On a conjecture of Grünbaum on longest cycles

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Abstract. Grünbaum conjectured that for any integer $k \geq 2$, there exists no n-vertex graph G of circumference $n - k$ in which the removal of any k vertices from G yields a hamiltonian graph. We show that for any positive integers c and k there is an infinite family of c-connected graphs of circumference k less than their order, in which the limit (as the graphs' order tends to infinity) of the ratio between the number of k-vertex sets whose removal yields a hamiltonian graph and the number of all k-vertex sets is 1. Motivated by a question of Katona, Kostochka, Pach, and Stechkin, it is proven that there exists an infinite family of non-hamiltonian graphs of increasing diameter d in which the removal of any two vertices at distance 1 or any distance at least $(d + 6)/2$ yields a hamiltonian graph.

Key words. Circumference; longest cycle; hamiltonian

MSC 2020. 05C38, 05C45

1 Introduction

Grünbaum introduced $\Gamma(j, k)$ in [6] as the family of all graphs whose circumference is exactly k less than their order, and in which any j vertices are missed by some longest cycle; *circumference* here means the length of a longest cycle. $\Gamma(1, 1)$ are exactly the *hypohamiltonian* graphs, i.e. non-hamiltonian graphs in which every vertex-deleted subgraph is hamiltonian. These have been studied extensively and various infinite families are known. Examples include Petersen's graph and Coxeter's graph. In 1974, Grünbaum [6] conjectured that $\Gamma(j, j)$ is empty for all $j \geq 2$. Very little is known about this conjecture. Thomassen [8] agrees with Grünbaum that $\Gamma(2, 2)$ is empty, and points out that every member of $\Gamma(2, 2)$ has the property that each of its vertex-deleted subgraphs lies in $\Gamma(1, 1)$. Clearly, a graph lying in $\Gamma(2, 2)$ would have to be 4-connected, and proving hamiltonian properties of 4-connected graphs is notoriously difficult. One reason is that many powerful tools rely on planarity, but by a celebrated theorem of Tutte planar 4-connected graphs are hamiltonian, so they certainly do not lie in $\Gamma(2, 2)$ (or $\Gamma(j, j)$ for any j for that matter).

A problem related to the question whether $\Gamma(2, 2)$ is empty was raised in 1989 by Katona, Kostochka, Pach, and Stechkin [5]. They asked whether an n-vertex graph in which every induced k-vertex subgraph is hamiltonian must itself be hamiltonian, where k is an integer satisfying $n/2 < k < n - 1$. We shall here be interested in the case $k = n - 2$, i.e. whether

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there exists a non-hamiltonian n-vertex graph in which any two vertices are missed by an $(n-2)$-cycle; this is a relaxation of Grünbaum's question. This problem, in a different but equivalent formulation, was also raised by van Aardt, Burger, Frick, Llano, and Zuazua, see [1, Question 1]. The only difference between Grünbaum's problem for $j = 2$ and the Katona et al. problem (for $k = n - 2$) is that the former does not allow $(n-1)$-cycles to be present, while the latter does.

Throughout this paper, for a given graph G the *distance* between vertices v and w in G is the length of a shortest path in G with endpoints v and w. We denote the *diameter* of G, i.e. the greatest distance between any two of its vertices, by $\mathrm{diam}(G)$, but sometimes it will be convenient to simply write d. Also, put $D := \{1, \ldots, \mathrm{diam}(G)\}$. We shall investigate metric relaxations of the aforementioned problems and will use to this end the following definitions. For an integer $k \geq 2$, let $\Gamma(2, k, S)$ be the family of all n-vertex graphs G of circumference $n - k$ in which for all $c \in S \subseteq D$, for any pair of vertices v and w in G at distance exactly c the graph $G - v - w$ contains an $(n-k)$-cycle. Clearly, $\Gamma(2, k) = \Gamma(2, k, D)$. For an integer $j \geq 3$, let $\Gamma(j, k, c)$ be the family of all n-vertex graphs G of circumference $n - k$ in which for any j-vertex set X in G composed exclusively of vertices at pairwise distance at least $c \in D$ the graph $G - X$ contains an $(n-k)$-cycle; observe that $\Gamma(j, k) = \Gamma(j, k, 1)$ and that these definitions only make sense for $k \geq j$. We emphasise that the definitions for $j = 2$ and $j \geq 3$ vary because removing pairs of vertices at distance exactly c for all $c \in D$ encompasses the choice of any pair of vertices, but removing e.g. triples of vertices at distance *exactly* c for all $c \in D$ does not encompass the choice of any triple of vertices, e.g. two adjacent vertices and a third vertex, far from the first two.

By replacing in the preceding two definitions the term "$(n-k)$-cycle" by "hamiltonian cycle", we define $\Gamma^*(2, k, S)$, $\Gamma^*(j, k, c)$ (for $j \geq 3$), and $\Gamma^*(j, k)$ (for $j \geq 2$); the latter is thus the family of all n-vertex graphs of circumference $n - k$ in which the removal of any j-vertex set yields a hamiltonian graph. These definitions only make sense for $j \geq k$. We note that $\Gamma(2, 2, S) = \Gamma^*(2, 2, S)$, $\Gamma(j, j, c) = \Gamma^*(j, j, c)$ (for $j \geq 3$), and in particular $\Gamma(j, j) = \Gamma^*(j, j)$ (for $j \geq 2$). In this notation, the Katona et al. problem is equivalent to the question whether $\Gamma^*(2, 1) \cup \Gamma(2, 2)$ is empty or not.

The paper is structured as follows. We shall first show that for every positive integer k there exists an infinite family of graphs of circumference k less than their order, in which the limit (as the graphs' order tends to infinity) of the ratio between the number of k-vertex sets whose removal yields a hamiltonian graph and the number of all k-vertex sets is 1; thereafter, metric relaxations of Grünbaum's conjecture and the Katona et al. problem are given. These complement the theorem from [10] stating that there exists an infinite family $\mathcal{G} \subset \Gamma^*(2, 1, \{1\})$ with

$$\sup_{G \in \mathcal{G}} \frac{|\{v \in V(G) : G - v \text{ is non-hamiltonian}\}|}{|V(G)|} = \frac{1}{4}$$

as well as the result from [4] that $\Gamma(1, 1) \cap \Gamma^*(2, 1, \{1\})$ contains infinitely many polyhedral graphs.

Graphs in this paper are assumed to be connected, unless explicitly stated otherwise; for a possibly disconnected graph G, we write $\omega(G)$ for the number of connected components of G. The connectivity of a graph G will be denoted by $\kappa(G)$. In a graph G, a path with endpoint $v \in V(G)$ is called a *v-path* and a path between distinct vertices $v \in V(G)$ and $w \in V(G)$ is called a *vw-path*. A graph is *traceable* if it contains a hamiltonian path, and *hypotraceable* if it is non-traceable, but all of its vertex-deleted subgraphs are traceable. For a positive integer k, we write $[k] := \{1, \ldots, k\}$. For a vertex v in a graph G, we denote by $N_G(v)$ its set of neighbours; whenever the graph G is clear from the context, we simply write $N(v)$. For a set S and a positive integer k, we put $\binom{S}{k} := \{X \subseteq S : |X| = k\}$. For a graph G, we denote by $\overline{G}$ its complement.

A set of vertices whose removal disconnects a given graph is a *cut*, and a k-*cut* is a cut of cardinality k. Let G be a non-complete graph of connectivity k and order greater than k, X a k-cut in G, and C a component of $G - X$. Then $G[V(C) \cup X]$ is a k-*fragment* of G with *attachments* X, but sometimes we will suppress specifying the attachments, or shorten this and simply write X-*fragment*. A k-fragment is *trivial* if it contains exactly $k + 1$ vertices, and a cut X of a graph G is *trivial* if $G - X$ has exactly two components and X is the set of attachments of a trivial k-fragment. Let F, F' be disjoint 3-fragments of graphs of connectivity 3, and let F have attachments x_1, x_2, x_3 and F' have attachments x'_1, x'_2, x'_3. Identifying x_i with x'_i for all i, we obtain the graph $(F, \{x_1, x_2, x_3\}) \vdots (F', \{x'_1, x'_2, x'_3\})$. When the vertices that are being identified (always using a bijection) are clear from the context, we simply write $F \vdots F'$. In a 2-connected non-hamiltonian graph G, we call $\text{exc}(G) \subset V(G)$, which contains every vertex v of G such that $G - v$ is non-hamiltonian, the set of *exceptional vertices* of G. A 2-connected non-hamiltonian graph with exactly one exceptional vertex is *almost hypohamiltonian*.

2 Results

The first part of the next theorem aims at giving an asymptotic solution to Grünbaum's problem which is constructive, as well as pointing out that any graph may occur as an induced subgraph of the constructed graphs. The latter property is inspired by Chvátal's question whether every graph occurs as an induced subgraph of some hypohamiltonian graph (solved, affirmatively, in [11]). The second part of the next theorem is motivated as follows. In [10] it was observed that $\Gamma(2, 2, \{2\}) \neq \emptyset$ due to the join of K_t and $\overline{K}_{t+2}$. Note that here $D \setminus \{1\} = \{2\}$. In the same way, for any integers $j \geq 3$ and $t \geq 2$ we have that the join of K_t and $\overline{K}_{t+j}$ lies in $\Gamma(j, j, 2)$. It would be of interest to find similar results for graphs of diameter larger than 2. More precisely, to verify for $|D| \geq 3$ whether $\Gamma(2, 2, D \setminus \{1\})$ or $\Gamma(j, j, 2)$ (for $j \geq 3$) are empty or not. Another way to see $\Gamma(2, 2, D \setminus \{1\}) \neq \emptyset$ is as covering half of all distances, since here we have $|D| = 2$. This we *can* generalise, as shown below.

Theorem 1. (i) *For any positive integers j and κ, and possibly disconnected graph A, there exists an infinite family of κ-connected graphs $\mathcal{G}$ such that A is an induced subgraph of every graph in $\mathcal{G}$, every $G \in \mathcal{G}$ has circumference $|V(G)| - j$, and*

$$\lim_{\substack{|V(G)| \to \infty, \\ G \in \mathcal{G}}} \frac{\left|\left\{W \in \binom{V(G)}{j} : G - W \text{ is hamiltonian}\right\}\right|}{\left|\binom{V(G)}{j}\right|} = 1.$$

(ii) *Let $j \geq 3$ be an integer. The families $\Gamma(2, 2, \{\lceil d/2 \rceil + 1, \ldots, d\})$ and $\Gamma(j, j, \lceil d/2 \rceil + 1)$ contain for infinitely many values of d infinitely many graphs of connectivity 2 and diameter d as well as infinitely many graphs of connectivity 3 and diameter d.*

Proof. We will need the following construction. Let j be a given positive integer, and let $k > j$ be any integer. Consider pairwise disjoint graphs $H_1, \ldots, H_k$, each containing a distinguished vertex $x_i \in V(H_i)$ which we will call *special*. Let $G(H_i - x_i; k; B)$ be the graph obtained from pairwise disjoint graphs H_i, $i \in [k]$, and a disjoint (and possibly disconnected) graph B by joining each vertex of B with each vertex in $\bigcup_i N(x_i)$ and deleting vertices x_i, $i \in [k]$. The following auxiliary result will be useful.

Claim. *Consider positive integers j and k with $j < k$. For every $i \in [k]$, assume a graph H_i to have order at least $j + 2$ and to be hypohamiltonian, in which case its special vertex x_i can be chosen arbitrarily, or almost hypohamiltonian, in which case its special vertex x_i shall be*

its exceptional vertex. Then for any graph B on $2k-j$ vertices, connected or not,
(a) *the circumference of $G := (H_i - x_i; k; B)$ is $|V(G)| - j$; and*
(b) *for any $W \subset V(G) \setminus V(B)$ with $|W| = j$ and $|W \cap V(H_i - x_i)| \leq 1$ for every $i \in [k]$, we have that $G - W$ is hamiltonian.*

Proof of the Claim. Put $N_{H_i}(x_i) = \{x_{ij}\}_{j=1}^{\deg(x_i)}$ as well as $F_i := H_i - x_i$. From the hypohamiltonicity or almost hypohamiltonicity of H_i one immediately infers that in F_i for no $s, t \in [\deg(x_i)]$ there exists a hamiltonian $x_{is}x_{it}$-path, a property we call (P1), but for any $v \in V(F_i)$ there exist $p, q \in [\deg(x_i)]$ such that the graph $F_i - v$ contains a hamiltonian $x_{ip}x_{iq}$-path, a property we call (P2). Henceforth, we see each F_i as well as B as a subgraph of G.

Consider $W \subset V(G)$ as described in the statement. Without loss of generality we may assume that $|W \cap V(F_i)| = 1$ for all $i \in [j]$ and $|W \cap V(F_i)| = 0$ for all $i \in \{j+1, \ldots, k\}$. For $i \in [j]$, we denote the unique vertex contained in $W \cap V(F_i)$ by w_i.

By (P2), for every $i \in [j]$ there exist $p_i, q_i \in [\deg(x_i)]$ such that there is a hamiltonian $x_{ip_i}x_{iq_i}$-path $\mathfrak{p}_i$ in $F_i - w_i$ and for every $i \in \{j+1, \ldots, k\}$ there exist $p_i, q_i \in \{2, \ldots, \deg(x_i)\}$ such that there is a hamiltonian $x_{ip_i}x_{iq_i}$-path $\mathfrak{p}_i$ in $F_i - x_{i1}$. Denote the vertices of B by $v_1, \ldots, v_{2k-j}$ and put $v_{2k-j+1} := v_1$. Then

$$\bigcup_{i=1}^{k} \mathfrak{p}_i + \sum_{i=1}^{j} (v_i x_{ip_i} + v_{i+1} x_{iq_i}) + \sum_{i=j+1}^{k} (v_i x_{i1} + v_{i+1} x_{i1}) + \sum_{i=k+1}^{2k-j} (v_i x_{i-k+j, p_{i-k+j}} + v_{i+1} x_{i-k+j, q_{i-k+j}})$$

is a hamiltonian cycle in $G - W$, so $\operatorname{circ}(G) \geq |V(G)| - j$. Let $\mathfrak{c}$ be a longest cycle in G. Since $|V(F_i)| \geq j+1$ for every $i \in [k]$, the cycle $\mathfrak{c}$ must visit every F_i. Put

$$S := \{i : \omega(F_i \cap \mathfrak{c}) = 1\} \quad \text{and} \quad T := \{i : \omega(F_i \cap \mathfrak{c}) > 1\},$$

and denote the cardinalities of these sets by s and t, respectively.

If $\omega(F_i \cap \mathfrak{c}) = 1$, then $F_i \cap \mathfrak{c}$ is a path but it cannot be spanning by (P1). By construction there is a bijection between the components of $\bigcup_i (F_i \cap \mathfrak{c})$ and the components of $B \cap \mathfrak{c}$, so $2t + s \leq 2k - j$. Thus, as $s + t = k$, we have $s \geq j$. This yields

$$|V(\mathfrak{c})| \leq |V(B)| + \sum_{i \in S} (|V(F_i)| - 1) + \sum_{i \in T} |V(F_i)| = |V(G)| - s \leq |V(G)| - j,$$

which completes the proof of the Claim.

We now prove statement (i). As described in [9], let H be a 4-connected almost hypohamiltonian graph obtained by considering the join of a 3-connected hypotraceable graph T (of which there are infinitely many [7]) and $K_1 = (\{x\}, \emptyset)$, i.e. adding to T the vertex x and joining x to every vertex in T. The exceptional vertex of H is x, and we choose x to be the special vertex of H. We now choose k and H such that $2k - j \geq \max\{\kappa, |V(A)|\}$ and $|V(H)| \geq j + 2$. Consider pairwise disjoint copies $H_1, \ldots, H_k$ of H, and denote the copy of x in H_i by x_i. Let B be a (possibly disconnected) graph on $2k - j$ vertices containing A as an induced subgraph. Put $G := (H_i - x_i; k; B)$. Note that by the choice of k and the structure of H_i, due to which every vertex of $H_i - x_i$ is connected to every vertex of B, the graph G is κ-connected.

By Claim (a), the circumference of G is $|V(G)| - j$. Put $p := |V(H)| - 1$ so that the order of G is $pk + 2k - j$. Let $\mathcal{W} \subset \binom{V(G) \setminus V(B)}{j}$ be the set of all sets W with $|W \cap (V(H_i) \setminus \{x_i\})| \leq 1$ for every $i \in [k]$. By Claim (b) for every $W \in \mathcal{W}$ we have that $G - W$ is hamiltonian, so

$$\frac{\left|\left\{W \in \binom{V(G)}{j} : G - W \text{ is hamiltonian}\right\}\right|}{\left|\binom{V(G)}{j}\right|} \geq \frac{|\mathcal{W}|}{\left|\binom{V(G)}{j}\right|}$$

of which the right-hand side equals

$$\frac{p^j \cdot \binom{k}{j}}{\binom{pk+2k-j}{j}} = \prod_{i=0}^{j-1} \frac{1 - \frac{i}{k}}{1 + \frac{2}{p} - \frac{j+i}{pk}}.$$

The limit of this ratio, when $k \to \infty$ and $p \to \infty$, is 1. This concludes the proof of statement (i).

We now prove statement (ii), first treating the connectivity 2 case. We call a graph G *partially hypohamiltonian* if it contains two vertices v and w, which we will call *nice*, such that there is no hamiltonian vw-path in G, but for every vertex x in $G - v - w$ there exists a hamiltonian vw-path in $G - x$. Note that G is 2-connected and may be hamiltonian. Furthermore, as a locally hypohamiltonian graph may contain more than one pair of nice vertices, but later on in a given locally hypohamiltonian graph G we will want to work with a fixed such pair (v, w), once v and w have been fixed as the nice vertices of G we shall not call any other pair of vertices in G "nice". Petersen's graph is partially hypohamiltonian, but there are other such graphs, as we shall now see.

Let H be a cubic hypohamiltonian graph and let a be a vertex in H such that H contains a vertex b at distance $\mathrm{diam}(H)$ from a. It is easy to deduce from the hypohamiltonicity of H that $H - N(a)$ has exactly two components. From these we obtain two 3-fragments with attachments $N(a)$ in H: one on four vertices, which is thus trivial, and which contains the vertex a, and one on $|V(H)| - 1$ vertices not containing the vertex a. Since hypohamiltonian graphs have at least ten vertices [3], the latter fragment must contain at least nine vertices. (We recall that there exist infinitely many cubic hypohamiltonian graphs; see for instance [2].) We shall call this non-trivial $N(a)$-fragment F. We will use the following lemma due to Thomassen.

Lemma 1 (Thomassen; Lemma 1 from [8]). *Let F be a 3-fragment with at least five vertices of a hypohamiltonian graph, and let A be the attachments of F. Then (i) F has no hamiltonian path connecting two vertices of A; (ii) for every vertex v of F, the graph $F - v$ has a hamiltonian path connecting two vertices of A.*

Denote the neighbours of a by $A = \{x, y, z\}$. Add to F two new vertices v, w, and join both v and w to every vertex of A. We call this graph G'. We now prove that G' is partially hypohamiltonian with nice vertices v and w. By Thomassen's Lemma 1, for no $c, d \in A$ there exists a hamiltonian cd-path in F. Thus, there exists no hamiltonian vw-path in G'. Now let u be some vertex in $G' - v - w$. Then there is a hamiltonian pq-path in $G' - v - w - u$ for appropriate $p, q \in A$, again by Lemma 1. Adding to this path the edges pv and qw the desired hamiltonian vw-path in $G' - u$ is obtained. Every graph constructed in this manner is partially hypohamiltonian, as we have just proven, and will be called *suitable*. By construction, in a suitable partially hypohamiltonian graph G' with nice vertices v and w, there is a vertex v' at distance $\mathrm{diam}(G')$ from both v and w. We abbreviate this property by $(\star)$.

For integers s and j with $s \geq 2j \geq 4$, consider the complete graph K_s with vertex set $\{c_1, \ldots, c_s\}$ and j pairwise disjoint suitable partially hypohamiltonian graphs $L_1, \ldots, L_j$, all of the same diameter which we denote by d_L, and with pairs of nice vertices $(v_1, w_1), \ldots, (v_j, w_j)$, respectively.

Identify each v_i with c_{2i-1} and each w_i with c_{2i}. We obtain the graph G which clearly has connectivity 2. Put $d := \mathrm{diam}(G)$ and consider henceforth $L_1, \ldots, L_j$ and K_s to be subgraphs of G. As $\{v_1, w_1\}$ and $\{v_2, w_2\}$ are 2-cuts of G at distance 1 and by $(\star)$ for $i \in \{1, 2\}$ the graph L_i contains a vertex v_i' at distance d_L from both v_i and w_i, we have that the distance in G between v_1' and v_2' is $2d_L + 1$, so $d_L \leq (d-1)/2$. If W is any set of j vertices in G at pairwise distance at least $\lceil d/2 \rceil + 1$, then no two of its vertices can lie in the same L_i, as vertices therein lie at distance at most $d_L \leq (d-1)/2 < \lceil d/2 \rceil + 1$, and moreover $W \cap K_s = \emptyset$, as the distance between any vertex in K_s and any vertex in G is at most $d_L + 1 \leq (d+1)/2 < \lceil d/2 \rceil + 1$. Since $|W| = j$, we have $|W \cap V(L_i)| = 1$ for every i. Thus, as $\lceil d/2 \rceil + 1 \geq 2$, each of the graphs $L_1 - \{v_1, w_1\}, \ldots, L_j - \{v_j, w_j\}$ contains exactly one element of W.

We now show that $G - W$ is hamiltonian. We denote by x_i the unique vertex in $V(L_i) \cap W$. Since each L_i is locally hypohamiltonian, there is a hamiltonian $v_i w_i$-path $\mathfrak{p}_i$ in $L_i - x_i$. Then

$$\left(\bigcup_{i=1}^{j} V(\mathfrak{p}_i) \cup \{c_{2j+1}, \ldots, c_s\}, \bigcup_{i=1}^{j} E(\mathfrak{p}_i) \cup \{c_{2k} c_{2k+1}\}_{k=1}^{j-1} \cup \{c_\ell c_{\ell+1}\}_{\ell=2j}^{s-1} \cup \{c_s c_1\} \right)$$

is the desired hamiltonian cycle in $G - W$.

Each L_i is locally hypohamiltonian, so there is no hamiltonian $v_i w_i$-path in L_i and thus, the circumference of G is at most $|V(G)| - j$. As we have just provided a cycle of length $|V(G)| - j$, the circumference of G is exactly this quantity. There are infinitely many such graphs G of some fixed diameter—as advertised in the theorem's statement—because we can choose s freely as long as $s \geq 2j$, and this does not affect the graph's diameter.

We now discuss the connectivity 3 case which is based on the same construction as in the proof of (i). Consider integers j and k with $k > j \geq 2$ and let $H_1, \ldots, H_k$ be pairwise disjoint graphs, all of the same diameter d_H, and each H_i a cubic hypohamiltonian graph in which we choose the special vertex x_i of H_i, for every $i \in [k]$, such that it has in H_i a vertex at distance d_H.

Let $G := (H_i - x_i; k; K_{2k-j})$ and consider henceforth $H'_1 := H_1 - x_1, \ldots, H'_k := H_k - x_k$ and K_{2k-j} to be subgraphs of G. Arguments as given above yield that $\mathrm{diam}(G) =: d = 2d_H$. If W is any set of j vertices in G at pairwise distance at least $\lceil d/2 \rceil + 1$, then no two of its vertices can lie in the same H'_i, as vertices therein lie at distance at most $d_H = d/2 < \lceil d/2 \rceil + 1$; note that H'_i and one vertex of K_{2k-j} induce H. Hence $|W \cap V(H'_i)| \leq 1$ for every i. By Claim (a), G has circumference $|V(G)| - j$, and by Claim (b), $G - W$ is hamiltonian.

Thus, we have for $j = 2$ that $G \in \Gamma\left(2, 2, \{\lceil \mathrm{diam}(G)/2 \rceil + 1, \ldots, \mathrm{diam}(G)\}\right)$ and for $j \geq 3$ that $G \in \Gamma(j, j, \lceil \mathrm{diam}(G)/2 \rceil + 1)$. That indeed G has connectivity 3 is a routine argument using the fact that hypohamiltonian graphs must be 3-connected, and left to the reader. Finally, there are infinitely many such graphs G of some fixed diameter since we can choose k freely as long as $k > j$, and, as above, this does not affect the graph's diameter. $\square$

We do not know whether Theorem 1 (ii) holds for graphs of connectivity at least 4. Although 4-connected almost hypohamiltonian graphs have been described [9], a family of such graphs would be needed in which the diameter becomes arbitrarily large in order to solve the connectivity 4 case.

A natural intermediary question is to determine whether, for $j = 2$, we can find non-hamiltonian graphs in which the removal of vertices at a small distance as well as at a large distance yields hamiltonian graphs. We can answer this affirmatively, but first need two lemmas and a definition: a graph is called *K_2-hamiltonian* if the removal of any pair of adjacent vertices yields a hamiltonian graph.

Lemma 2 (Thomassen; Corollary 1 from [8]). *Let G_1 and G_2 be disjoint hypohamiltonian graphs. For $i \in \{1, 2\}$, let G_i contain a 3-cut X_i and X_i-fragments F_i and F'_i. Then, if both F_1 and F_2 are non-trivial, or both F_i and F'_{3-i} are trivial, $(F_1, X_1) : (F_2, X_2)$ is hypohamiltonian.*

Lemma 3 [10]. *Let G_1 and G_2 be disjoint non-hamiltonian K_2-hamiltonian graphs. For $i \in \{1, 2\}$, let G_i contain a 3-cut X_i and X_i-fragments F_i and F'_i such that for each $x \in X_i$*

there is a hamiltonian path in $F_i - x$ and in $F'_i - x$ between the two vertices of $X_i - x$. This is fulfilled e.g. when X_i is non-trivial, or $\mathrm{exc}(G_i) \cap X_i = \emptyset$. Then, if both F_1 and F_2 are non-trivial, or both F_i and F'_{3-i} are trivial, $(F_1, X_1) \vdots (F_2, X_2)$ is K_2-hamiltonian, but not hamiltonian.

Theorem 2. *For infinitely many diameters d, we have*

$$\Gamma(1,1) \cap \Gamma^*(2, 1, \{1, \lfloor (d+6)/2 \rfloor, \dots, d\}) \neq \emptyset.$$

Proof. Consider a cubic K_2-hamiltonian hypohamiltonian graph H (infinitely many such graphs exist as proven in [10]) of diameter d_H, and a vertex v therein such that there exists in H a vertex at distance d_H from v. Take two copies of the 3-fragment $H - v$ which we denote by F and F', their respective attachments being X and X'. Let G be the graph given in Fig. 1.

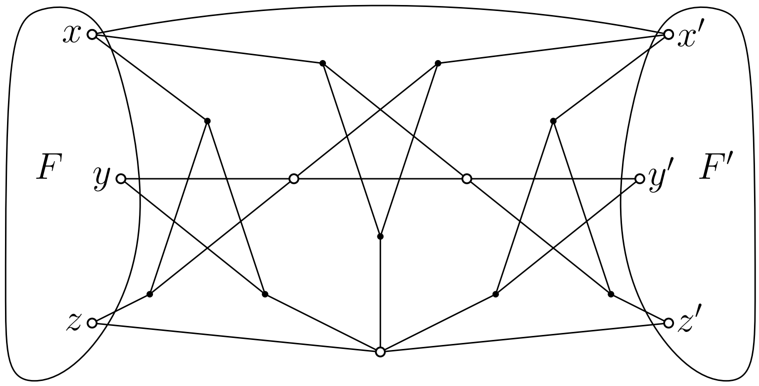

Fig. 1: The construction used in the proof of Theorem 2. White vertices indicate vertices contained in 3-cuts being identified when applying the operation $\vdots$.

G can be obtained as

$$F \vdots P \vdots P' \vdots P \vdots F',$$

where P and P' are Petersen graphs with two vertices at distance 2 removed (and the triples of vertices identified when applying $\vdots$ are omitted; they are given in Fig. 1). Since Petersen's graph and H are hypohamiltonian, Lemma 2 yields that G is hypohamiltonian—in particular, its circumference is $|V(G)| - 1$. Thus, as Petersen's graph and H are K_2-hamiltonian, by Lemma 3 we have that G is K_2-hamiltonian. So $G \in \Gamma^*(2, 1, \{1\})$.

Henceforth, we consider F, F', and the aforementioned copies of Petersen's graph to be subgraphs of G. For the rest of the proof note that, by construction, all arguments given for F hold analogously for F'. In G, the distance between any two vertices in X is at most 3, so the distance between any two vertices residing in F is at most $d_H + 1$: any shortest path S between vertices in H either does not use v (as defined in this proof's first paragraph) and thus exists in F, and therefore has length at most d_H, or uses v, in which case S can be altered to a path in G between the same vertices and of length at most $|E(S)| + 1$.

Assuming H to have sufficiently large diameter, by the choice of v it is thus clear that for vertices q, q' at maximum distance in G we must have $q \in V(F)$ and $q' \in V(F')$. Note that the distance between q and X must be $d_H - 1$ due to the fact that X is a 3-cut of G and by the condition from this proof's first paragraph. So for at least one vertex in X, the distance between q and that vertex is $d_H - 1$. Since the distance between x and x' is 1, between y and y' is 3, and between z and z' is 2 (vertices x, x', y, y', z, z' as defined in Fig. 1), we have $d := \mathrm{diam}(G) \leq 2d_H + 1$. Moreover, by construction, there exists a vertex $r \in V(F)$ at distance $d_H - 1$ from X, so the distance between r and x is at least $d_H - 1$, whence $d \geq 2d_H - 1$. Summarising, the diameter d of G is at least $2d_H - 1$ and at most $2d_H + 1$.

Let w and w' be vertices in G at distance at least $\lfloor (d+6)/2 \rfloor$. We have shown earlier that two vertices in F (and analogously in F') lie at distance at most $d_H + 1 < \lfloor (d+6)/2 \rfloor$, so w and w' do not both reside in F or F'. We may assume without loss of generality that $w \in V(F)$.

We use the drawing of G given in Fig. 1. Consider the central vertical axis A of G and let L (R) be the graph induced by all vertices on A or to the left of A (on A or to the right of A). It is easy to see that if a shortest path between w and a vertex in L uses vertices outside of L, it can always be rerouted to a path of the same length lying entirely in L.

By previously given arguments, we have $d = 2d_H + \rho$ with $\rho \in \{-1, 0, 1\}$. Then the distance between w and a vertex in L must be at most $d' := (2d_H + \rho)/2$, by the rerouting argument we have just given and the fact that if d' would be greater than $(2d_H + \rho)/2$, the diameter of G would exceed $2d_H + \rho$, a contradiction. By extending paths in L, for any vertex $\tilde{w}$ in $G - F - F'$ a $w\tilde{w}$-path can be described of length at most

$$d' + 2 = \frac{2d_H + \rho}{2} + 2 = d_H + 2 + \rho/2 < \left\lfloor \frac{2d_H + 6 + \rho}{2} \right\rfloor = \left\lfloor \frac{d+6}{2} \right\rfloor.$$

So $w' \in V(F')$. We now describe a hamiltonian cycle in $G - w - w'$.

As H is hypohamiltonian, for any $u \in V(F)$ there exist two vertices a, b in X such that there is a hamiltonian ab-path in $F - u$ (in particular, if $u \in X$, then $F - u$ contains a hamiltonian path between the two vertices in $X - u$); the same holds for F'. There are, ignoring symmetric cases, six situations to deal with. These can be deduced from the properties of the Petersen graph, but it is better to simply illustrate the necessary cycles, see Fig. 2. Moreover, it is easy to modify the cycles given in Fig. 2 in order to address the cases when the removed vertices lie in X or X'. □

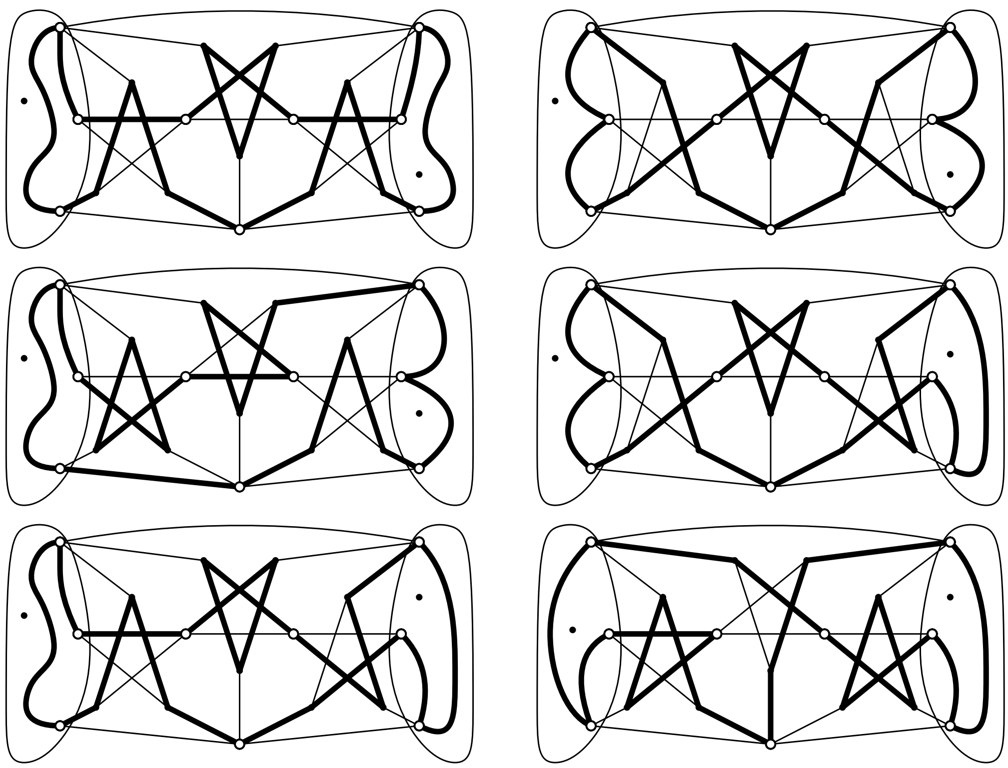

Fig. 2: The six relevant cases in the proof of Theorem 2. Each fragment contains a black vertex indicating that a vertex is being omitted from the cycle in that fragment.

In [6], Grünbaum was particularly interested in the planar case. Unfortunately, the graphs from Theorem 2 are non-planar. In fact, due to Tutte's classic theorem that every planar 4-connected graph is hamiltonian, $\Gamma^*(2,1) \cup \Gamma(2,2)$ cannot have planar members. However, there might be planar graphs if we restrict ourselves to all distances *except* 2. A partial answer—again in the spirit of trying to find pairs of vertices at small *and* large distances whose removal yields a hamiltonian graph—is given by the following result.

Proposition. *There exists a polyhedral graph in* $\Gamma^*(2,1,\{1,d\})$.

Proof. One of the graphs from [4], namely G_{48} (reproduced in Fig. 3), yields the statement. In that paper it is shown that G_{48} is hypohamiltonian, so it is of circumference $|V(G_{48})|-1$, as well as K_2-hamiltonian, so for every pair of adjacent vertices a and b in G_{48}, the graph $G_{48}-a-b$ is hamiltonian. (The proof of the result containing the latter fact as a special case, given in [4], uses a new method inspired by an old idea of Chvátal, namely his so-called *flip-flop* graphs [2].) Excluding symmetric cases, there is only one pair of vertices in G_{48} at distance $\text{diam}(G_{48}) = d = 8$, denoted in Fig. 3 by x and y. Note that the only vertex at distance 8 from x is y, and the only vertex at distance 8 from y is x. That $G_{48}-x-y$ is indeed hamiltonian is shown in Fig. 3. □

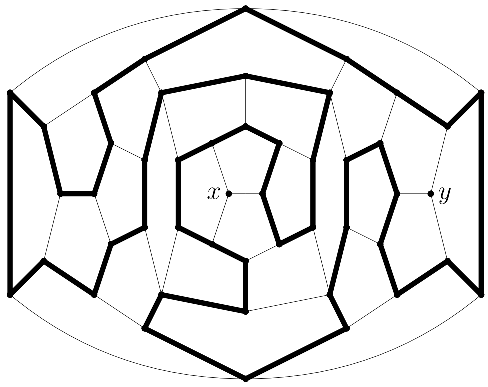

Fig. 3: The graph G_{48}. Therein, two vertices x and y are labelled; these are at distance $\text{diam}(G_{48}) = 8$. A hamiltonian cycle in $G_{48}-x-y$ is also emphasised.

Unfortunately G_{48} contains two vertices at distance $\text{diam}(G_{48}) - 1 = d-1$ whose removal yields a non-hamiltonian graph, so $G_{48} \notin \Gamma^*(2,1,\{1,d-1,d\}) = \Gamma^*(2,1,\{1,7,8\})$. As mentioned earlier, in [4] it was proven that $\Gamma(1,1) \cap \Gamma^*(2,1,\{1\})$ contains infinitely many polyhedral graphs—by the last proposition we know that $\Gamma(1,1) \cap \Gamma^*(2,1,\{1,d\})$ contains a polyhedral graph, but whether there are infinitely many remains an open question.

Acknowledgements. Thanks are due to Kenta Noguchi and Jarne Renders as well as the two anonymous referees, for carefully reading the manuscript and offering constructive suggestions. The research presented in this paper was supported by a Postdoctoral Fellowship of the Research Foundation Flanders (FWO).

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