Polynomial Gyárfás-Sumner conjecture for graphs of bounded boxicity

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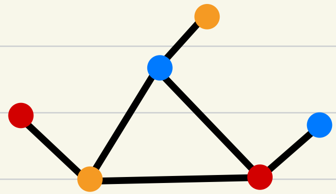
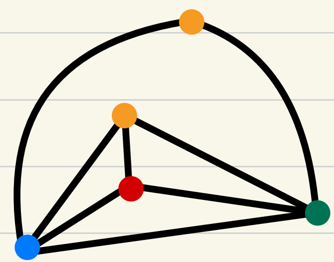
Fact:

$$\chi(G) \geq \omega(G)$$

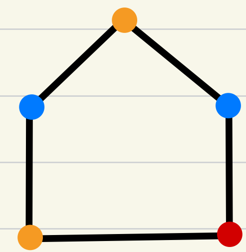
the chromatic number of G | the clique number of G



$\chi(G) = \omega(G) = 3$



$\chi(G) = \omega(G) = 4$



$\chi(G) = 3$
$\omega(G) = 2$

It is not true that $\chi(G) \leq \omega(G)$ $\forall G$.

Question:

Is there a function f such that
$$\chi(G) \leq f(\omega(G)) \quad \forall G\,?$$

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Question:

Is there a function f such that

$$\chi(G) \leq f(\omega(G)) \quad \forall G\,?$$

No.

Theorem (Erdős 1959):

For every $k \in \mathbb{N}$, there exists a graph with girth at least k and $\chi(G) \geq k$.

Question:

Is there a function $f_{\mathcal{G}}$ such that

$$\chi(G) \leq f_{\mathcal{G}}(\omega(G))$$

$\forall G \in \mathcal{G}$ for some class of graphs $\mathcal{G}$?

Question:

Is there a function $f_{\mathcal{C}}$ such that $\chi(G) \leq f_{\mathcal{C}}(\omega(G))$ $\forall G \in \mathcal{C}$ for some class of graphs $\mathcal{C}$?

Conjecture (Gyárfás-Sumner '80):

Yes, if $\mathcal{C} = \mathcal{C}_T$ is the class of graphs which do not contain the tree T as an induced subgraph.

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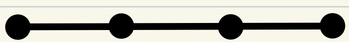
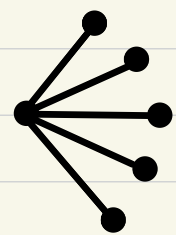
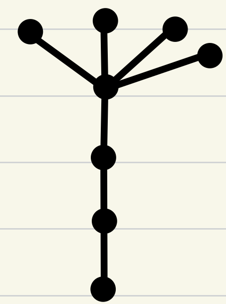


paths



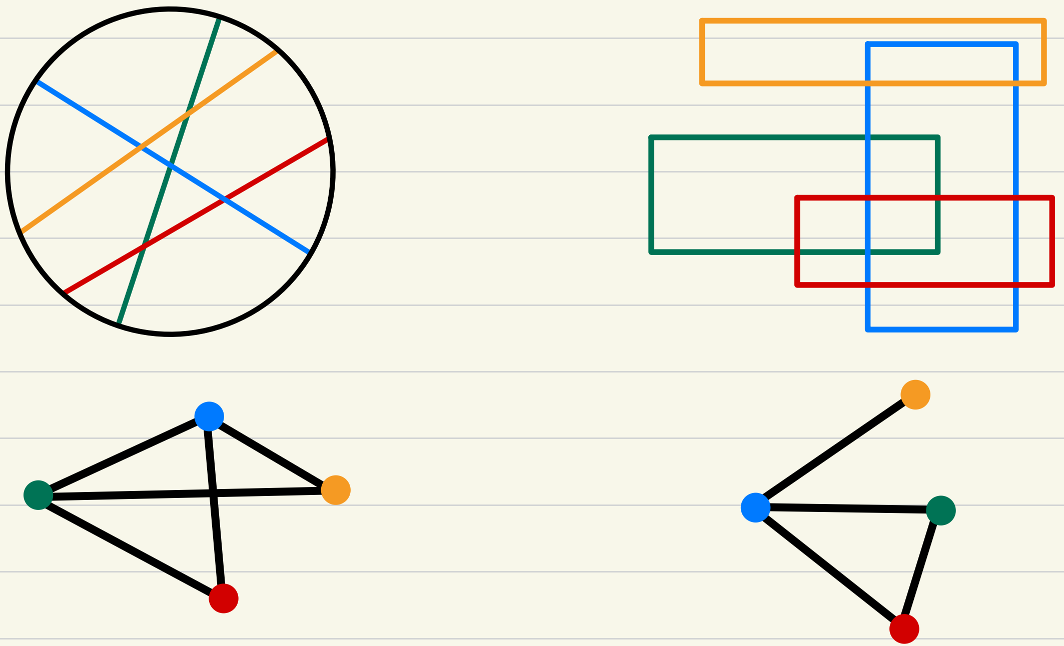
stars



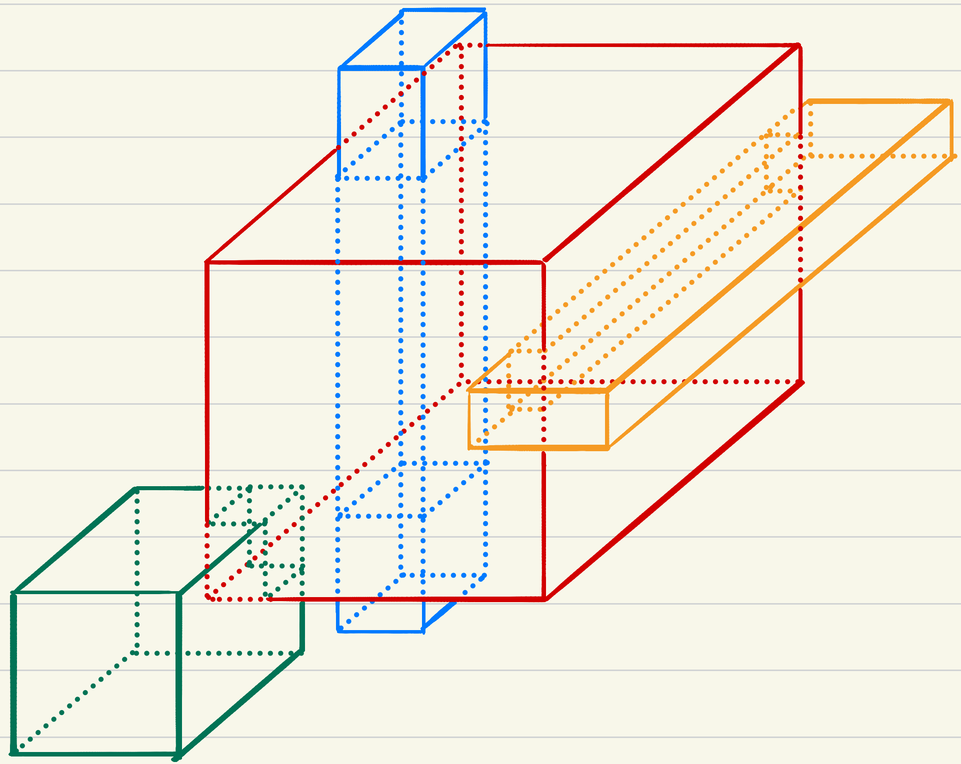
"brooms"

...

$\mathcal{G}$ is a class of geometrically defined graphs:



Axis-aligned boxes:



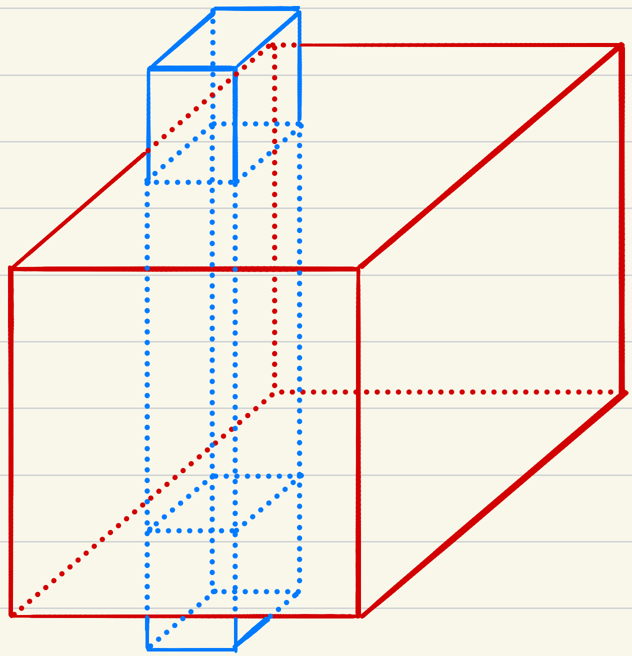
Our result:

Theorem (Davies, Y. 2024): $\forall d \in \mathbb{N}$ and a tree T, the class of intersection graphs of axis-aligned boxes in $\mathbb{R}^d$ with no induced copy of T is polynomially χ-bounded.

Proof idea: (in $\mathbb{R}^3$)

Partition the edges into a few types.

Type 1:



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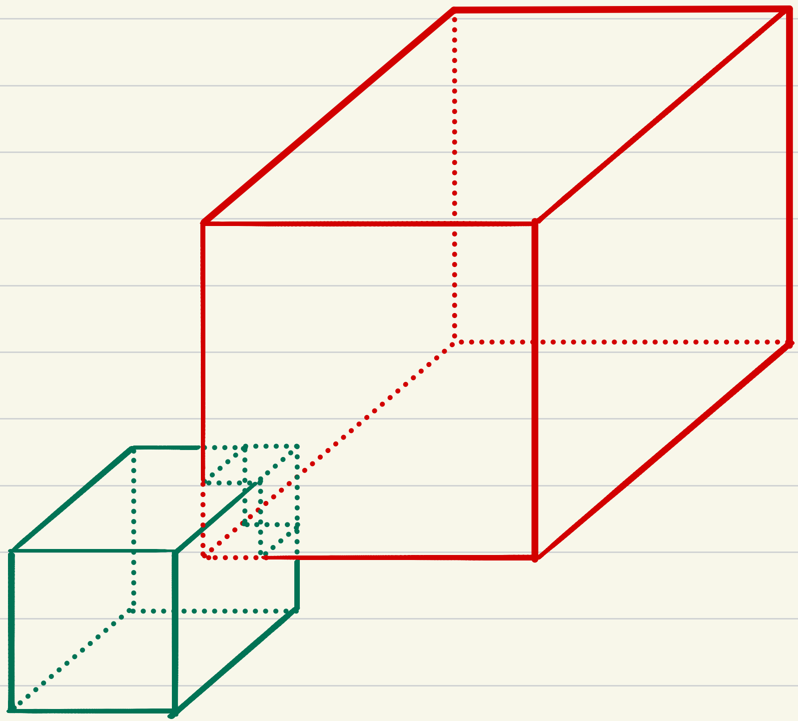

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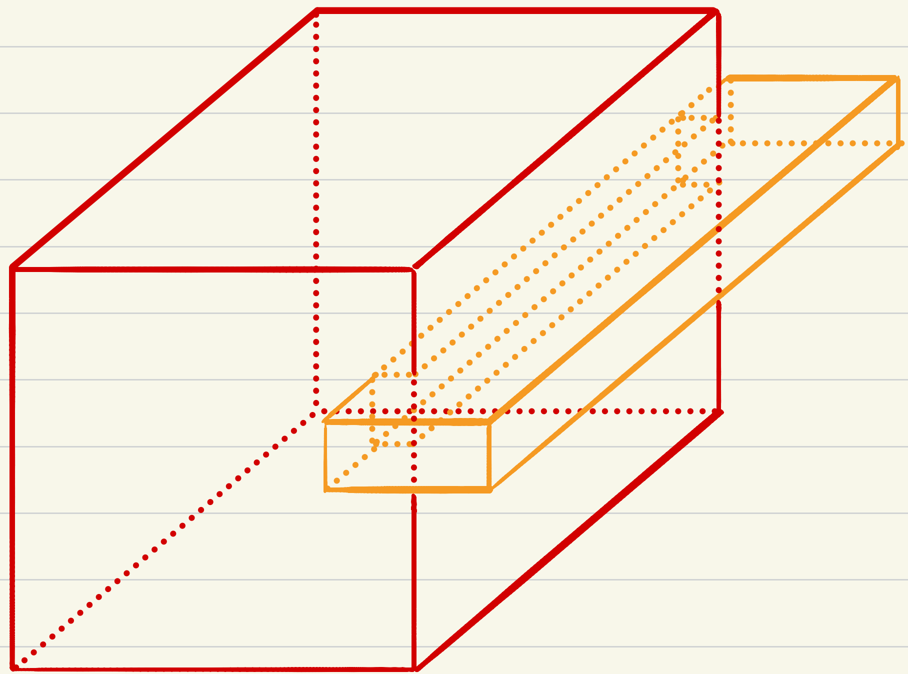

Open questions:

Does the Gyárfás-Sumner conjecture hold for the following classes of graphs:

- Intersection graphs of lines in $\mathbb{R}^d$? segments in $\mathbb{R}^d$?

- Hasse diagrams? Disjointness graph of strings in the plane?

Thank you!