Bad circuit graphs in Hamilton-like problems
--3-trees in 3-connected planar graphs--

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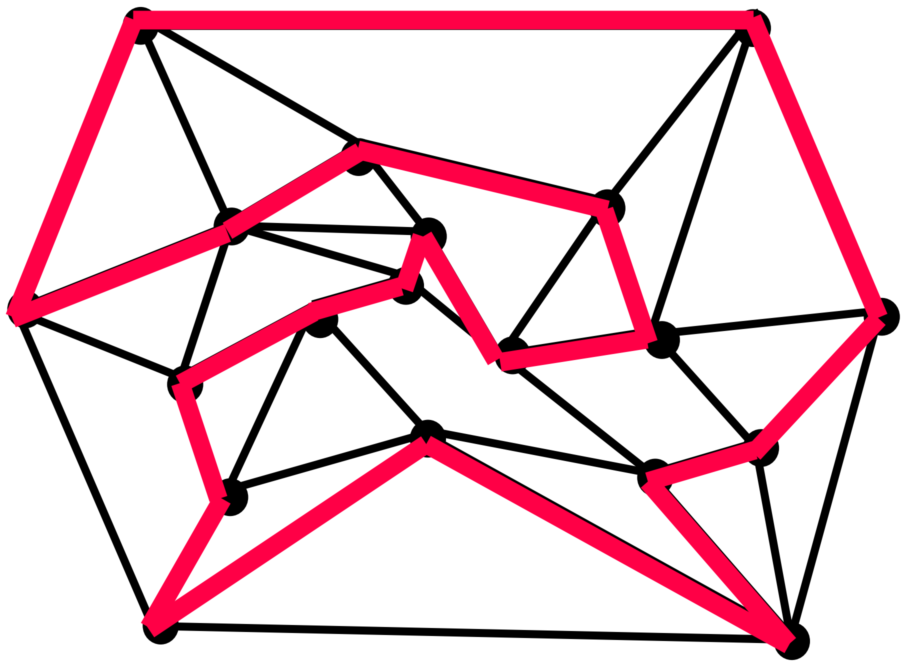
Theorem 1（Tutte, 1956）

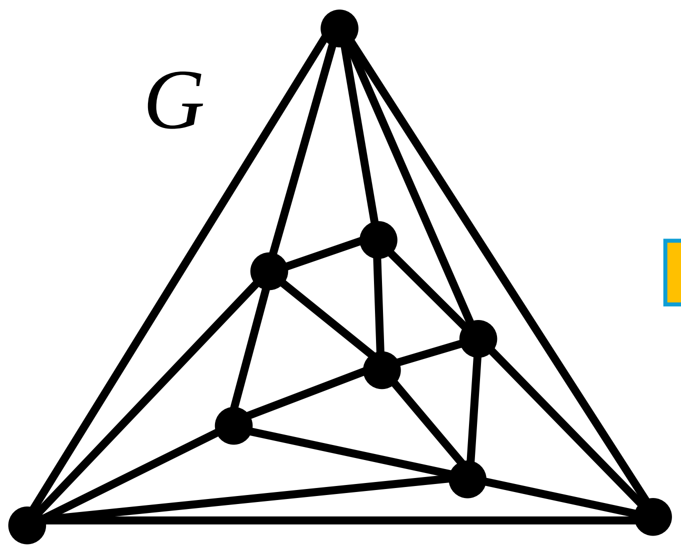
Every 4-connected planar graph has a Hamiltonian cycle.

G : k-connected $\Leftrightarrow$ for any k-1vertices S, $G - S$ is connected



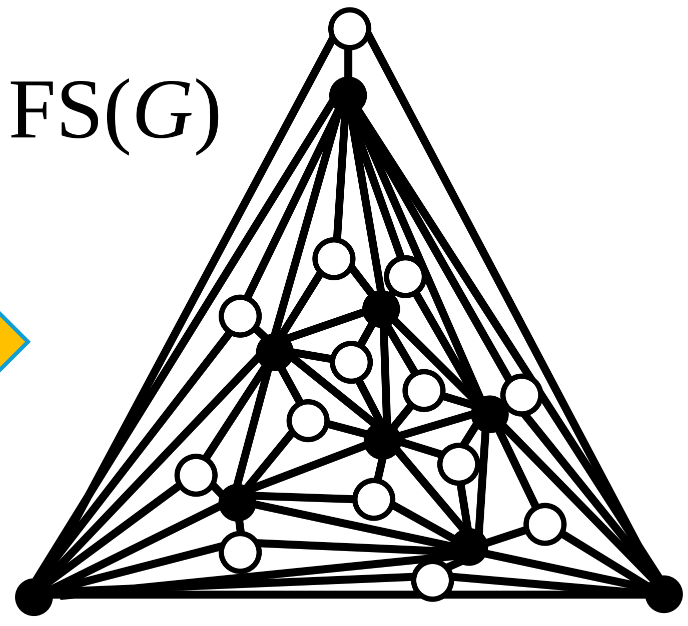
There exists a 3-connected planar graph with no Hamiltonain cycle.

FS(G) has no Hamiltonian cycle



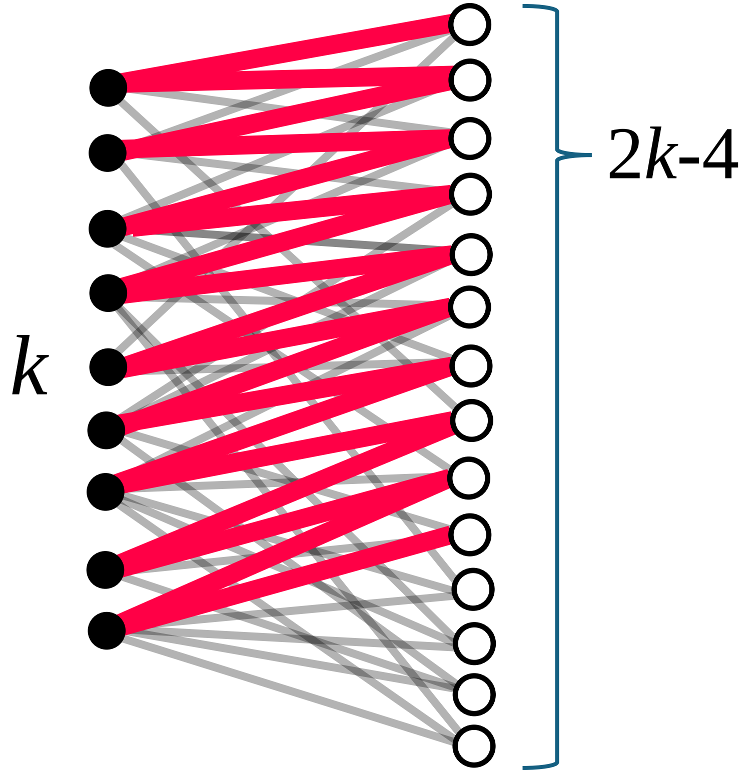
k-vertex triangulation

By Euler's formula,
$|F| = 2k-4$



The face subdivision of G

$|\text{black}| = k$, $|\text{White}| = 2k-4$
White vertices are independent



k black vertices
$2k$-4 white vertices

Hamilton-like properties

(Barnette, 1966)
Every 3-connected planar graph has a 3-*tree*, which is a stanning tree with maximum degree at most 3, and N., Oda, Ota improved it to bound the number of 3-vertices of a 3-tree.

(Barnette, 1994)
Every 3-connected planar graph has a 2-connected spanning subgraph of maximal degree at most 15, and Gao improved the bound of maximum degree to 6.

(Kaneko, Kawarabayashi, N., Ota, Richter, Yoshimoto, 2003)
Every 3-connected planar graph has a 2-connected spanning subgraph of size at most 3(n-1)/4.

(Gao and Richter, 1994)
Every 3-connected planar graph has a spanning 2-walk, which is a closed walk visiting every vertex at most twice. (# of vertices visited twice is not estimated.)

(Kawarabayashi and Ozeki, 2014)
Every 3-connected planar graph has a path of length $n^{\log_3 2}$.

and so on.

3-trees in 3-connected planar graphs

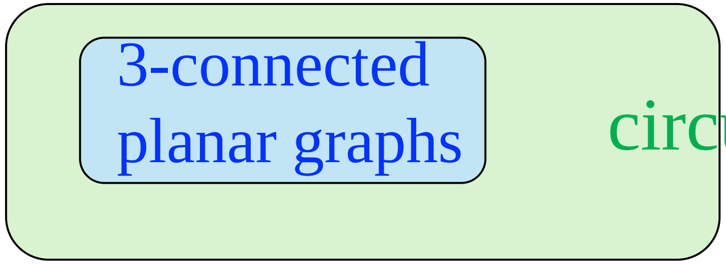
Theorem 2 (N., Oda & Ota, 2009)

G : 3-connected planar, $|V(G)| = n \geq 7$
$\Rightarrow \exists T \subset G$: 3-tree s.t. $|V_3| \leq \frac{n-7}{3}$

a spanning tree of maximal degree at most 3

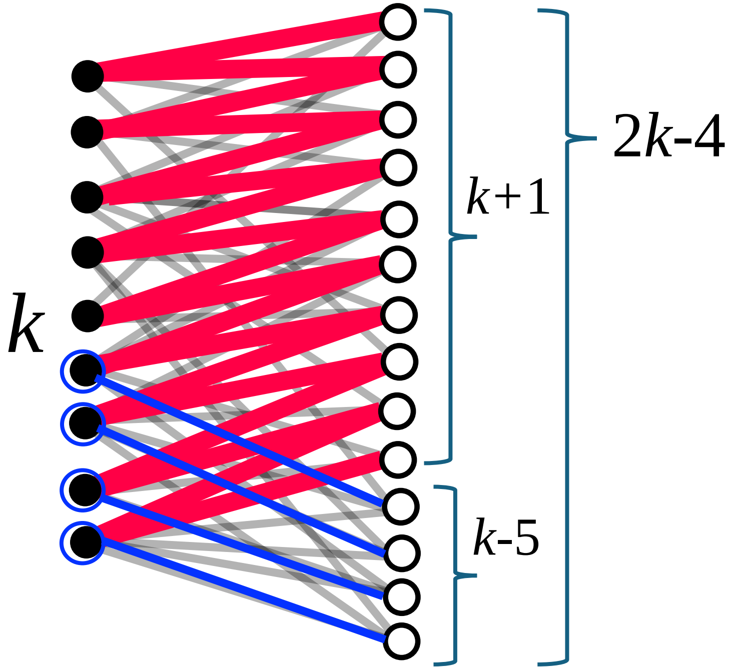
Theorem 4 (N., Oda & Ota, 2009)

G : circuit graph, $|V(G)| = n \geq 7$
$\Rightarrow \exists T \subset G$: 3-tree s.t. $|V_3| \leq \frac{n-7}{3}$ *best*



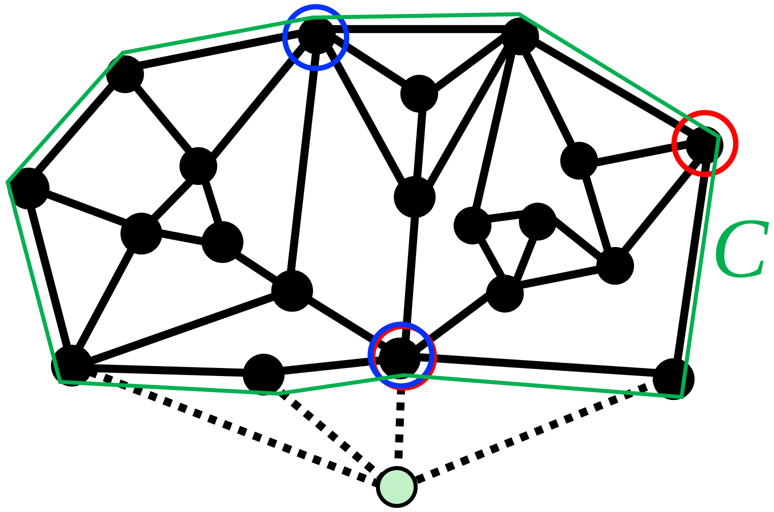
Conjecture 3 (N., Oda & Ota, 2009)

G : 3-connected planar, $|V(G)| = n \geq 11$
$\Rightarrow \exists T \subset G$: 3-tree s.t. $|V_3| \leq \frac{n-11}{3}$



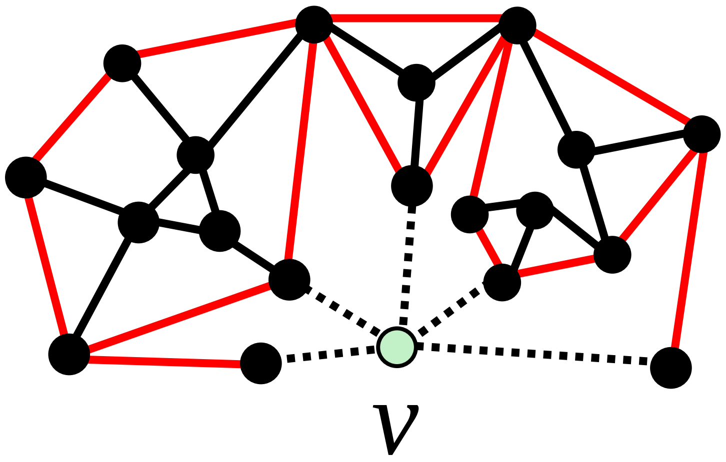
k-5 = (n-11)/3 ← n = 3k-4

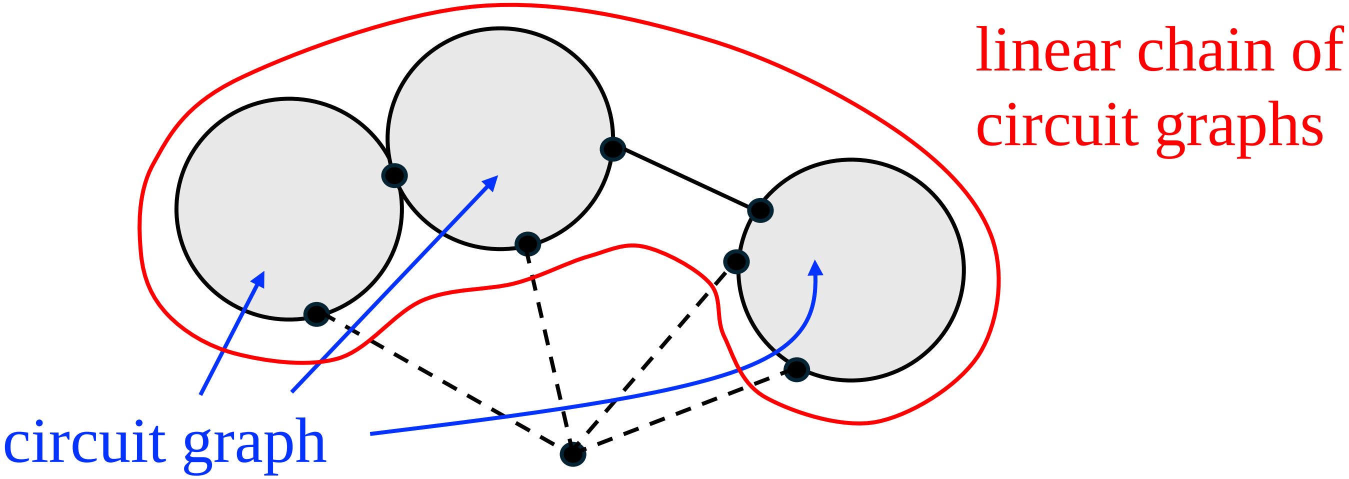
Circuit graphs



(G, C) : circuit graph $\Leftrightarrow$
G is a 2-connected plane graph with outer cycle C such that every inner vertex x has three internally-disjoint paths to C.

Hence a 3-connected plane graph with outer cycle C is a circuit graph (G, C).





For $v \in V(C)$, G-v is a linear chain of circuit graphs or K_2
$\Rightarrow$ this can be used in a proof for 3-connected plane graphs

Theorem 5

G : circuit graph, $n \geqq 7$

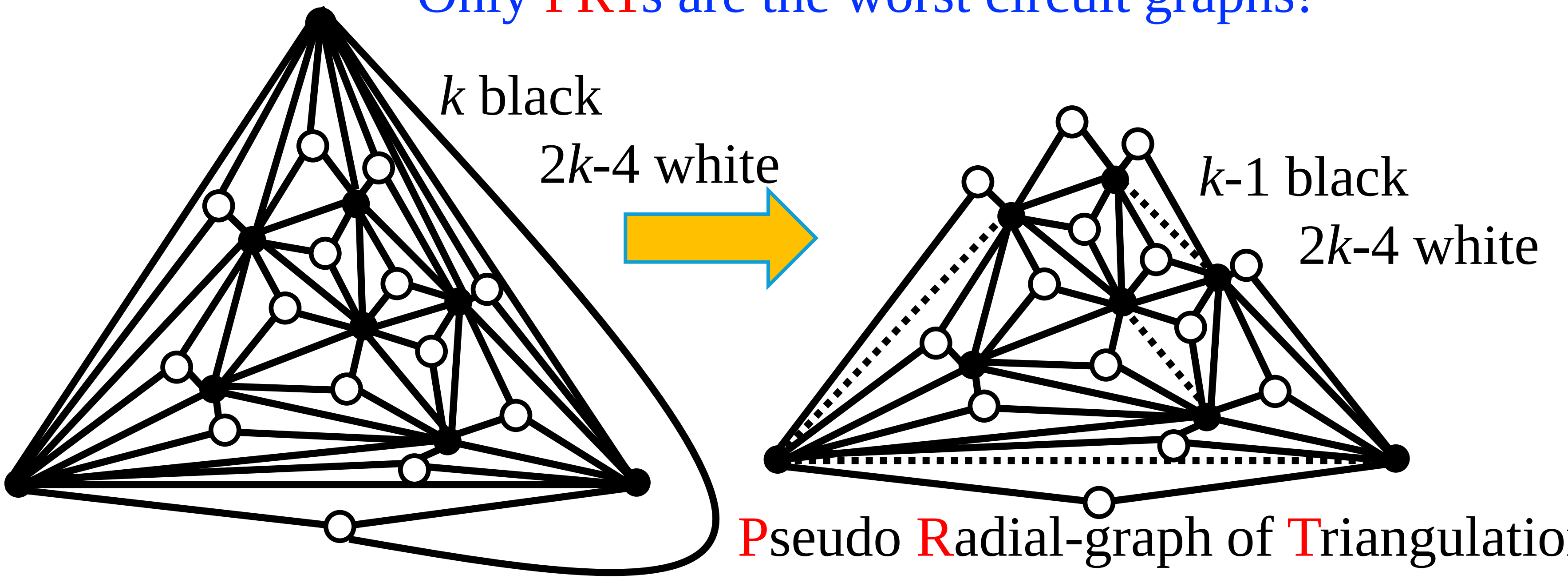
$\Rightarrow \exists T \subset G$: 3-tree s.t. $|V_3| \leq \frac{n-7}{3}$ (N., Oda & Ota, 2009)

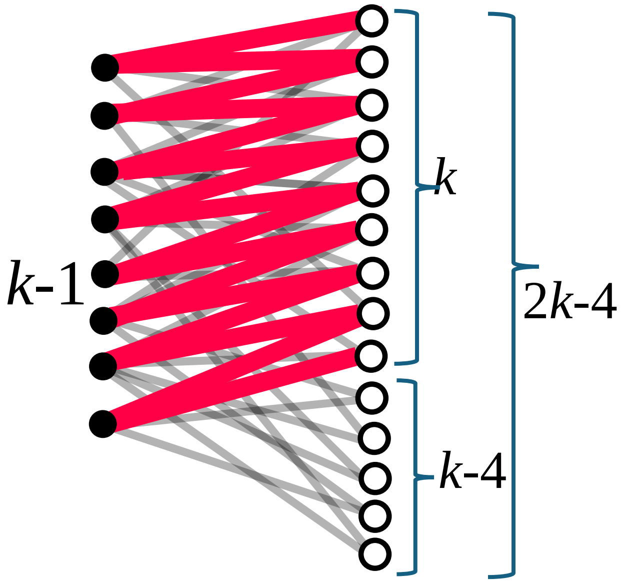
In particular, if G is not a PRT, then $|V_3| \leqq \frac{n-8}{3}$

A PRT attains this bound.

Only PRTs are the worst circuit graphs!

k-4 = (n+5)/3 – 4 = (n-7)/3





Pseudo Radial-graph of Triangulation

PRT

Corollary 5. G : 3-connected planar $\Rightarrow \exists T \subset G$: 3-tree s.t. $|V_3| \leq \frac{n-8}{3}$

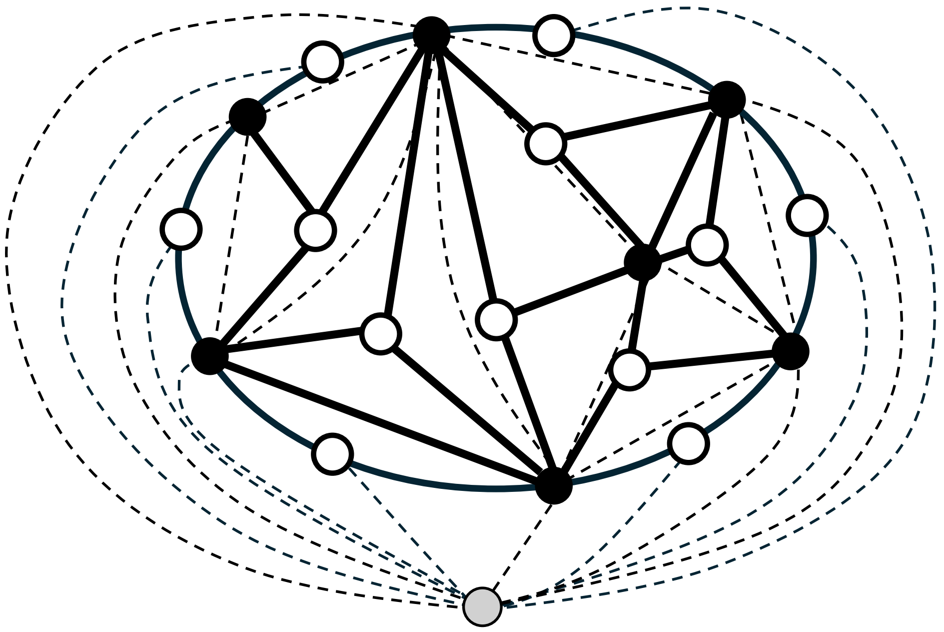
Conclusion

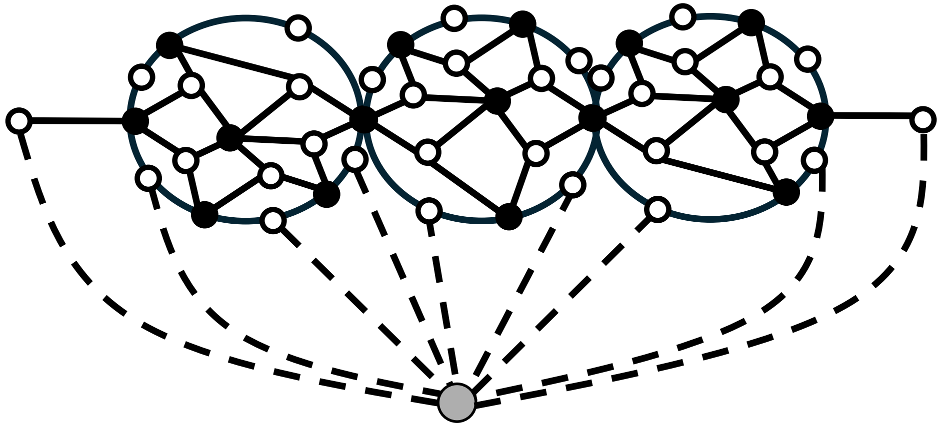
(1) Hamilton-like problems were popular in about 30 years (1990--), and using circuit graphs, researchers solved many problems.

(2) We have proved that a face subdivision of triangulations is only the worst graph in the theorems on 3-trees and 2-connected spanning subgraph with bounded number of edges.

(3) We have recently got a similar result on projective plane, torus and Klein bottle (but not checked them yet). We want to consider what we can say for other Hamilton-like problems.

Idea for the proof





Lemma 6

(G, C) : circuit graph

G is a PRT

$\Leftrightarrow$ $\exists$spartition $B \cup W = V(G)$ s.t.

$|B| \geq 2$, $|W| = 2|B| - 2$, W is indep.

$\forall w \in W \cap V(C)$, $\deg(w) = 2$,

$\forall w \in W - V(C)$, $\deg(w) = 3$,

black and white are alternate on C.

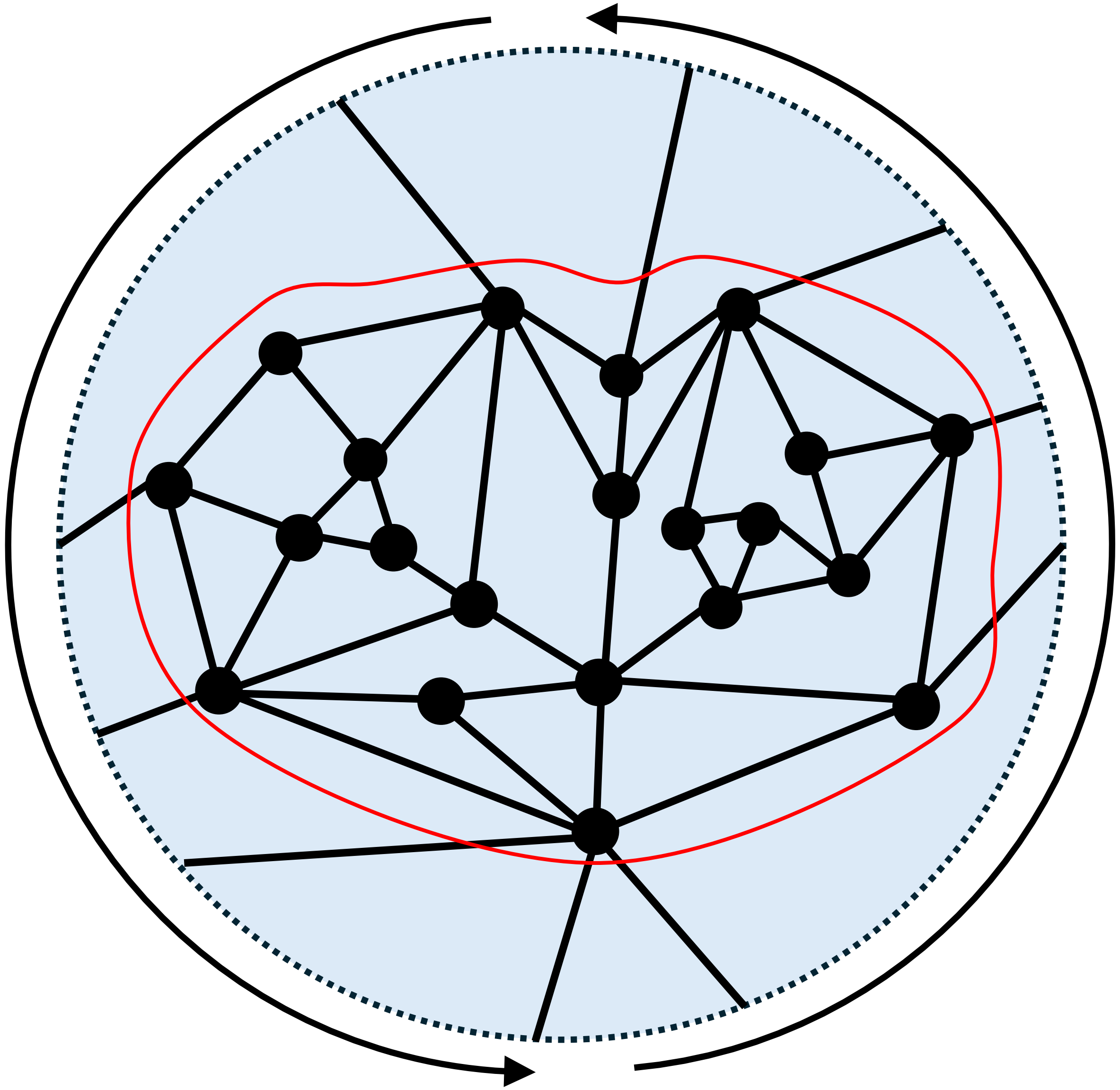
Theorem 7

G : circuit graph, $n \geq 7$

$\Rightarrow \exists T$: 3-tree s.t. $|V_3| \leq \frac{n-7}{3}$

$G \neq$ PRT $\Rightarrow |V_3| \leq \frac{n-8}{3}$

3-connected graphs on the projective plane



projective plane

Theorem 8

G : 3-connected, projective planar

(1) ⇒ ∃ 3-tree (Gao & Richter, 1994)

(2) ⇒ $\exists T$: 3-tree s.t. $|V_3| \leq \frac{n-7}{3}$ *best*

(N., Oda & Ota, 2009)

A face subdivision of projective-planar triangulations attains this equality.

Theorem 5

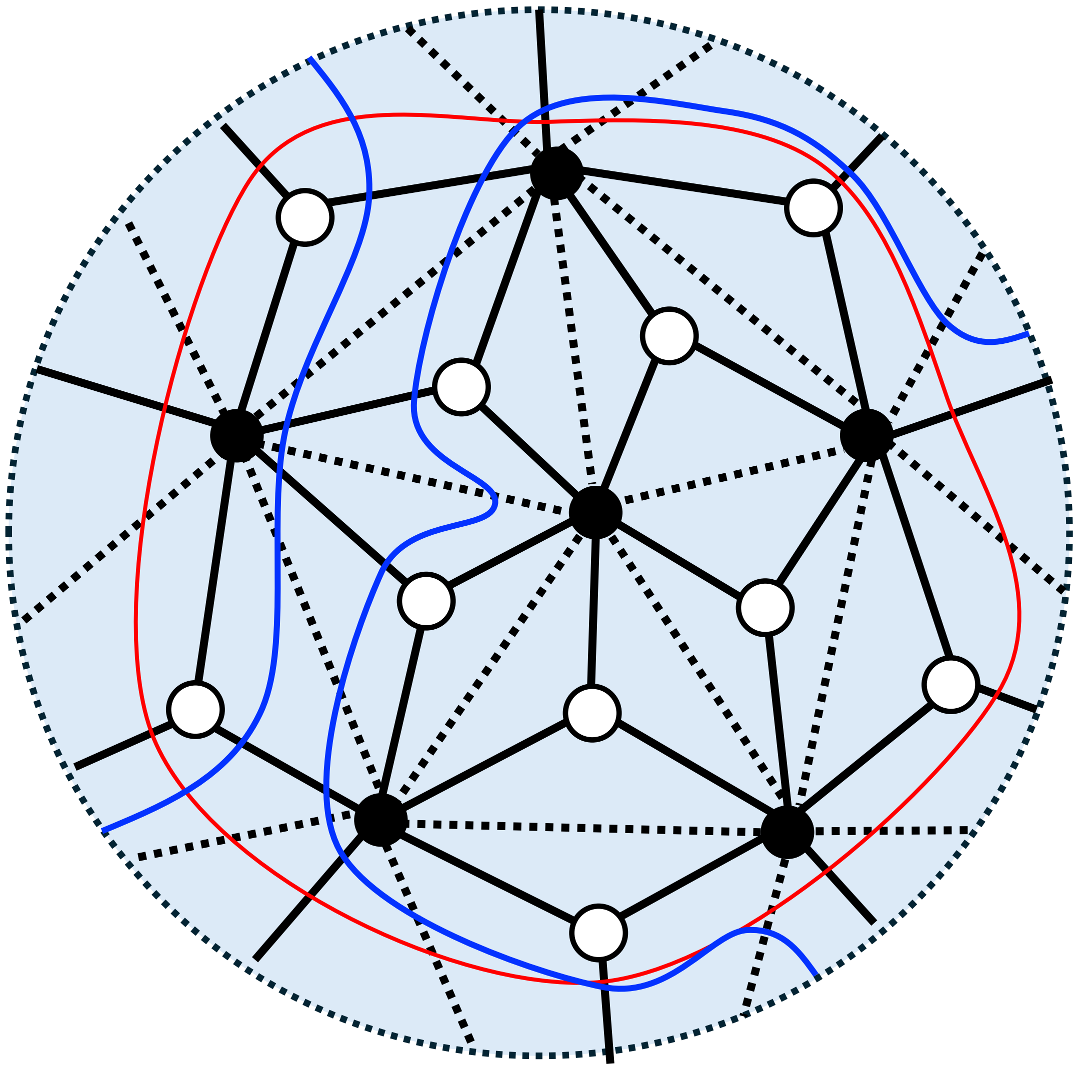
D : circuit graph, $n \geq 7$

⇒ $\exists T$: 3-tree s.t. $|V_3| \leq \frac{n-7}{3}$

$D \neq$ PRT ⇒ $|V_3| \leq \frac{n-8}{3}$

Can we characterize the worst graphs G ?

3-connected graphs on the projective plane



Theorem 8

G : 3-connected, projective planar

(1) $\Rightarrow \exists$ 3-tree (Gao & Richter, 1994)

(2) $\Rightarrow \exists T$: 3-tree s.t. $|V_3| \leq \frac{n-7}{3}$ *best*

(N., Oda & Ota, 2009)

$\forall T \subset G$: 3-tree, $|V_3| \geq \frac{n-7}{3}$

$\Leftrightarrow$ every spanning circuit subgraph D is a PRT

$\Leftrightarrow$ G : face subdivision of a triangulation

Theorem 9

G : 3-connected projective planar, $n \geq 7$

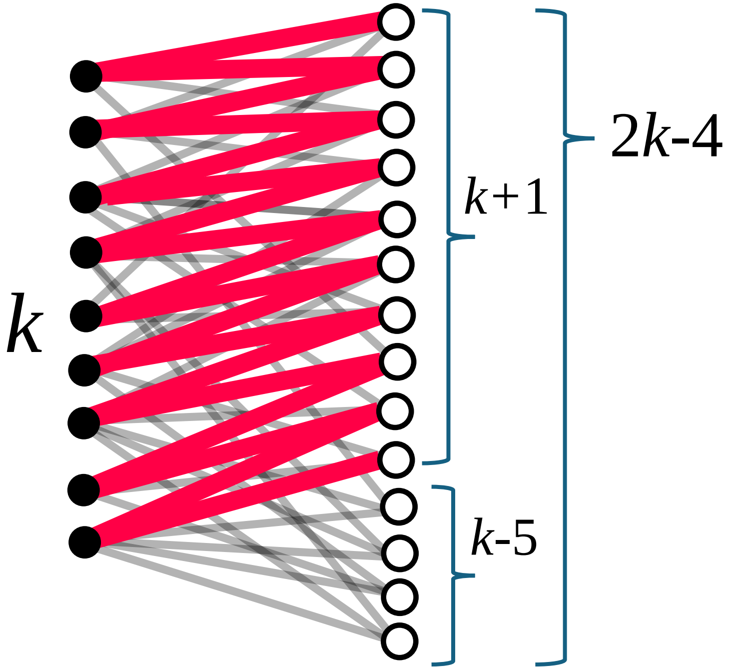
$\Rightarrow \exists T$: 3-tree s.t. $|V_3| \leq \frac{n-7}{3}$

$G \neq$ face subdivision of a triangulation

$\Rightarrow \exists T$: 3−tree s.t. $|V_3| \leq \frac{n-8}{3}$

Conjecture 3 (N., Oda & Ota, 2009)

G : 3-connected planar, $|V(G)| = n \geq 11$
$\Rightarrow \exists T \subset G$: 3-tree s.t. $|V_3(T)| \leq \frac{n-11}{3}$



k-5 = (n-11)/3

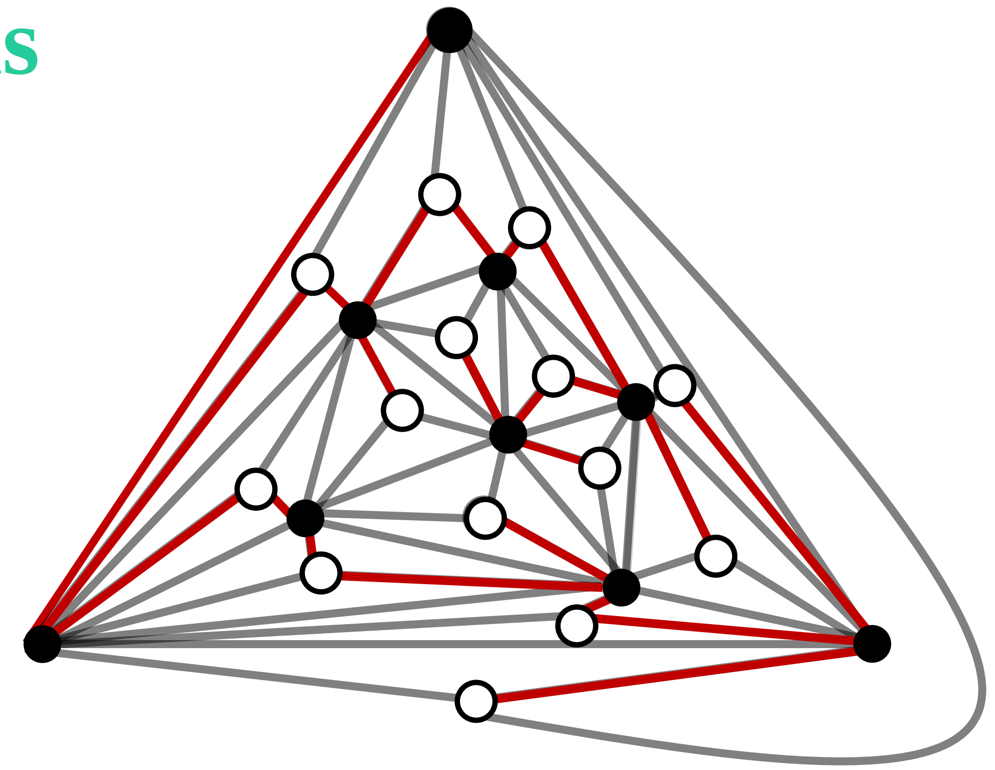
n = 3k-4

3-trees in 3-connected planar graphs

3-tree : a spanning tree of maximal degree at most 3

Then a Hamiltonian path is a 2-tree.

A face subdivision of a triangulation is a 3-connected planar graph with no Hamiltonain path.



Theorem 2

G : 3-connected planar
(1) $\Rightarrow \exists T \subset G$: 3-tree (Barnette, 1966)
(2) $\Rightarrow \exists T \subset G$: 3-tree s.t. $|V_3| \leq \frac{n-7}{3}$ (N., Oda & Ota, 2009)

Conjecture.
This should be $\frac{n-11}{3}$

However, this is best possible for circuit graphs

Hamilton-like properties

(Barnette, 1966)
Every 3-connected planar graph has a spanning 3-*tree*, which is a tree with maximum degree at most 3, and N., Oda, Ota improved it to bound the number of 3-vertices of a 3-tree.

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a face subdivision of a triangulation is a worst graph!

and so on.