform one into the other by a prescribed operation.

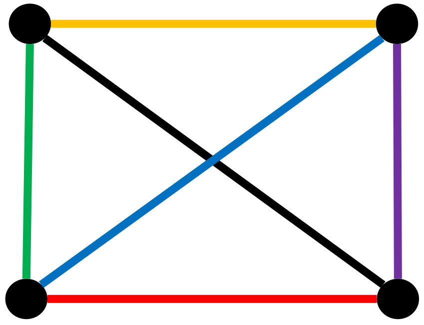
Edge-colored graph and rainbow

Definition (edge-colored graph and rainbow)

- An **edge-colored graph** is a graph with an edge coloring (not necessarily proper coloring).
- An edge-colored graph is **rainbow** if no two edges have the same color.



Edges in the same color may share an end-vertex

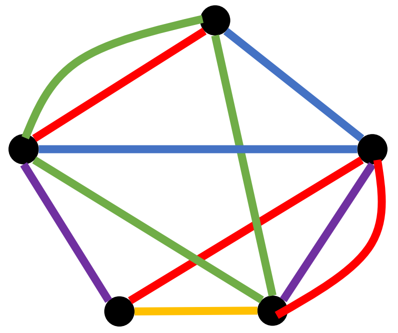


Rainbow

Rainbow spanning tree reconfiguration

Object : Edge-colored multigraph　　Operation : Edge flip

Rule : All intermediate results remain rainbow spanning trees



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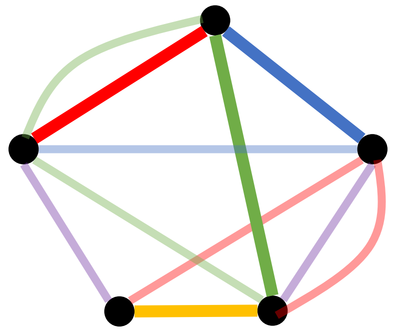

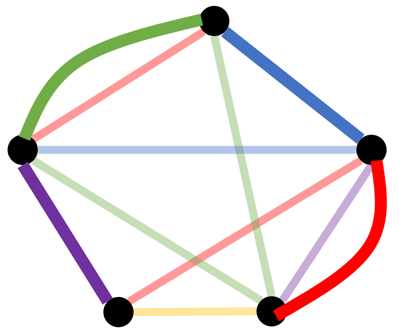

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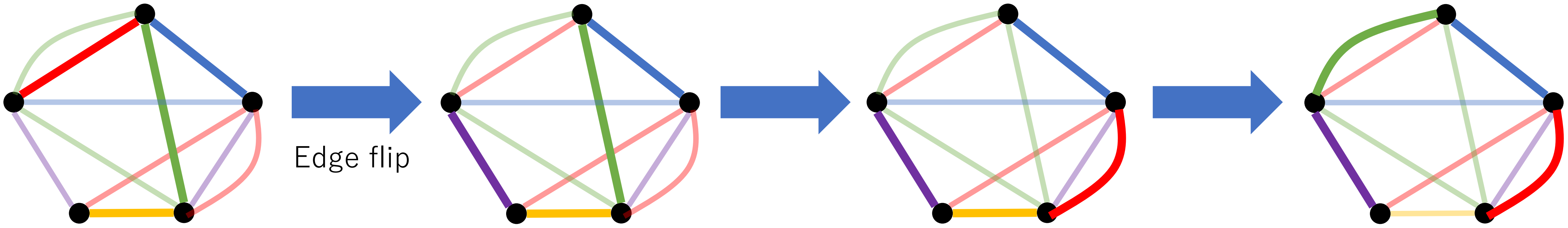

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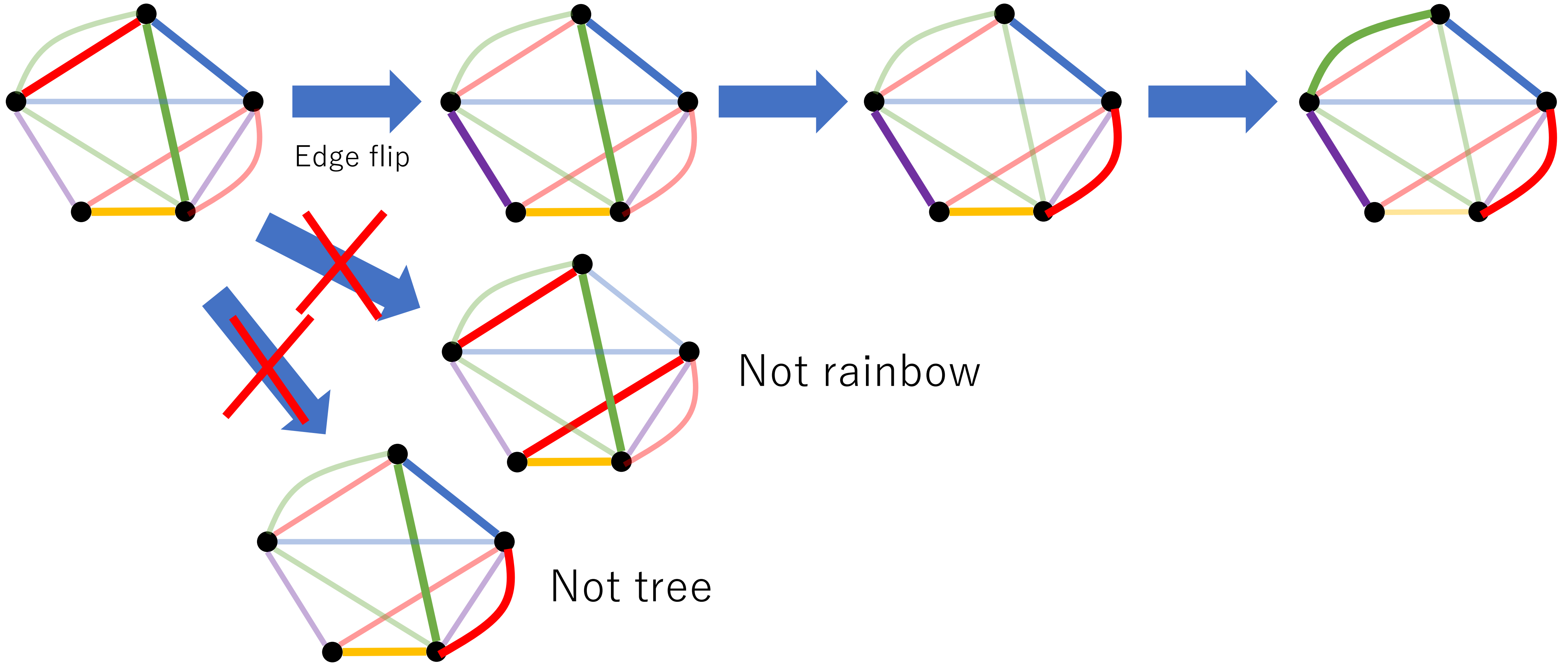

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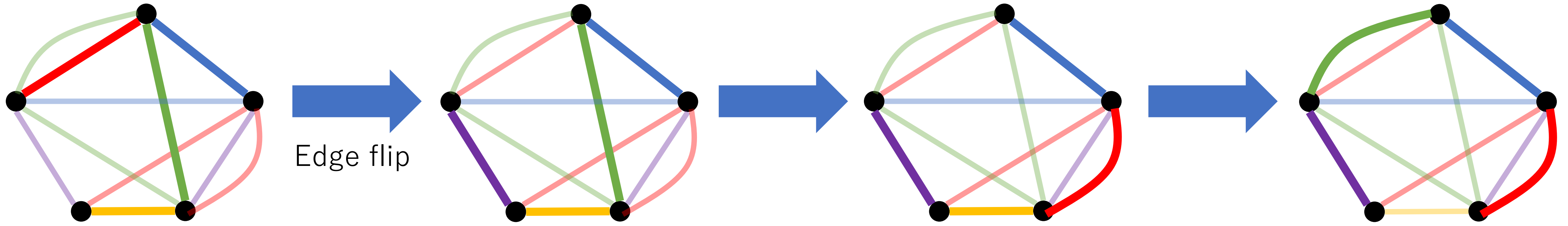


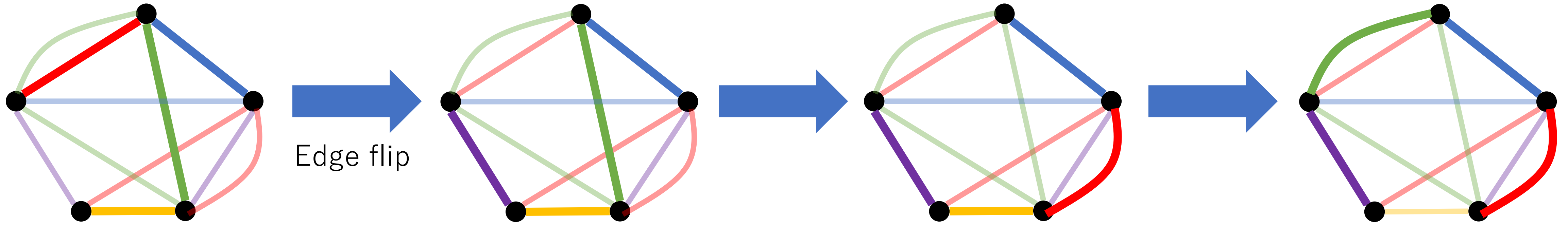
Definition (Rainbow spanning tree reconfiguration)

T, T' : Rainbow spanning trees of an edge-colored multigraph G

A **reconfiguration sequence** between T and T' is a sequence of rainbow spanning trees $(T_0, T_1, \ldots, T_k)$ in G with $T_0 = T$ and $T_k = T'$ s.t.

T_{i+1} is obtained from T_i by edge flip i.e. $T_{i+1} = T_i - e + f$.

Rainbow spanning tree reconfiguration



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What are conditions for G to always result in "yes"?
How about reconfiguration graph?

Problem (Rainbow spanning tree reconfiguration problem)

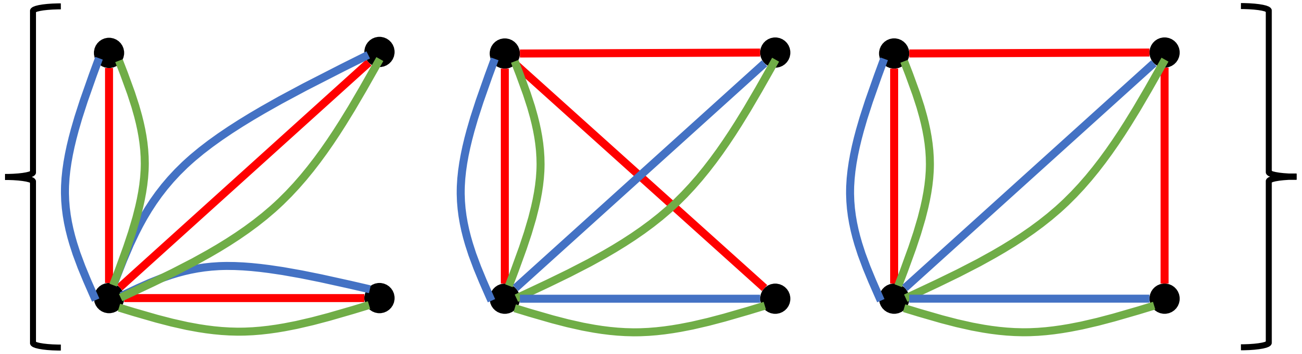
Is there a polynomial-time algorithm for the following decision problem?

Input : edge colored multigraph G, rainbow spanning trees T, T' of G

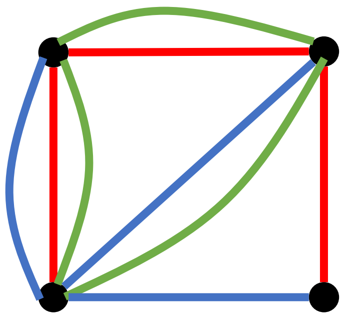
Output : whether there is a reconfiguration sequence between T and T'

Special edge-colored graph

$\mathcal{H}_n$: the set of edge-colored graphs with n vertices satisfying that edges colored with a color c induce a connected spanning graph for each color c in the graph



$\subseteq \mathcal{H}_4$



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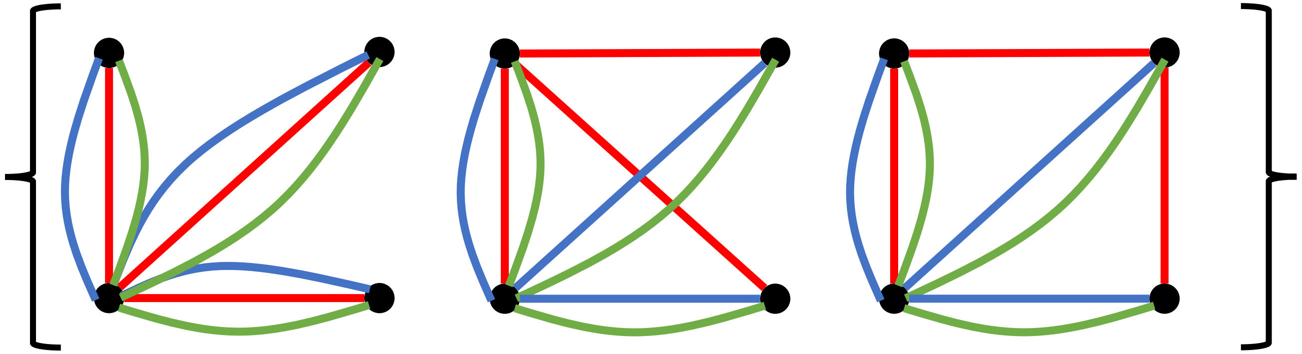
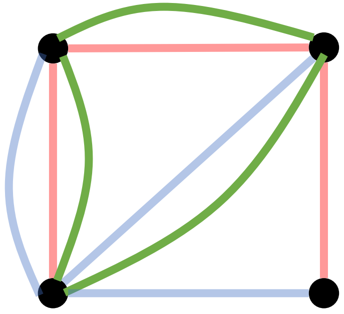
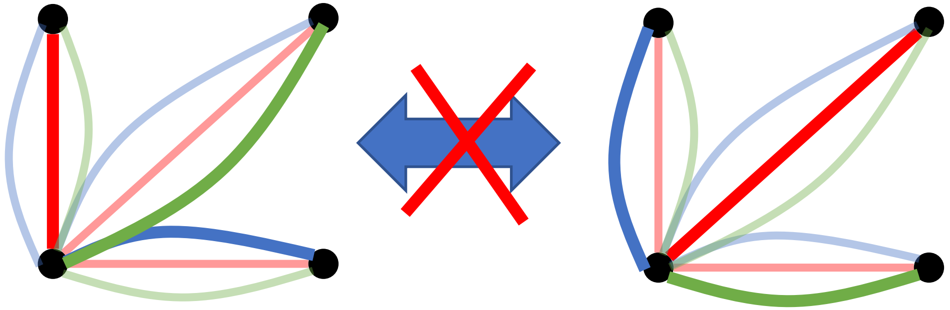
 $\subseteq \mathcal{H}_4$

 $\notin \mathcal{H}_4$

Is there reconf. sequence between any two rainbow spanning trees in $G \in \mathcal{H}_n$?

⇒No



Number of colors is too few…

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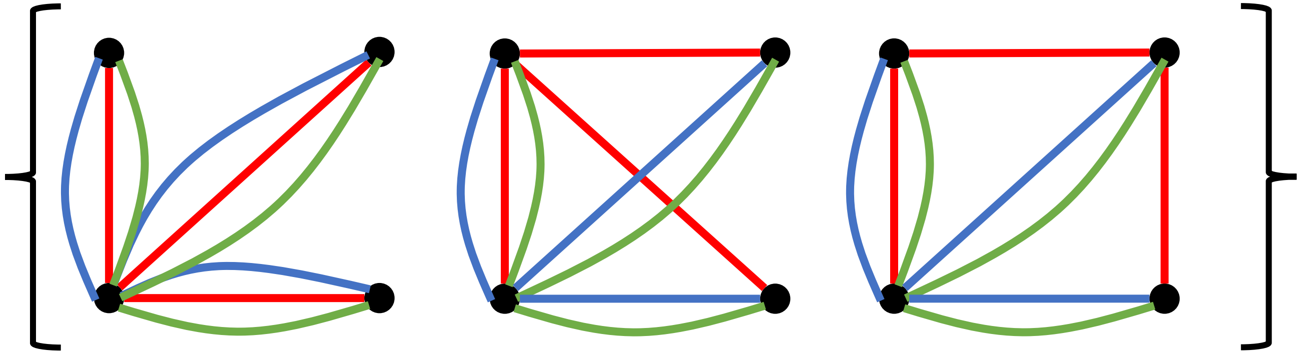

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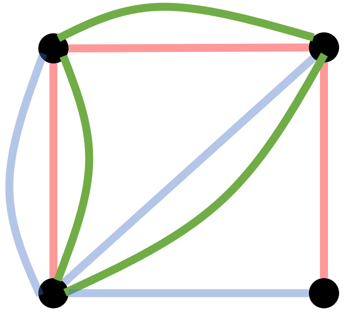

$\notin \mathcal{H}_4$

Theorem 1 (M and Yamaguchi)
For any two rainbow spanning trees in an edge-colored graph $G \in \mathcal{H}_n$ having at least n colors, there is a reconfiguration sequence between them and the length of the sequence is at most $\frac{3}{2}(n-1)$.

The bound is tight.

Special ed

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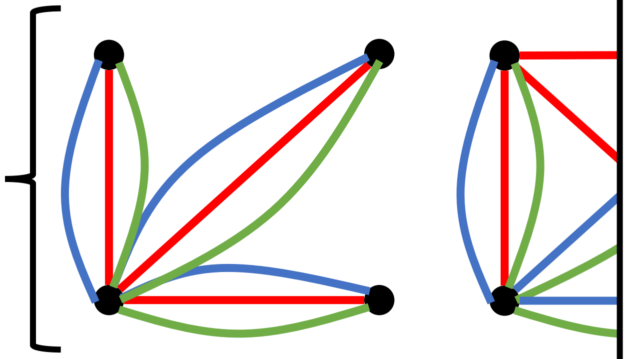
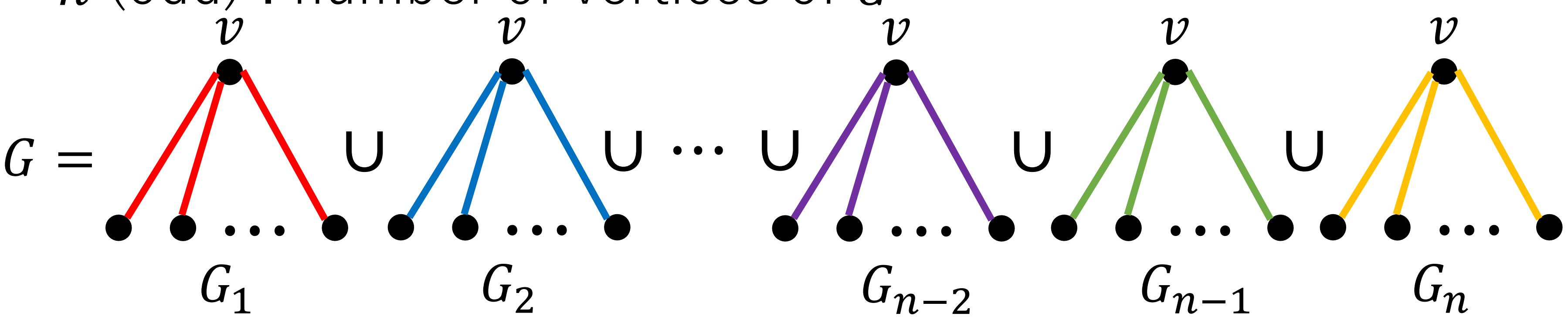
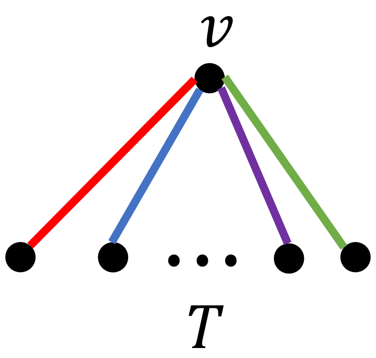
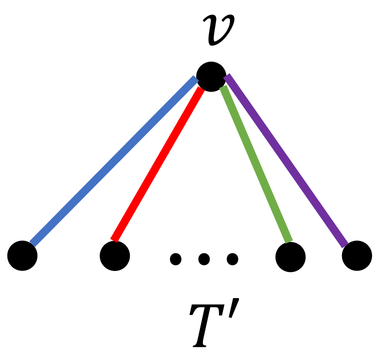


n (odd) : number of vertices of G







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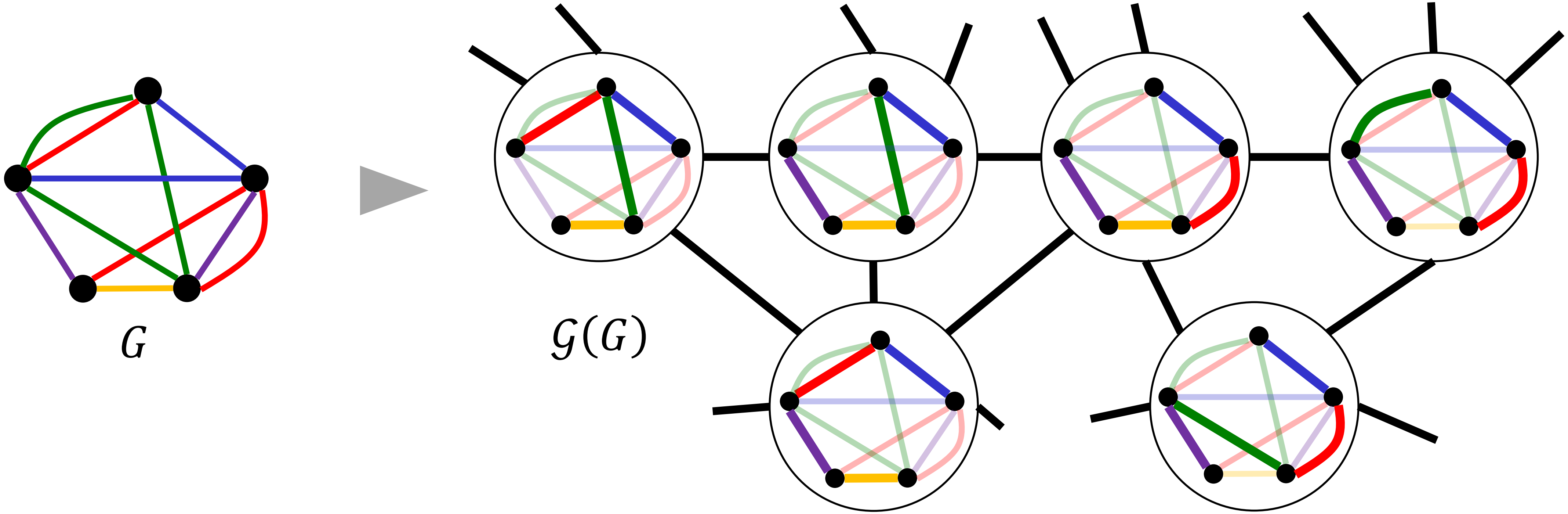
Reconfiguration graphs

Definition (Reconfiguration graph induced by rainbow spanning trees)

For an edge-colored multigraph G, the reconfiguration graph $\mathcal{G}(G)$ induced by rainbow spanning trees of G as follows:

$$V(\mathcal{G}(G)) = \{T : rainbow\ spanning\ tree\ T\ in\ G\},$$

two rainbow spanning trees are adjacent in $\mathcal{G}(G) \Leftrightarrow$
one is obtained from the other by exchanging a single edge.



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Theorem 1 (rephrase)

For an edge-colored graph $G \in \mathcal{H}_n$ having at least n colors, the diameter of $\mathcal{G}(G)$ is at most $\frac{3}{2}(n-1)$.

What other properties might the reconfiguration graphs have?

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Definition (Reconfiguration graph induced by spanning trees)

For a multigraph G, the reconfiguration graph $\mathcal{G}'(G)$ induced by spanning trees of G as follows:

$$V(\mathcal{G}'(G)) = \{T : spanning\ tree\ T\ in\ G\},$$

two spanning trees are adjacent in $\mathcal{G}'(G) \Leftrightarrow$
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Theorem A (Holzmann and Harary, 1972)

For a multigraph G, $\mathcal{G}'(G)$ is hamiltonian.

Reconfiguration graphs

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Problem 1

For any edge-colored multigraph $G \in \mathcal{H}_n$ having at least $n(\geq 3)$ colors, is $\mathcal{G}(G)$ hamiltonian?

Theorem 2 (M and Yamaguchi)

For any edge-colored multigraph $G \in \mathcal{H}_n$ having at least n colors, $\mathcal{G}(G)$ is 2-connected.

Problem 2

For any edge-colored multigraph $G \in \mathcal{H}_n$ having at least n colors, is $\mathcal{G}(G)$ 1-tough?

Conclusion Thank you very much!!

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