Coloring normal quadrangulations of projective spaces

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Joint work with

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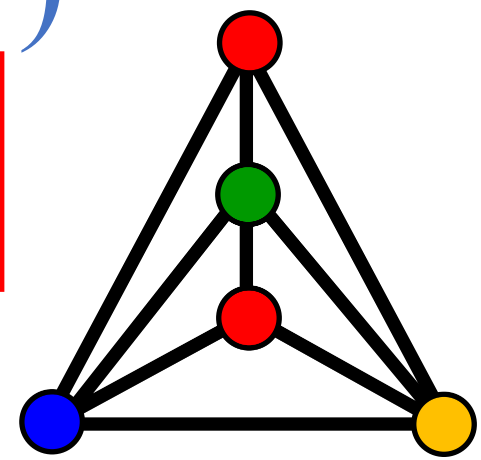
4 Color Theorem (4CT)

Thm. (Appel & Haken, `77)

Every spherical triangulation has a 4-coloring



4-coloring

k-coloring of a graph

$\overset{\text{def.}}{\Leftrightarrow}$ Assignment of one of k colors to each vertex so that no 2 adjacent vertices have the same color

- ✓ Proved using a computer
- ✓ Many studies have been emerged from 4CT.

Non-spherical case

Today's targets: Quadrangulations of d-dimensional projective space

Each face is quadrilateral

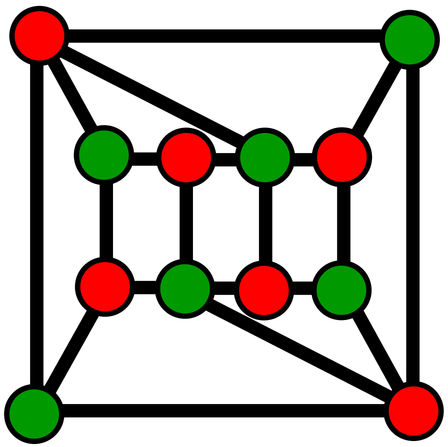
Higher dimensional case

Quadrangulation

Quadrangulation

def. ⇔ having a 2-coloring

Prop. Every planar quadrangulation is bipartite



Any quadrangulation can be locally 2-colorable.

In the planar case, any closed curve is trivial

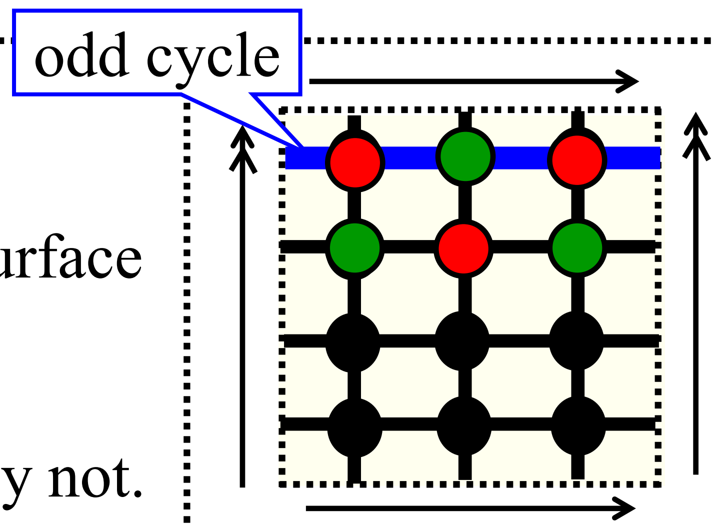
so, it is also globally 2-colorable.

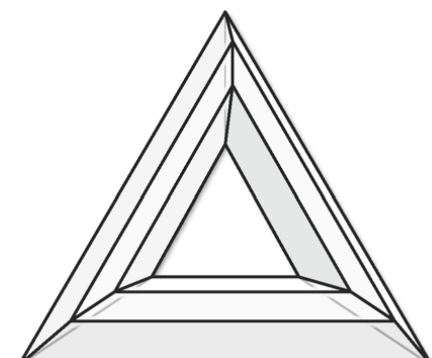
Non-bip. quad. exist

for non-spherical surface

Locally 2-colorable,

but globally not.





J. Erickson, Efficiently Hex-meshing things with topology, DiscreteComp. Geom. 52 (2014), 427–449

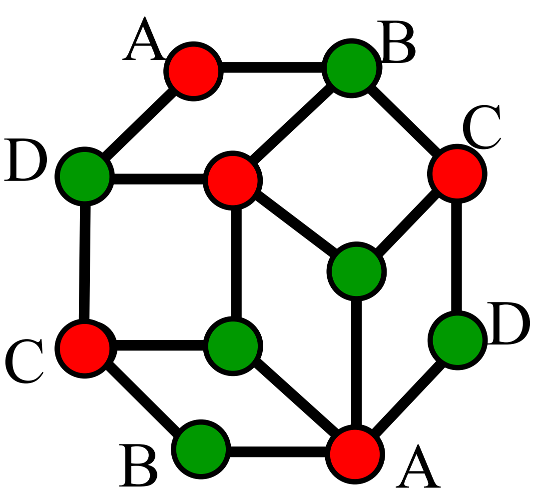
Projective planar case

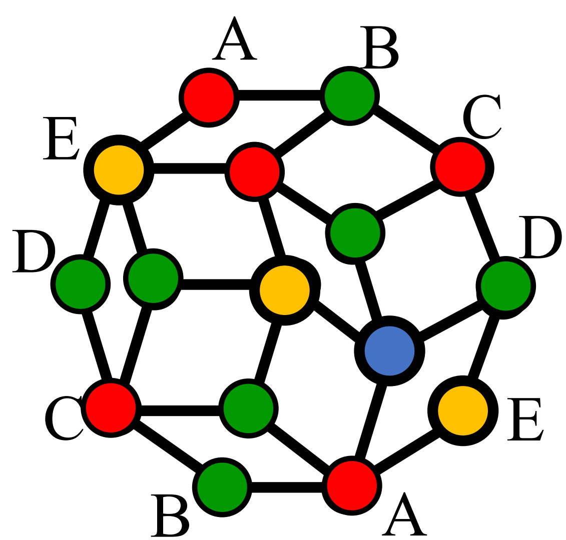
Thm (Hutchinson `75)

Every quadrangulation of a surface F^2 has a k-coloring, where $k = \frac{5+\sqrt{25-16\varepsilon(F^2)}}{2}$

$\varepsilon(F^2)$:Euler characteristic of F^2

Real projective plane





We obtain $\mathbb{RP}^2$ by identifying the antipodal points

When F^2 is $\mathbb{RP}^2$,

$\varepsilon(F^2) = 1$ and $k = 4$

So, every quad. on $\mathbb{RP}^2$ is 4-colorable.

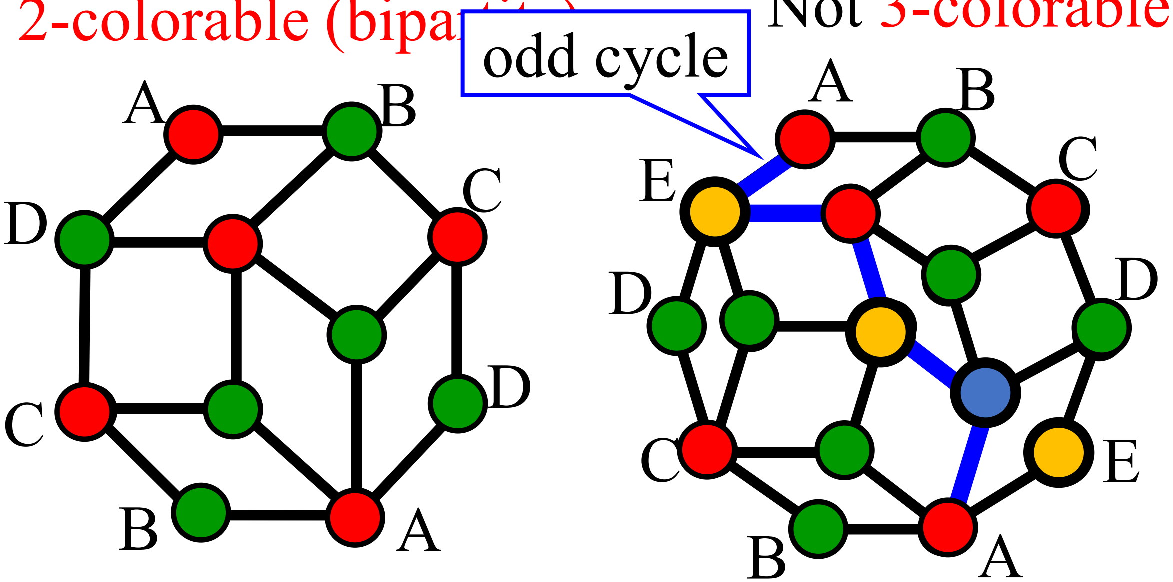
Projective planar case

Real projective plane

Thm. (Youngs, `96)

Every non-bip. quadrangulation on $\mathbb{R}P^2$ has NO 3-coloring

2-colorable (bipartite)   Not 3-colorable



We obtain $\mathbb{R}P^2$ by identifying the antipodal points

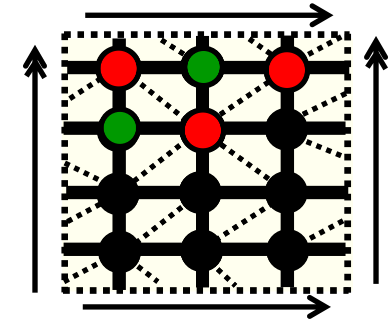
If we need 3rd color, 4th color is needed, too!

Purpose of this talk:

Extend this theorem to higher dimensional case

Higher dimensional case

KS-quadrangulation

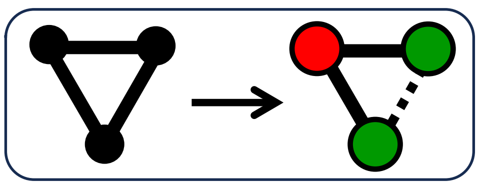


What is a ``quadrangulation'' of a topological space?

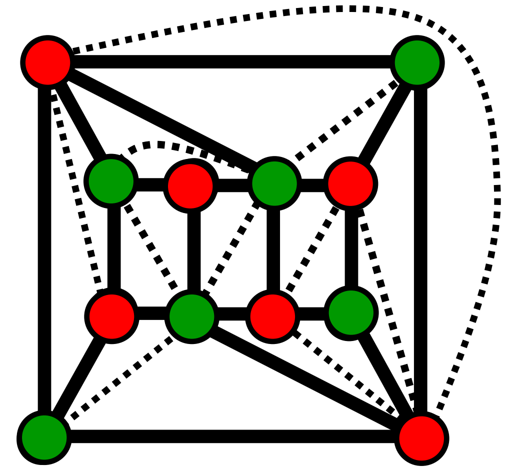
2-dim case:

Obtained from a triangulation of a surface

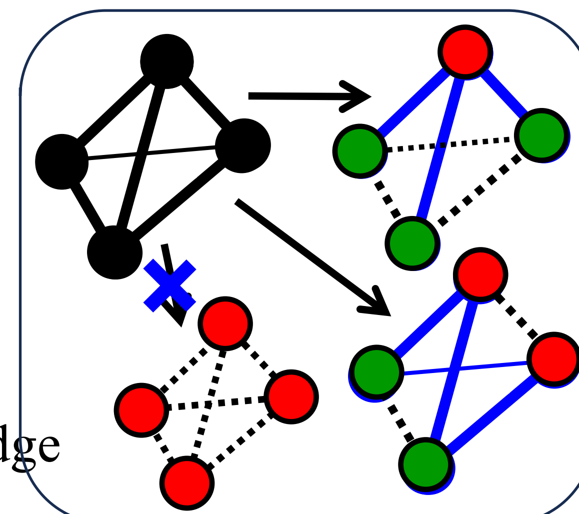
by removing 1 edge from each triangular face



locally 2-colorable



Higher dim case: defined by Kaiser & Stehlik `15



G: KS-quadrangulation of a topological space

def. $\Leftrightarrow$ G: obtained from a triangulation

by removing some edges from $\forall$simplex

to make it a complete bip. subgraph with an edge

KS-quadrangulation

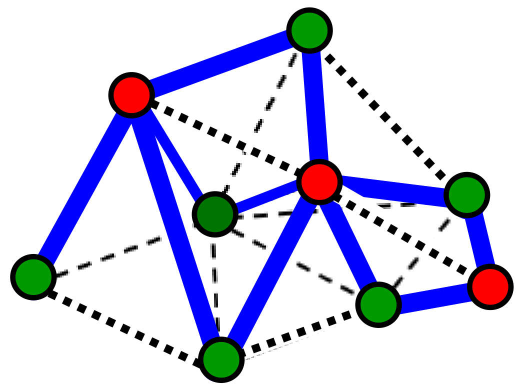
Thm. (Kaiser & Stehlik, `15)

$d \geq 2$

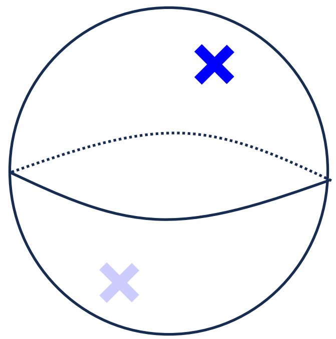
G : non-bip. KS-quad. of $\mathbb{RP}^d$

$\Rightarrow$ G has NO $(d+1)$-coloring

3-dim case:



Identify the antipodal points



Higher dim case: defined by Kaiser & Stehlik `15

G: KS-quadrangulation of a topological space

$\overset{\text{def.}}{\Leftrightarrow}$ G: obtained from a triangulation
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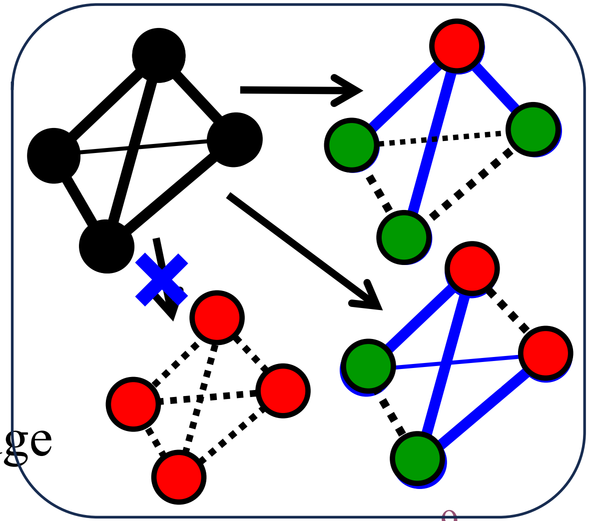
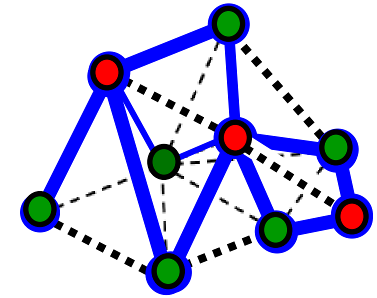


KS-quadrangulation



KS-quad.

G : non-bip. KS-quad. of $\mathbb{RP}^d$

$\chi(G)$: min. # colors needed for coloring of G

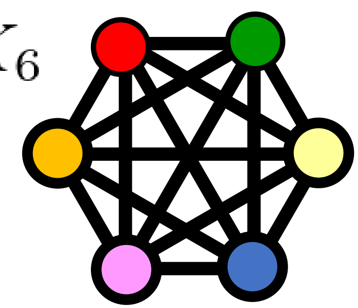
	$d=2$	$d=3$	$d=4$	$\cdots$	d
KS-quad.	$\chi(G)=4$	$\chi(G)\geq 5$ No upper bound	$\chi(G)\geq 6$	$\cdots$	$\chi(G)\geq d+2$ No upper bound

Thm. (Kaiser & Stehlik, `15)

$d \geq 3$, $t \geq 5$ s.t. $t-d$: even

$\Rightarrow K_t$ can be embedded in $\mathbb{RP}^d$ as a KS-quad

K_6



complete graph

So, fixed # colors are NOT enough for coloring of KS-quad. of $\mathbb{RP}^d$

Normal quadrangulations

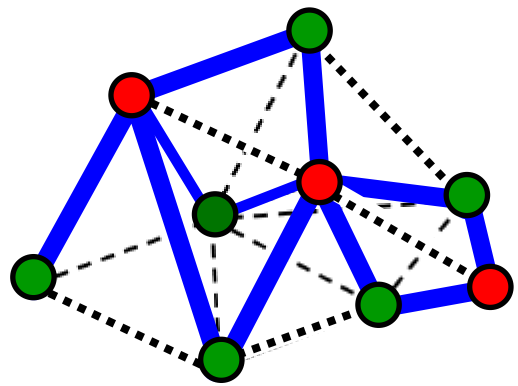
KS-quadrangulation

Can we really call them QUADrangulations?

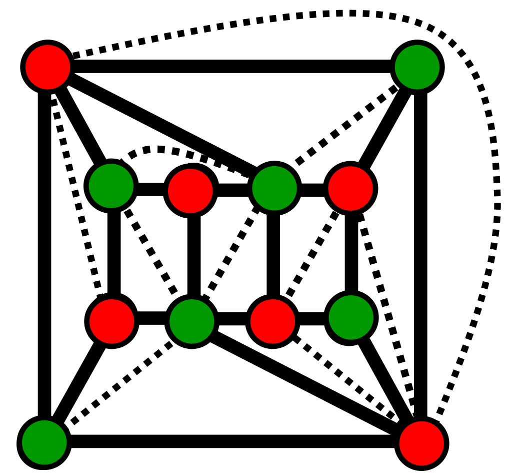
QUAD means four, 4 of…

Where can we see ``four''?

3-dim case:



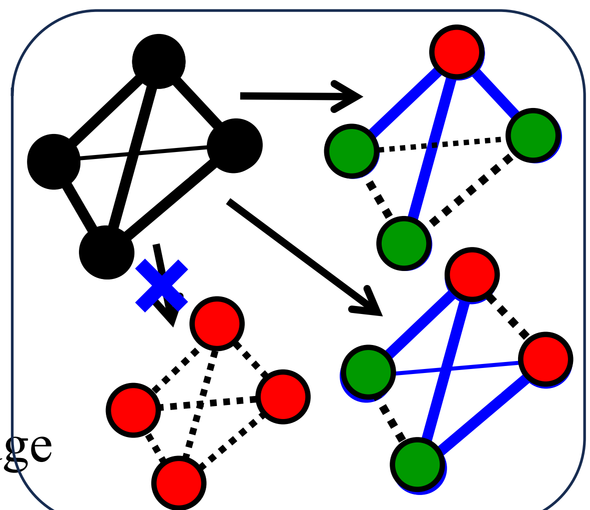
2-dim case:



Higher dim case: defined by Kaiser & Stehlik `15

G: KS-quadrangulation of a topological space

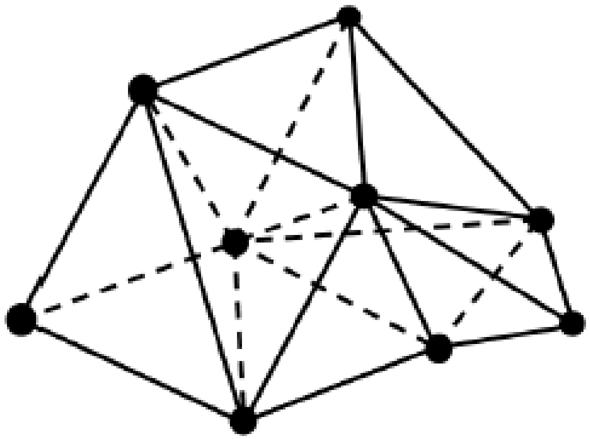
def. ⇔ G: obtained from a triangulation by removing some edges from $^{\forall}$simplex to make it a complete bip. subgraph with an edge



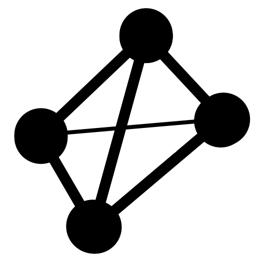
Normal quadrangulation

3-dim case:

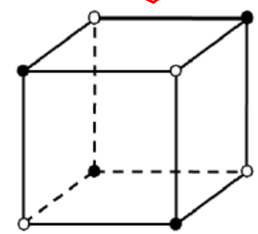
Can be seen as 3-dim. quadrilateral



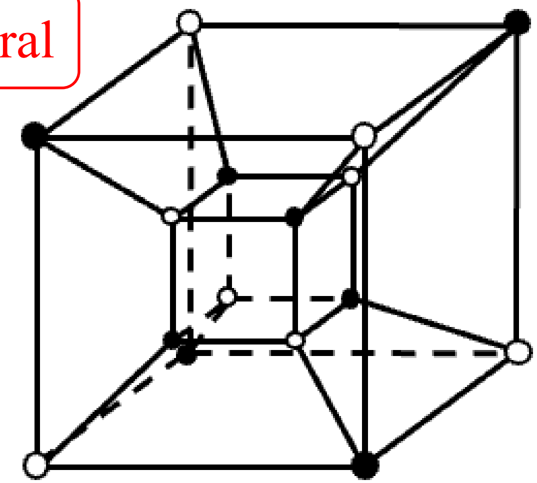
simplicial complex



simplex



cube



$\mathcal{C}$:cubical complex

$\overset{\text{def.}}{\Longleftrightarrow}$ (0) Each face in $\mathcal{C}$ is a cube (of any dim.) or its faces

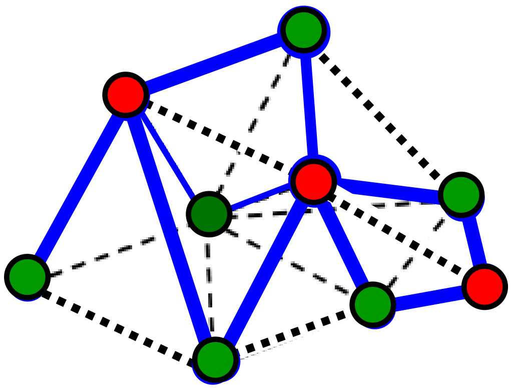
(i) $\emptyset \in \mathcal{C}$ (ii) $P \in \mathcal{C} \Rightarrow$ all faces of P are in $\mathcal{C}$

(iii) $P, Q \in \mathcal{C} \Rightarrow P \cap Q$ is a face of P and Q

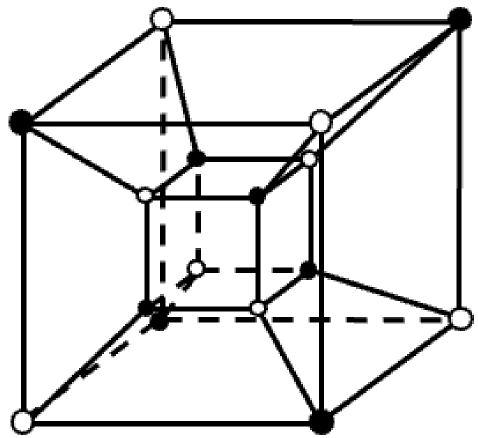
G: normal quadrangulation $\overset{\text{def.}}{\Longleftrightarrow}$ G: 1-skeleton of a cubical complex

KS vs. Normal

3-dim case:

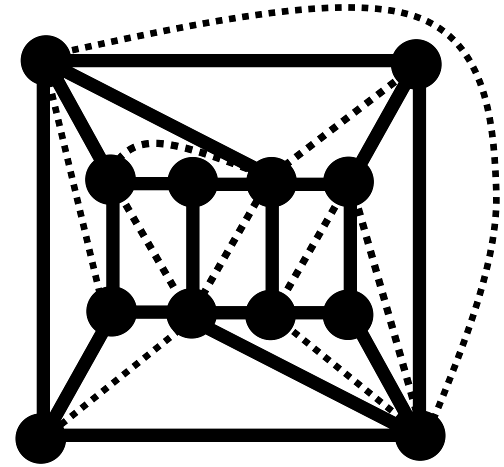


KS-quad.



normal quad.

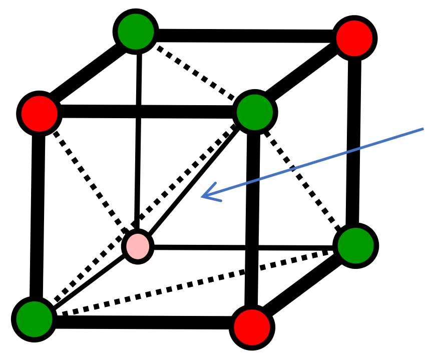
2-dim case:



KS-quad. = normal quad.

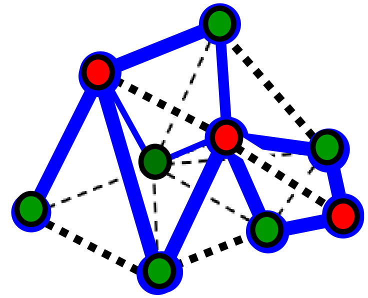
Prop. (Hachimori, Nakamoto, Oz.)

Every normal quad. can be extended to KS-quad. by adding some edges



Normal quad. seems to be easier to color than KS-quad.

Coloring quad.



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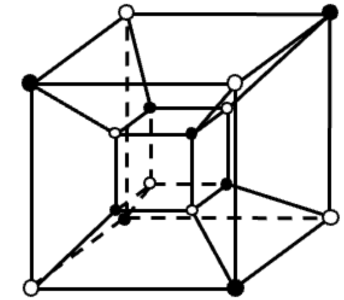


normal quad.

G : non-bip. quad. of $\mathbb{R}\mathrm{P}^d$

$\chi(G)$: min. # colors needed for coloring of G

	$d=2$	$d=3$	$d=4$	$\cdots$	d
KS-quad.	$\chi(G)=4$	$\chi(G)\geq 5$ No upper bound	$\chi(G)\geq 6$	$\cdots$	$\chi(G)\geq d+2$ No upper bound
normal quad.		$\chi(G)\geq 4$ No upper bound	$\chi(G)\geq 4$		$\chi(G)\geq 4$

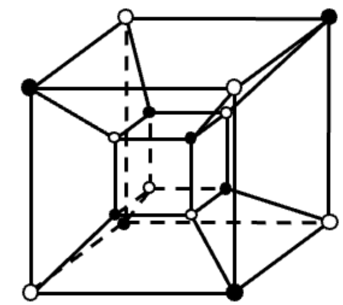
Thm. (Kaiser, Lo, Nakamoto, Nozaki, Oz., 24+)

Every non-bip. normal quad. of $\mathbb{R}\mathrm{P}^d$ has NO 3-coloring.

For any $t\geq 5$, $\exists$ non-bip. normal quad. of $\mathbb{R}\mathrm{P}^3$ containing K_t

High dimensional case

High dim. case



normal quad.

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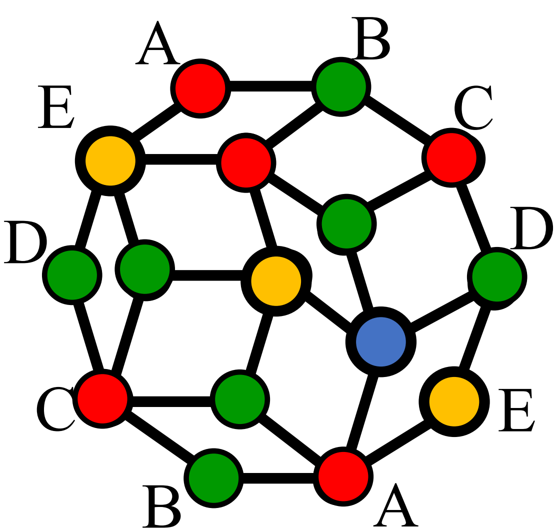
Every non-bip. normal quad. of $\mathbb{RP}^d$ has NO 3-coloring

Thm. (Youngs, `96)

Every non-bip. quadrangulation on $\mathbb{RP}^2$ has NO 3-coloring

Not 3-colorable



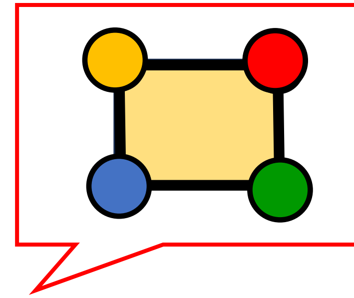
Youngs proved that using ``winding number''

(of each face after cut open into a plane)

We give a new proof to Youngs' thm.

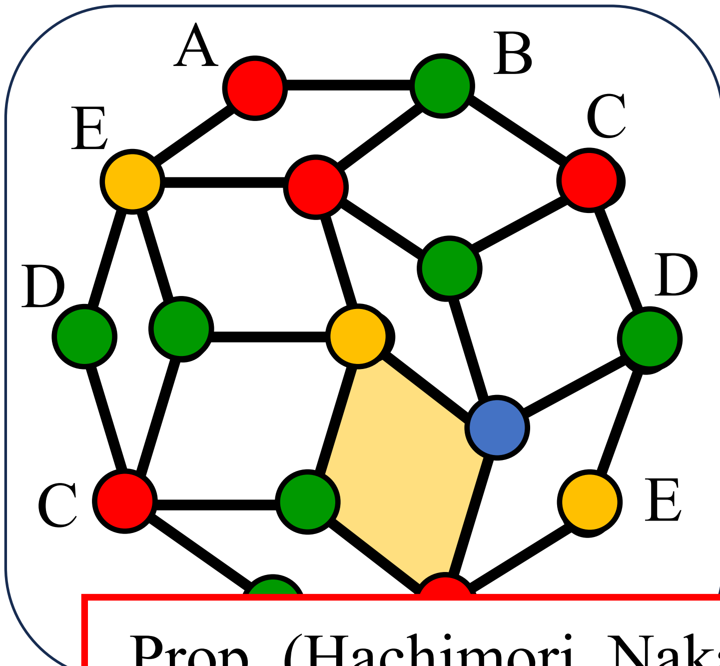
and extend it to *d*-dimensional case.

2- and 3-dim. case



G : non-bip. quad. of $\mathbb{R}P^2$

We show that for any 4-coloring, there is a rainbow quadrilateral

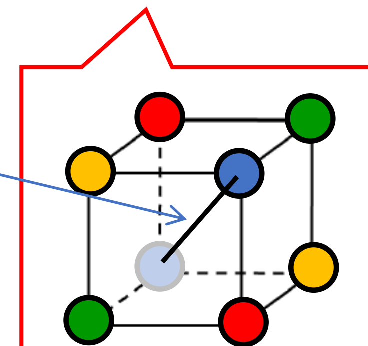


We show that

G : non-bip. normal quad. of $\mathbb{R}P^3$

$\Rightarrow$ for any 4-coloring,

there is a rainbow symmetric cube



Prop. (Hachimori, Nakamoto, Oz.)

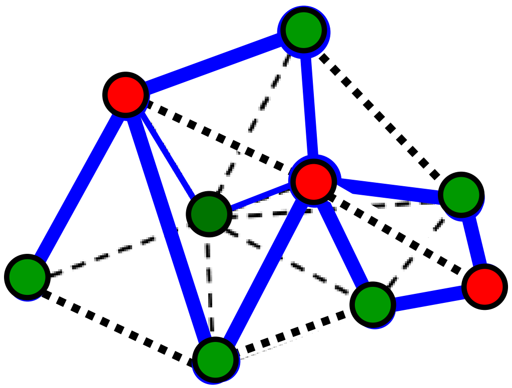
Every normal quad. can be extended to KS-quad. by adding some edges

This edge forbids KS-quad. to be 4-colorable

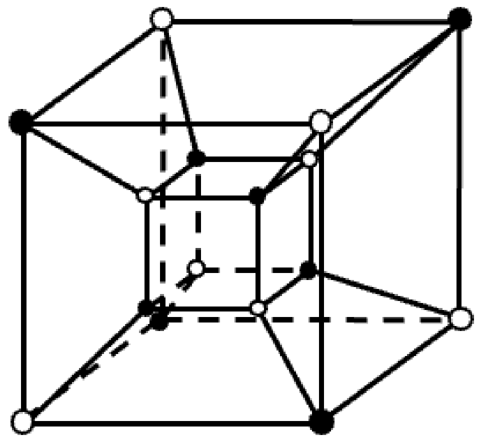
Summary

KS vs. Normal

3-dim case:

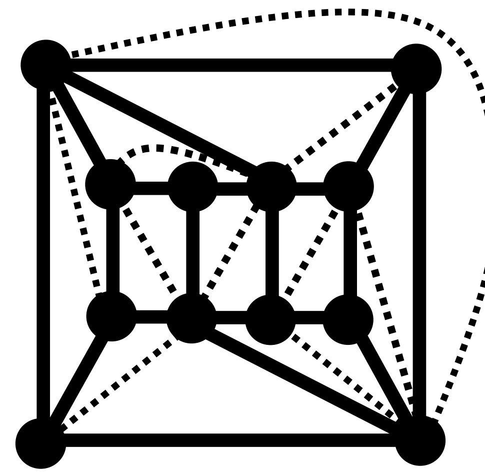


KS-quad.



normal quad.

2-dim case:



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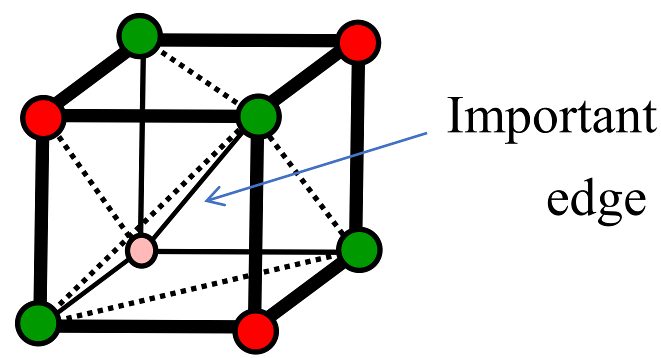

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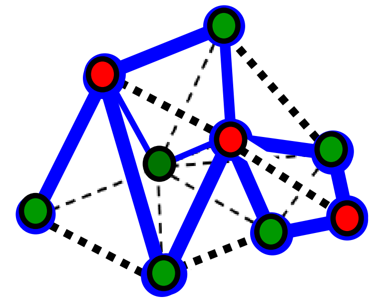
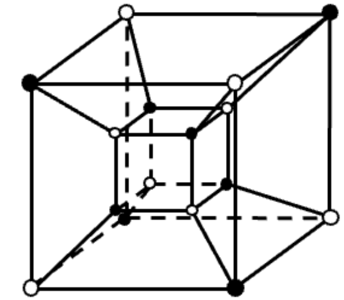


KS-quad.



normal quad.

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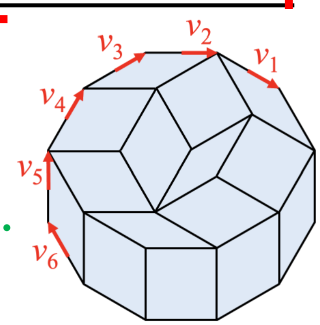
		No upper bound			No upper bound
normal quad.		$\chi(G)\geq 4$	$\chi(G)\geq 4$		$\chi(G)\geq 4$
		No upper bound	??		??

Thm. (Kaiser, Lo, Nakamoto, Nozaki, Oz., 24+)



Zonotopal quad.

Thm. (Hachimori, Nakamoto, Oz., 24)

For any $d\geq 3$, if G satisfies some geometrical condition, then $\chi(G)=4$