Graphs with nonnegative resistance curvature

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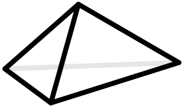
1. Graphs from simplices
2. Graph-theoretic characterization
3. Three results
4. Three questions

1. Graphs from simplices



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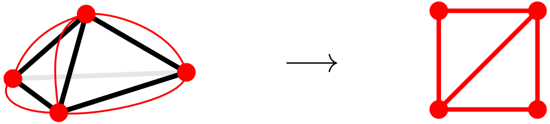

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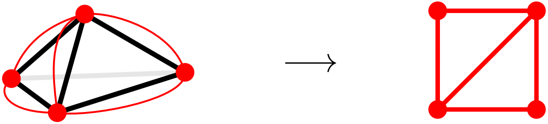

The acute skeleton of a simplex S is the graph $G(S)$ with

$$V = \{\text{vertices of S}\}$$
$$E = \{uv : \text{angle between } S\backslash u \text{ and } S\backslash v \text{ is acute } (< 90^\circ)\}$$

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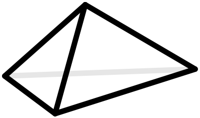

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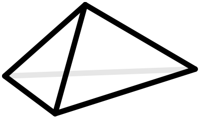

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▷ **non-obtuse** if all angles are non-obtuse ($\leq 90°$)

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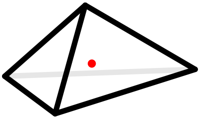

A simplex S is ...

▷ **non-obtuse** if all angles are non-obtuse $(\leq 90^\circ)$

▷ **centered** if it is non-obtuse and S contains its circumcenter

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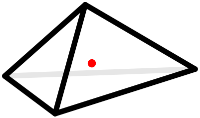

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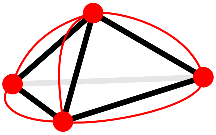

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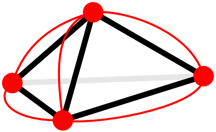

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The skeleton of a well-centered simplex S is 1-tough:

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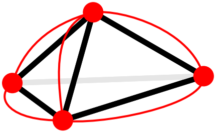

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The skeleton of a well-centered simplex S is 1-tough: to disconnect $G(S)$ in $k \geq 2$ components, at least k vertices must be removed from $G(S)$.

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- The ingredients (1)

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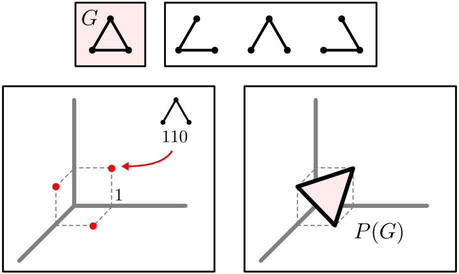

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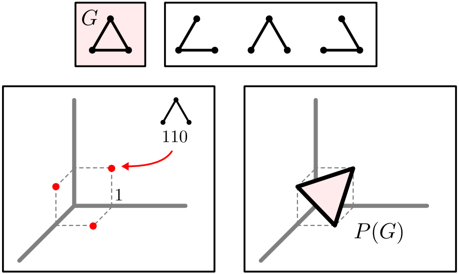

A **spanning tree** of G is a maximal acyclic subgraph $T \subseteq G$.

Spanning tree polytope $P(G)$ is the convex hull of $\chi_{E(T)}$ over all STs.

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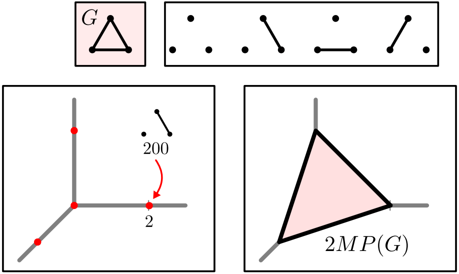

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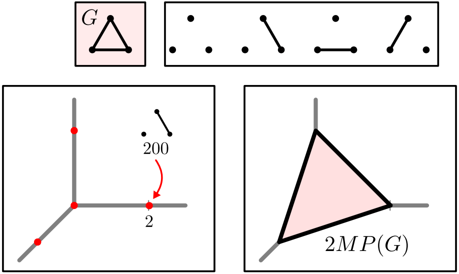

A **matching** of G is a subgraph $M \subseteq G$ with maximum degree 1.

2-matching polytope is the convex hull of $2\chi_{E(M)}$ over all matchings.

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3. *$P(G)^\circ \cap 2MP(G)$ is nonempty.*

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We call these graphs **resistance nonnegative (RN)**.

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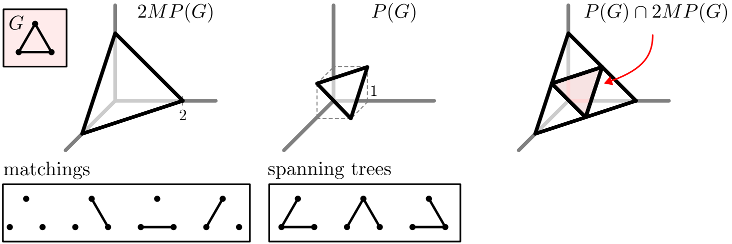

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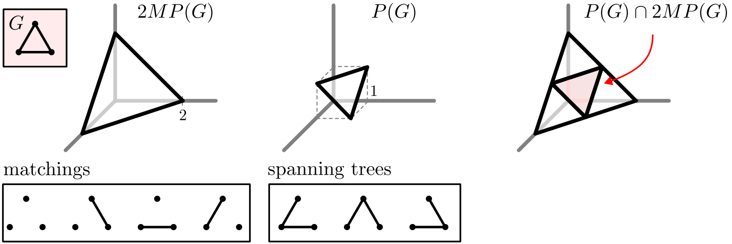
We call these graphs resistance positive (RP).



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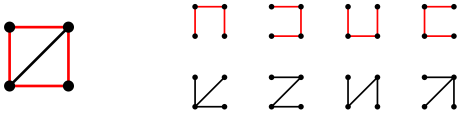
Proof idea:



$(1-\varepsilon) \times$ uniform distribution on Hamiltonian paths
$\varepsilon \times$ uniform distribution on all other trees

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If G is RN and even/odd, then G has a perfect/near perfect matching.

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Other suggestions or comments are welcome!

Thank you!

Questions?

karel.devriendt@maths.ox.ac.uk

"Graphs with nonnegative resistance curvature"
Karel Devriendt, arXiv:2410.07756 [math.CO]