Seven problems on hypohamiltonian and almost hypohamiltonian graphs

Carol T. Zamfirescu*

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Here, all graphs are undirected, finite, connected, and neither contain loops nor multiple edges. Let G be a graph. If a vertex of G has degree 3, then we call the vertex *cubic*. If every vertex of G is cubic, we call G *cubic*. G is *hamiltonian* if it contains a hamiltonian cycle, i.e. a cycle passing through all vertices of the graph. G is *hypohamiltonian* if G does not contain a hamiltonian cycle but for any vertex v of G, the graph $G - v$ does contain a hamiltonian cycle. If we replace "cycle" by "path", we obtain the definition of a *hypotraceable* graph.

Let n_0 (c_0) be the smallest natural number such that there exists a planar (planar cubic) hypohamiltonian graph of order n for every $n \geq n_0$ (for every even $n \geq c_0$).

The solution to the first problem we want to present would constitute an improvement to the main result from [4]. We consider it to be very interesting, since many consequences concerning problems on longest paths and cycles arise, but also very difficult. Similarly, we consider Problem 4 to be especially intriguing.

1. In [4], together with Jooyandeh, McKay, Östergård, and Pettersson, we showed that there exist 25 planar hypohamiltonian graphs of order 40. Despite the progress made in [4], there still is a considerable gap between the order of the smallest known planar hypohamiltonian graph, which is 40, and the best lower bound known for the order of the smallest such graphs, which is 18, see [1].

(a) Do planar hypohamiltonian graphs on less than 40 vertices exist?

(b) What is n_0? (It is known that $n_0 \leq 42$, see [4].)

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2. Araya and Wiener showed that there exists a planar cubic hypohamiltonian graph on 70 vertices [2].
(a) Is there such a graph with fewer vertices?
(b) Correspondingly, we ask for c_0. (Presently, we know that $c_0 \leq 74$, see [6].)

A graph G is *almost hypohamiltonian* if G is non-hamiltonian, and there exists a vertex w, which we will call *exceptional*, such that $G - w$ is non-hamiltonian, yet for any vertex $v \neq w$ the graph $G - v$ is hamiltonian. Define n_0' to be the smallest natural number such that there exists a planar almost hypohamiltonian graph of order n for every $n \geq n_0'$.

3. (a) We know that there exists an almost hypohamiltonian graph on 17 vertices [6]. Are there smaller examples?
(b) Do almost hypohamiltonian graphs of order $n \in \{18, 19, 21, 24\}$ exist?

4. Thomassen's question from 1978 whether 4-connected hypohamiltonian graphs exist remains open [5]. We ask here:
(a) Do 4-connected hypotraceable graphs exist? (Horton [3] showed that 3-connected hypotraceable graphs exist.)
(b) Do 5-connected almost hypohamiltonian graphs exist? (We know that 4-connected almost hamiltonian graphs do exist [6].)
Note that, by results from [6], a positive answer to (a) would imply a positive answer to (b), but not necessarily vice-versa.

5. Is there an almost hypohamiltonian graph with a cubic exceptional vertex and all other vertices of degree at least 4? Solving this would answer, by using a result from [6], Thomassen's question whether hypohamiltonian graphs with minimum degree 4 exist [5].

6. (a) Is there a planar almost hypohamiltonian graph with fewer than 39 vertices? (An example with 39 vertices is available [6].)
(b) What is n_0'? (So far, it is known that $n_0' \leq 76$, see [6].)

7. Due to certain gluing results, planar almost hypohamiltonian graphs with cubic exceptional vertex are of special interest. In [6] it is shown that there exists a planar almost hypohamiltonian graph of order 47 whose exceptional vertex is cubic. Are there smaller examples? (This would help improve the bound for n_0'.)

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Carol T. Zamfirescu

Department of Applied Mathematics, Computer Science and Statistics,

Ghent University, Krijgslaan 281 - S9, 9000 Ghent, Belgium.

czamfirescu@gmail.com