Cospectral mates for generalized Johnson and Grassmann graphs

Joint work with Aida Abiad, Willem H. Haemers and Robin Simoens

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Graphternoon 2024

Overview

1. Cospectral mates
2. How to find cospectral graphs
3. Which graphs did we check?
4. What is known?
5. Three new results

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Cospectral mates

Figure: Saltire pair

Both graphs have spectrum $\{-2, 0, 0, 0, 2\}$.

Cospectral mates

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1 Cospectral mates

Definition.
Graphs with the same spectrum are **cospectral**.
Cospectral nonisomorphic graphs are **cospectral mates**.

Definition.
A graph is **determined by its spectrum (DS)** if it has no cospectral mate. Otherwise, we say that it is **not determined by its spectrum (NDS)**.

Cospectral mates

Conjecture (Haemers).
Almost all graphs are determined by their spectrum.

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- Interesting for complexity theory

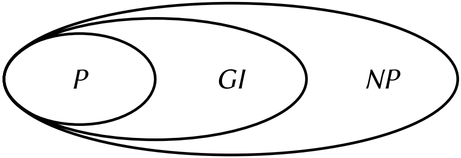

Figure: Is graph isomorphism an easy problem? Is it NP-complete?

Cospectral mates

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Almost all graphs are determined by their spectrum.

- Computational evidence (Brouwer and Spence, 2009)
- Interesting for complexity theory
- Interesting for chemistry

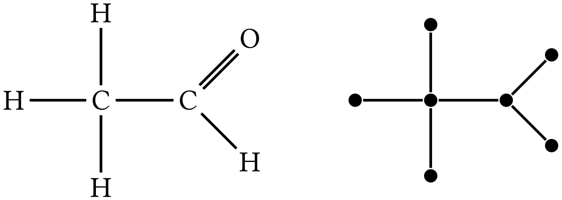

Figure: The molecular graph of acetaldehyde (ethanal).

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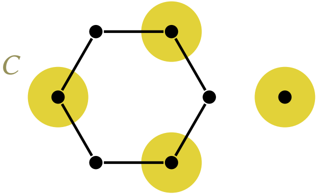
How to find cospectral graphs

Theorem (Godsil and McKay, 1982).
Let Γ be a graph with a subgraph C such that:

- C is regular.
- Every vertex outside C has 0, $\frac{1}{2}|C|$ or $|C|$ neighbours in C.

For every $v \notin C$ that has exactly $\frac{1}{2}|C|$ neighbours in C, reverse its adjacencies with C. The resulting graph is cospectral with Γ.



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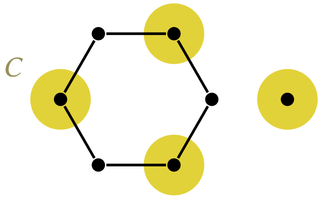

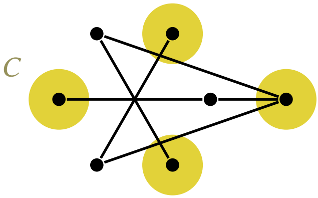

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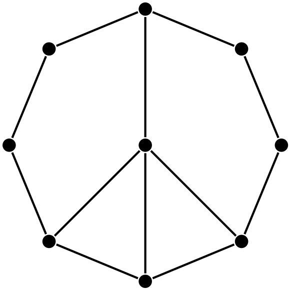

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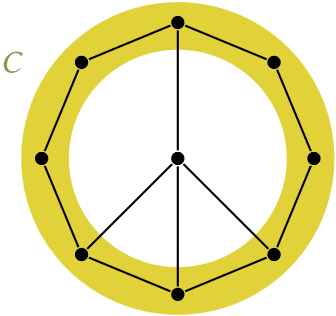

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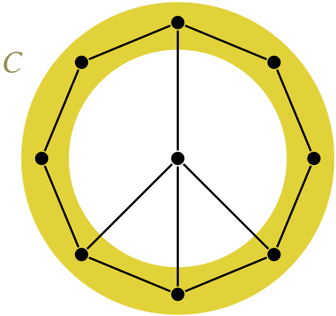
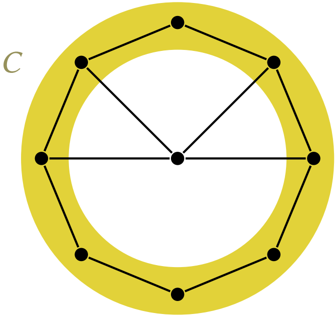




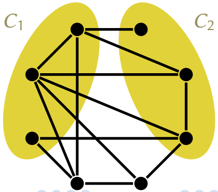
How to find cospectral graphs

Theorem (Wang, Qiu and Hu, 2019).

Let Γ be a graph with disjoint subgraphs C_1, C_2 such that:

- $|C_1| = |C_2|$.
- There is a constant c such that, for every vertex of C_i, the number of neighbours in C_i minus the number of neighbours in C_j, is c.
- Every vertex outside $C_1 \cup C_2$ has either:
  - 0.1 0 neighbours in C_1 and $|C_2|$ in C_2,
  - 0.2 $|C_1|$ neighbours in C_1 and 0 in C_2,
  - 0.3 equally many neighbours in C_1 and C_2.

For every $v \notin C_1 \cup C_2$ for which 1 or 2 holds, reverse its adjacencies with $C_1 \cup C_2$. The resulting graph is cospectral with Γ.



2 How to find cospectral graphs

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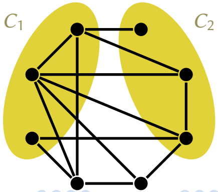

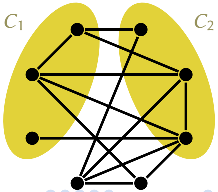

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3 Which graphs did we check?

Definition.

Let $S \subseteq \{0, 1, \ldots, k-1\}$. The *generalized Johnson graph* $J_S(n, k)$ has as vertices the k-subsets of $\{1, \ldots, n\}$, where two vertices are adjacent if their intersection size is in S.

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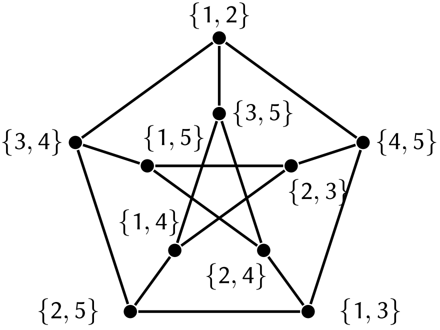
- $J_{\{0\}}(n, k)$ is the *Kneser graph* $K(n, k)$.



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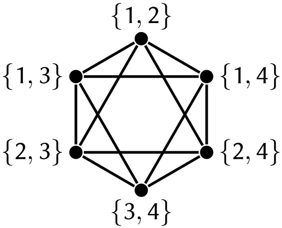
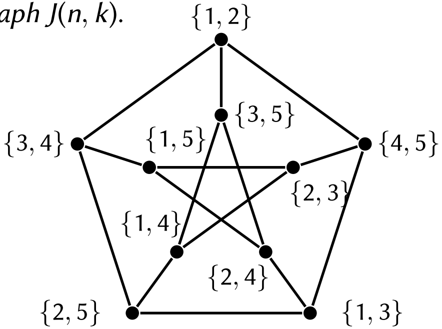
- $J_{\{0\}}(n,k)$ is the *Kneser graph* $K(n,k)$.
- $J_{\{k-1\}}(n,k)$ is the *Johnson graph* $J(n,k)$.





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Definition.
Let $S \subseteq \{0, 1, \ldots, k-1\}$. The *generalized Grassmann graph* $J_{q,S}(n, k)$ has as vertices the k-subspaces of $\mathbb{F}_q^n$, where two vertices are adjacent if their intersection dimension is in S.

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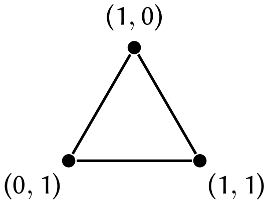
- $J_{q,\{0\}}(n, k)$ is the *q-Kneser graph* $K_q(n, k)$.



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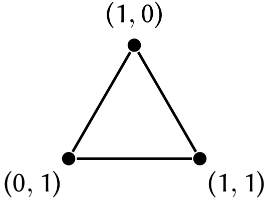
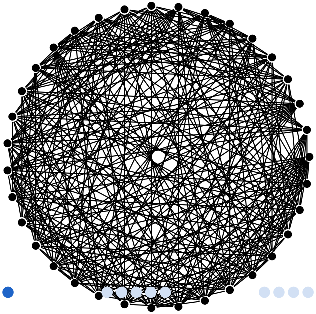
- $J_{q,\{0\}}(n, k)$ is the *q-Kneser graph* $K_q(n, k)$.
- $J_{q,\{k-1\}}(n, k)$ is the *Grassmann graph* $J_q(n, k)$.





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What is known?

$J_S(n, 2)$		S
		$\{0\}$
n	4	DS
	5	DS
	6	DS
	7	DS
	8	NDS
	9	DS

$J_S(n, 3)$		S		
		$\{0\}$	$\{1\}$	$\{2\}$
n	6	DS	NDS	NDS
	7	DS	NDS	NDS
	8	NDS	NDS	NDS
	9	?	NDS	NDS
	10	?	NDS	NDS
	11	?	NDS	NDS

Legend: Trivial | Hoffman/Chang (1959) | Huang, Liu (1999) | Van Dam et al. (2006) | Haemers, Ramezani (2010)

What is known?

$J_S(n,4)$		S						
		$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$
n	8	DS	?	?	NDS	?	NDS	?
	9	DS	?	?	NDS	NDS	NDS	?
	10	?	?	?	NDS	?	NDS	?
	11	NDS	?	?	NDS	?	NDS	?
	12	?	?	?	NDS	?	NDS	?
	13	?	?	?	NDS	?	NDS	?

Legend: Trivial | Huang, Liu (1999) | Van Dam et al. (2006) | Haemers, Ramezani (2010) | Cioabă et al. (2018)

What is known?

$J_S(n,4)$		S						
		$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$
n	8	DS	?	NDS	NDS	?	NDS	?
	9	DS	?	NDS	NDS	NDS	NDS	?
	10	?	?	NDS	NDS	?	NDS	?
	11	NDS	NDS	NDS	NDS	?	NDS	?
	12	?	?	NDS	NDS	?	NDS	?
	13	?	?	NDS	NDS	?	NDS	?

Legend: Trivial | Huang, Liu (1999) | Van Dam et al. (2006) | Haemers, Ramezani (2010) | Cioabă et al. (2018) | New result: $J_{\{2\}}(n,4)$ is NDS | Sporadic result

What is known?

$J_{q,S}(n,2)$		$q=2$, $S=\{0\}$	$q=3$, $S=\{0\}$	$q=4$, $S=\{0\}$
n	4	NDS	NDS	NDS
	5	NDS	NDS	NDS
	6	NDS	NDS	NDS
	7	NDS	NDS	NDS
	8	NDS	NDS	NDS
	9	NDS	NDS	NDS

Legend: Van Dam, Koolen (2005) Ihringer, Munemasa (2019)

What is known?

$J_{q,S}(n,3)$		$q=2$			$q=3$			$q=4$		
		S			S			S		
		$\{0\}$	$\{1\}$	$\{2\}$	$\{0\}$	$\{1\}$	$\{2\}$	$\{0\}$	$\{1\}$	$\{2\}$
n	6	?	?	NDS	?	?	NDS	?	?	NDS
	7	?	?	NDS	?	?	NDS	?	?	NDS
	8	?	?	NDS	?	?	NDS	?	?	NDS
	9	?	?	NDS	?	?	NDS	?	?	NDS
	10	?	?	NDS	?	?	NDS	?	?	NDS
	11	?	?	NDS	?	?	NDS	?	?	NDS

Legend: Van Dam et al. (2006)

What is known?

$J_{q,S}(n,3)$		$q=2$			$q=3$			$q=4$		
		S			S			S		
		$\{0\}$	$\{1\}$	$\{2\}$	$\{0\}$	$\{1\}$	$\{2\}$	$\{0\}$	$\{1\}$	$\{2\}$
n	6	NDS	?	NDS	?	?	NDS	?	?	NDS
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	8	NDS	?	NDS	?	?	NDS	?	?	NDS
	9	NDS	?	NDS	?	?	NDS	?	?	NDS
	10	NDS	?	NDS	?	?	NDS	?	?	NDS
	11	NDS	?	NDS	?	?	NDS	?	?	NDS

Legend: Van Dam et al. (2006) New result: $K_2(n,k)$ is NDS

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Three new results

Theorem.
$J_{\{2\}}(n, 4)$ is NDS if $n \geq 8$.

Theorem.
$J_{\{1,2,\ldots \frac{k-1}{2}\}}(2k, k)$ is NDS if $k \geq 5$, k odd.

Theorem.
$K_2(n, k)$ is NDS.

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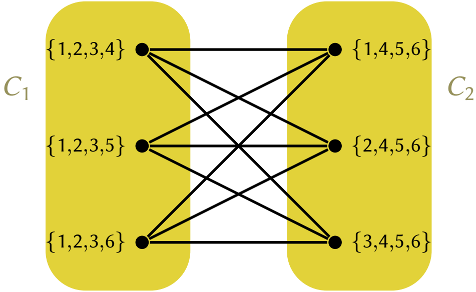
- WQH-switching



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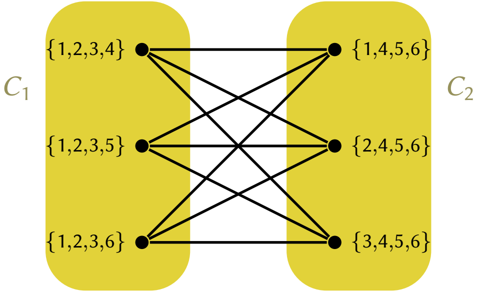

- $J_{\{2\}}(n,4)$ is edge-regular, the new graph is not

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- $\binom{2k}{k} = \binom{2k-1}{k-1} + \binom{2k-1}{k} = 2\binom{2k-1}{k-1}$
- adjacency matrix $A = \begin{pmatrix} A' & \bar{A}' \\ \bar{A}' & A' \end{pmatrix}$

Theorem (Cioabă et al. (2018)).
$J_{\{0,1,\dots,\frac{k-3}{2}\}}(2k-1,k-1)$ is NDS if $k \geq 5$, k odd.

Three new results

$J_{\{1\}}(6,3)$ has vertices $\{1,2,3\}, \{1,2,4\}, \ldots, \{1,4,6\}, \{1,5,6\},$
$\{4,5,6\}, \{3,5,6\}, \ldots, \{2,3,5\}, \{2,3,4\}$

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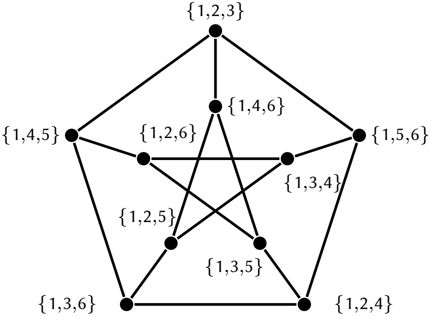

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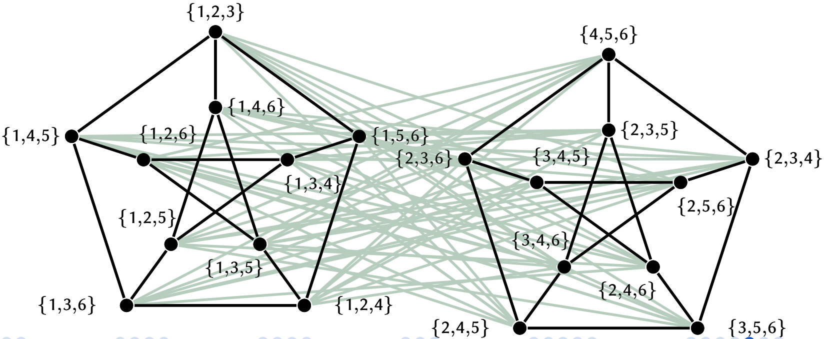

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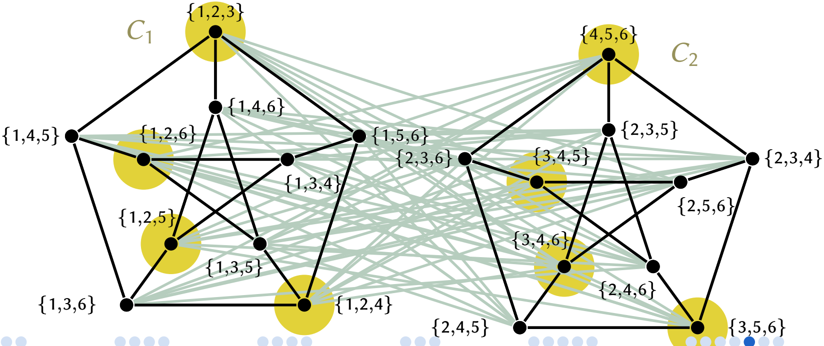

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Theorem.
$K_2(n, k)$ is NDS.

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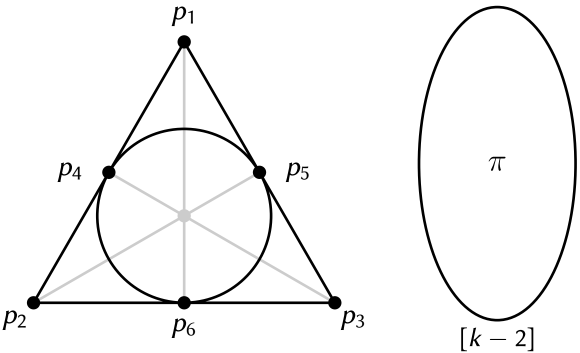

- GM-switching set $C := \{p_1p_2\pi,\ p_1p_3\pi,\ p_2p_3\pi,\ p_4p_5\pi\}$

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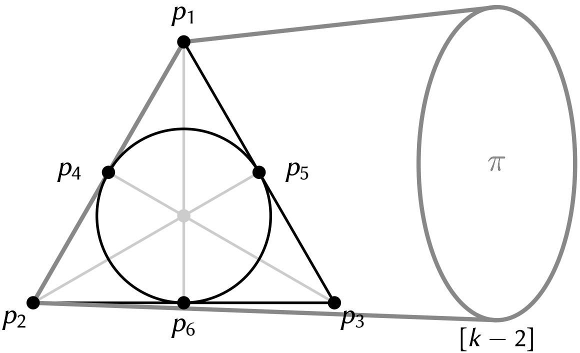

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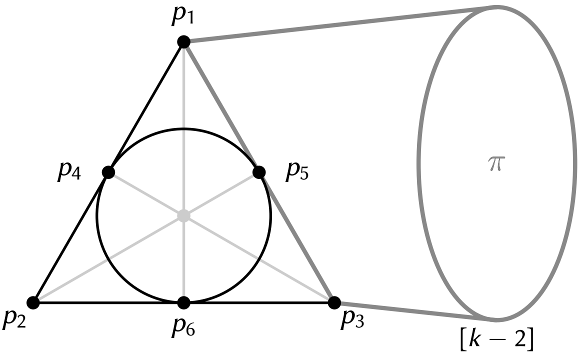

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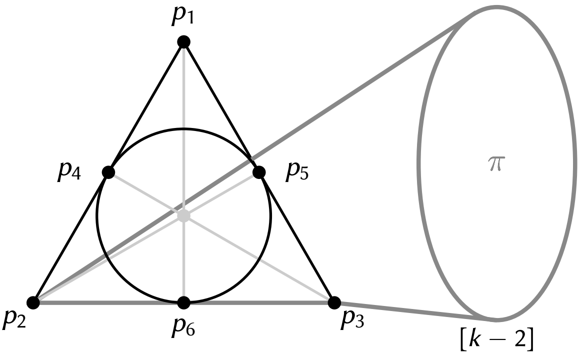

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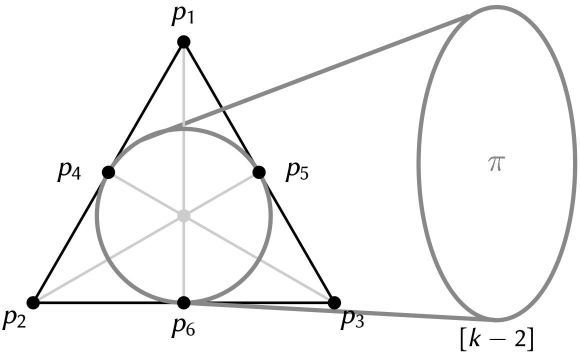

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Thank you for listening!