On the hamiltonicity of a planar graph and its vertex-deleted subgraphs

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Abstract. Tutte proved that every planar 4-connected graph is hamiltonian. Thomassen showed that the same conclusion holds for the superclass of planar graphs with minimum degree at least 4 in which all vertex-deleted subgraphs are hamiltonian. We here prove that if in a planar n-vertex graph with minimum degree at least 4 at least $n-5$ vertex-deleted subgraphs are hamiltonian, then the graph contains two hamiltonian cycles, but that for every $c < 1$ there exists a non-hamiltonian polyhedral n-vertex graph with minimum degree at least 4 containing cn hamiltonian vertex-deleted subgraphs. Furthermore, we study the hamiltonicity of planar triangulations and their vertex-deleted subgraphs as well as Bondy's meta-conjecture, and prove that a polyhedral graph with minimum degree at least 4 in which all vertex-deleted subgraphs are traceable, must itself be traceable.

Keywords. Hamiltonian, planar, vertex-deleted subgraph, hypohamiltonian, 1-hamiltonian, traceable

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1 Introduction

Tutte proved in 1956 that all 4-connected planar graphs are hamiltonian [32]—see also [11, 31]—, generalising Whitney's result that 4-connected triangulations of the plane are hamiltonian [33]. One of the first extensions of Tutte's celebrated theorem is due to Nelson, who observed that in a planar 4-connected graph every vertex-deleted subgraph is hamiltonian [19]. This was then strengthened substantially by Thomas and Yu, who confirmed the conjecture of Plummer that the graph obtained when deleting any *pair* of vertices from a planar 4-connected graph is hamiltonian [26], as well. Viewing Nelson's observation from a different perspective, Thomassen showed in 1978 the following strengthening of Tutte's theorem.

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Theorem 1 (Thomassen [29])**.** *A planar graph with minimum degree at least* 4 *in which every vertex-deleted subgraph is hamiltonian, must itself be hamiltonian.*

In a subsequent paper [30], Thomassen showed that the statement is not true if 4 is replaced with 3. Of great interest but unknown is whether planarity can be dropped as restriction, which was asked by Thomassen four decades ago [29]. We emphasise that, seeing planar graphs of minimum degree at least 4 in which every vertex-deleted subgraph is hamiltonian as a superclass of planar 4-connected graphs, Theorem 1 answers the natural question whether the members of this superclass are hamiltonian or not. We see the present work as a first step in studying the general question asking which results on planar 4-connected graphs extend to this superclass, and which do not.

Inferring hamiltonian properties of a graph from those of its vertex-deleted subgraphs is challenging; an example of a general statement which follows from a remarkable result of Thomason [27] is that if a cubic graph contains a vertex-deleted subgraph which has an odd number of hamiltonian cycles, then the graph itself must be hamiltonian (and that if a cubic graph has an odd number of hamiltonian cycles, all of its vertex-deleted subgraphs are hamiltonian). However, such general results are rare—Bermond and Thomassen summarise in [4]: "Undirected hypohamiltonian graphs have been studied to a large extent [...], and the richness of such graphs shows that it is difficult to obtain a sufficient condition for a Hamiltonian cycle in a graph in terms of Hamiltonian properties of vertex-deleted subgraphs." (A graph is *hypohamiltonian* if it is non-hamiltonian, yet all of its vertex-deleted subgraphs are hamiltonian.)

Can we deduce, in an arbitrary graph, from the fact that all of its vertex-deleted subgraphs are hamiltonian, that the graph itself must be hamiltonian? (One can see this as a qualitative version of the question behind the Kelly-Ulam Reconstruction Conjecture for hamiltonicity.) In general, we can guarantee an affirmative answer only for graphs on at most 9 or exactly 11, 12, 14, or 17 vertices, as for all other orders both 1-hamiltonian and hypohamiltonian graphs of the same order exist [2]. (A graph is 1-*hamiltonian* if the graph itself as well as all of its vertex-deleted subgraphs are hamiltonian.) Even restricted to planar graphs, the situation is similar, as proven by Thomassen [28] (see also [16]), rebutting a conjecture of Grünbaum [13, p. 37]. However, on the positive side, by Theorem 1, among planar graphs of minimum degree at least 4 (or, in this context equivalently, without cubic vertices), if all vertex-deleted subgraphs are hamiltonian, then the graph itself must be so as well. The author recently extended Theorem 1 in three directions.

Theorem 2 [39]**.** *A graph G is hamiltonian, if*

(i) *G is planar and contains no cubic vertices, and if all but at most one vertex-deleted subgraphs of G are hamiltonian, or*

(ii) *G is planar and contains at most three cubic vertices and each vertex-deleted subgraph of G is hamiltonian, or*

(iii) *G has crossing number at most* 1 *and contains no cubic vertices, and if all vertex-deleted subgraphs of G are hamiltonian.*

Ozeki and the author [22] recently strengthened (iii) by showing that if a graph with crossing number 1 in which all vertex-deleted subgraphs are hamiltonian contains at most one cubic vertex, then it is hamiltonian. It is unknown whether (iii) can be extended to crossing number 2. A crucial ingredient of (ii) is the theorem that every polyhedral graph with at most three 3-cuts is hamiltonian [9]—it has turned out to be a very challenging problem to extend this to 'four' or even 'five' (extending it to 'six' or more is impossible, as non-hamiltonian examples are known [21]), so an extension of (ii) seems, at least with this strategy, out of reach.

One of our present goals is to extend (i). To this end, the following partition of all 2-connected n-vertex graphs of circumference $n-1$ introduced in [38] will prove to be useful and will constitute the common theme throughout this article. Consider such a graph G and let $\text{exc}(G) \subset V(G)$ be the set of all vertices w in G such that the graph $G - w$ is non-hamiltonian. We shall frequently call a non-hamiltonian graph G with $|\text{exc}(G)| = k$ a *k-graph*. A vertex from $\text{exc}(G)$ is *exceptional*. A 0-graph is hypohamiltonian, and a 1-graph is *almost hypohamiltonian*—their interplay has been investigated in [38]. For a survey on hypohamiltonian graphs, see Holton and Sheehan [14]. A subclass of k-graphs in which the exceptional vertices form an independent set is being studied in [12]. Various famous graphs have circumference one less than their order and are thus k-graphs, e.g. Petersen's and Coxeter's graph (both $k = 0$), Tietze's graph ($k = 3$), Grinberg's graph ($k = 4$, see Grünbaum [13]), Herschel's graph and the Goldner-Harary graph (both $k = 5$, see Corollary 2), Kirkman's graph ($k = 6$), the Tutte graph ($k = 13$), the Lederberg-Bosák-Barnette graph ($k = 14$, see Neyt [20]), Meredith's graph ($k = 40$), and many but not all small snarks [7].

We introduce further notation used throughout this article. For a graph G, we denote with $\overline{G}$ the complement of G. For $M \subset E(\overline{G}) \cup E(G)$ we denote by $G + M$ the graph obtained when adding all edges in $M \cap E(\overline{G})$ to G. All cuts in this paper are assumed to be vertex-cuts. Let G be a non-complete graph of connectivity k, X a k-cut in G, and C a component of $G - X$. Then $G[V(C) \cup X]$ is a *k-fragment* of G with *attachments* X. (Note that we here use Wiener's definition [36] which slightly differs from Thomassen's [28].) A k-fragment is *trivial* if it contains exactly $k+1$ vertices. A cut X of G is *trivial* if $G - X$ has exactly two components and X is the set of attachments of a trivial k-fragment. A path with endvertex v is a *v-path*, and a v-path with endvertex $w \neq v$ is a *vw-path*.

The article is structured as follows. In the next section we extend Thomassen's Theorem 1 in several directions and discuss consequences. In Section 3 we treat the hamiltonicity of plane triangulations and their vertex-deleted subgraphs. In Section 4 we give a result on Bondy's meta-conjecture concerning the cycle spectrum. In Section 5 we prove a natural analogue of Thomassen's Theorem 1 regarding hamiltonian paths. The paper concludes with Section 6, which discusses open problems.

2 Extending Thomassen's Theorem

Thomassen asked in the seventies whether hypohamiltonian graphs of minimum degree at least 4 exist [29]. This problem remains open. A natural relaxation of Thomassen's question is to ask for which k there exist k-graphs with minimum degree at least 4. It follows from a recent result of Wiener on so-called path-critical graphs (see [34, 35] for details) and the not difficult to prove fact that the join of K_k and a $(k+1)$-path-critical graph is a k-graph, that for every $k \geq 1$ we can answer this question affirmatively, even providing 4-connected examples. On the other hand, Theorem 2 (i) implies that every *planar* 1-graph must contain a cubic vertex. However, there exists at least one planar 2-graph with no cubic vertices, namely the join of K_2 and $3K_1$, but no further examples are known. The author asked in [39] whether every polyhedral 2-graph contains a cubic vertex. We now answer this question in the affirmative and strengthen Thomassen's Theorem 1, the proof of which uses an entirely different strategy than what follows.

Theorem 3. *If in a planar graph G of order n and minimum degree at least 4 at least $n-5$ vertex-deleted subgraphs are hamiltonian, then G contains at least two hamiltonian cycles.*

Proof. The only graph on fewer than six vertices which has minimum degree at least 4 is K_5, but K_5 is not planar, so we may consider henceforth G to have order at least 6. As G has minimum degree at least 4 and G contains at least one vertex-deleted subgraph that is hamiltonian, its connectivity cannot be 1. By the theorem of Tutte [32] stating that planar 4-connected graphs are hamiltonian, it remains to deal with the cases when the connectivity of G is 2 or 3.

The connectivity 2 case: Let G contain a 2-cut X. Since the minimum degree of G is at least 4, each component of $G - X$ has at least four vertices—in particular, it is non-trivial, so by a straightforward toughness argument $G - X$ consists of exactly two non-trivial components C_1, C_2. Combining this with the fact that the vertices in X are exceptional, each C_i contains a non-exceptional vertex v_i, so there is a hamiltonian cycle $\mathfrak{h}_i$ of $G - v_i$. Put $F_i = G[V(C_i) \cup X]$. Then $(\mathfrak{h}_1 \cap F_2) \cup (\mathfrak{h}_2 \cap F_1)$ is a hamiltonian cycle in G.

We may assume that G has connectivity 3. It was proven in [9] that a polyhedral graph of connectivity 3 must contain a 3-cut $X = \{x, y, z\}$ such that for at least one of the 3-fragments F, F' with attachments X (there are exactly two such fragments since $K_{3,3}$ is non-planar), say F, the graph $F_\Delta = F + xy + yz + zx$ is either K_4 or 4-connected. Since G has minimum degree at least 4, the former case is impossible, so F_Δ is 4-connected.

The smallest planar 4-connected graph, the 1-skeleton of the octahedron, has six vertices, so F_Δ contains a non-exceptional vertex v. Since X is a 3-cut, the hamiltonicity of $G - v$ yields, ignoring analogous cases, that there exists either a hamiltonian yz-path $\mathfrak{p}'$ in $F' - x$ (for $v = x$) or F' (for $v \notin X$). We now treat these two situations.

Case 1. *There is a hamiltonian yz-path $\mathfrak{p}'$ in $F' - x$.* We invoke the "Three Edge Lemma" (see (2.7) in [26] or [23]) as stated by Sanders in [24]: Every 2-connected plane graph H has a Tutte cycle through any three edges of the facial k-cycle of the unbounded face of H whenever $k \geq 4$. (For the definition of Tutte cycles and further details thereon, we refer to [21].) We consider F_Δ to be embedded in the plane—since F_Δ is 3-connected this embedding is unique by a classic theorem of Whitney. Let Δ be the facial triangle of F_Δ with vertices x, y, z, and R_{xy} (R_{xz}) the face of F_Δ sharing the edge xy (xz) with Δ. In F_Δ, we add a vertex u on xy and a vertex w on xz, join u with w, and connect u (w) with all vertices in $R_{xy} - x - y$ ($R_{xz} - x - z$). We obtain a plane graph H. The verification, for instance with Menger's theorem, that H is 4-connected is left to the reader. We re-embed H in the plane such that the quadrilateral $yuwz$ is its unbounded face and apply the Three Edge Lemma to yu, uw, wz. We obtain a hamiltonian yz-path $\mathfrak{p}$ (since every Tutte cycle in a 4-connected graph is hamiltonian) in F_Δ using no edge of Δ. Then $\mathfrak{p} \cup \mathfrak{p}'$ is a hamiltonian cycle in G.

CASE 2. *There is a hamiltonian yz-path $\mathfrak{p}'$ in F'.* We use Sanders' theorem stating that in a planar 4-connected graph there exists a hamiltonian cycle through any pair of edges [24]. Thus, we obtain a hamiltonian cycle in F_Δ using the edges xy, zx and thus a hamiltonian yz-path $\mathfrak{p}$ in $F_\Delta - x$. Since none of the edges xy, yz, zx lie in $\mathfrak{p}$, the path $\mathfrak{p}$ lies in $F - x$. As above, $\mathfrak{p} \cup \mathfrak{p}'$ is a hamiltonian cycle in G.

Alternatively, [9, Lemma 14] can be used to deal with Cases 1 and 2, but we believe that the proof given above is of interest as well. Finally, Bondy and Jackson [6] showed that a planar graph containing exactly one hamiltonian cycle has at least two "small" vertices—they call a vertex *small* whenever its degree is 2 or 3. With the prior arguments and this theorem the statement of Theorem 3 follows. □

Combining the following lemma with Theorem 3 we can give another strengthening of Thomassen's Theorem 1, essentially stating that if the right planar 3-fragment is present in a (not necessarily planar) 3-connected graph G, then we can guarantee the hamiltonicity of G without any additional information on the rest of G's structure—in particular, G might be of large genus.

Lemma 1 [39]**.** *Let $i \in \{1,2\}$. Consider the disjoint graphs G_1 and G_2 such that G_i is a k_i-graph with exceptional vertices W_i. Let F_i be a non-trivial 3-fragment of G_i, with the set X_i of attachments of F_i disjoint from W_i. Put $U_i = W_i \cap V(F_i)$. Then the graph Γ obtained from $F_1 \cup F_2$ by identifying X_1 with X_2 using a bijection is a $(|U_1| + |U_2|)$-graph. If G_1 and G_2 are planar, then so is Γ.*

Corollary 1. *Let G be a 3-connected graph containing a planar 3-fragment F with attachments $\{x_1, x_2, x_3\}$ such that F has minimum degree (in G) at least 4, every vertex in $V(F) \cap \bigcup_i N[x_i]$ is non-exceptional, and F contains at most two exceptional vertices. Then G is hamiltonian.*

Proof. Suppose G is non-hamiltonian (reductio ad absurdum). Put $X = \{x_1, x_2, x_3\}$ and let $i \in \{1,2,3\}$. As G is 3-connected, x_i has at least one neighbour in $F - X$. Assume it has exactly one, which we call v. As every vertex in $V(F) \cap N[x_i]$ is non-exceptional, v is non-exceptional, so there is a hamiltonian $x_j x_k$-path $\mathfrak{p}'$ in F', where F' is the 3-fragment with attachments X different from F (since $G - x_i$ is hamiltonian, there are exactly two such fragments) and i, j, k are pairwise different. The vertex x_i is non-exceptional as well, so there is a hamiltonian $x_j x_k$-path $\mathfrak{p}$ in $F - x_i$. But then $\mathfrak{p}' \cup \mathfrak{p}$ is a hamiltonian cycle in G, a contradiction. In consequence, x_i has at least two neighbours in $F - X$.

Consider two disjoint copies F_1, F_2 of F and identify, using a bijection, their respective attachments, in order to obtain a graph Γ. By our argument from the first paragraph and Lemma 1, Γ is a planar non-hamiltonian graph of minimum degree at least 4 in which at least $|V(\Gamma)| - 4$ vertex-deleted subgraphs are hamiltonian—but this contradicts Theorem 3. $\square$

We present a further strengthening of Thomassen's Theorem 1. As already mentioned, it follows directly from a theorem of Bondy and Jackson [6] that hamiltonian planar graphs with minimum degree at least 4 contain at least *two* hamiltonian cycles.

Theorem 4. *A planar graph of minimum degree at least* 4 *in which every vertex-deleted subgraph is hamiltonian, contains at least three hamiltonian cycles.*

Proof. Let G be a graph with the given properties. We use the notation from the proof of Theorem 3. Since now no vertex of G is exceptional, each vertex $v \in X$ yields a hamiltonian cycle $\mathfrak{h}_v$ in $G - v$. As in the proof of Theorem 3, using the Three Edge Lemma we modify the path $F \cap \mathfrak{h}_v$ to a hamiltonian path in F with the same endvertices. $\square$

For a long time, even for planar 4-connected graphs the published literature contained only constant lower bounds on the number of hamiltonian cycles [21]. However, in a recent breakthrough, Brinkmann and Van Cleemput [8] proved that there is a linear (in the graphs' order) number of hamiltonian cycles in planar 3-connected graphs containing at most one 3-cut. It is generally believed that for planar 4-connected graphs a quadratic bound is true—this would be best possible due to double wheels. It would certainly be interesting to investigate whether such results can be established for planar graphs of minimum degree at least 4 in which every vertex-deleted subgraph is hamiltonian.

The following is an asymptotic counterpart to Theorem 3.

Theorem 5. *For every* $c < 1$ *there exists a polyhedral non-hamiltonian* n*-vertex graph of minimum degree* 4 *with at least* cn *hamiltonian vertex-deleted subgraphs.*

Proof. In [39] we showed that if $\overline{\mathcal{H}}_n$ denotes the family of all planar hypohamiltonian graphs of order n, and $V_3(G)$ the set of all cubic vertices in a graph G, then

$$\frac{1}{n} \cdot \left(\min_{G \in \overline{\mathcal{H}}_n} |V_3(G)| \right) \to 0 \quad \text{as} \quad n \to \infty.$$

By replacing every cubic vertex (exceptional or non-exceptional)—of which there are, asymptotically, a vanishing number—by the 1-skeleton of an octahedron, as shown in Fig. 1, we obtain the statement. The proof that this replacement indeed performs as desired is straightforward and therefore omitted. $\square$

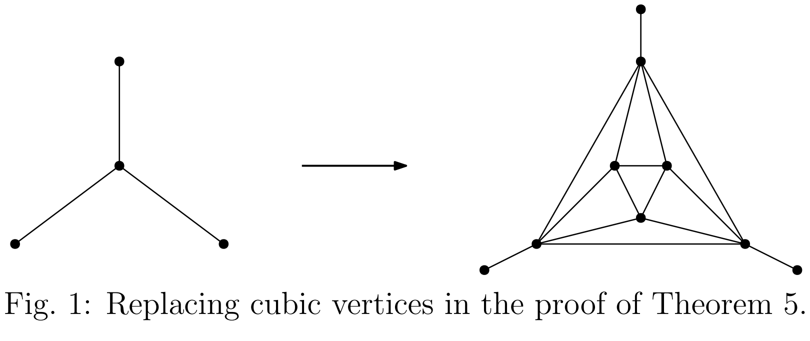

Fig. 1: Replacing cubic vertices in the proof of Theorem 5.

3 Triangulations

Thomassen's Theorem 1 addresses planar graphs—what if we strengthen this requirement to *maximal planar*, i.e. planar graphs in which the addition of any edge causes the graph to become non-planar, but drop the degree condition? Recall that such graphs are also called *triangulations* (of the plane), since every face of their (unique) plane embedding is triangular. In the following, a *triangle* is simply a 3-cycle; if it is facial, then we say so. In order to proceed, we need the following concept introduced by Jackson and Yu [15].

A plane triangulation containing a separating triangle Δ can be split into two triangulations by considering the subgraphs inside and outside of Δ, with a copy of Δ contained in both. Iteratively applying this procedure to a plane triangulation T with k separating triangles, we obtain a collection of $k+1$ triangulations that are either 4-connected or K_4, i.e. without separating triangles. These *pieces* form the vertices of the *decomposition tree* of T, and two such vertices share an edge if the corresponding pieces share a separating triangle in T. It follows from the Cunningham-Edmonds decomposition theory that T is indeed an acyclic graph, as well as unique; for further details we refer to [15] and [21]. We require the following result.

Lemma 2 (Jackson and Yu [15])**.** *Let T be a plane triangulation whose decomposition tree D has maximum degree at most three. Let H be a piece of T corresponding to a vertex of D of degree at most two, and Δ be a facial triangle of both H and T. Then T has a hamiltonian cycle through two edges of Δ.*

Theorem 6. *There exists a non-hamiltonian n-vertex triangulation with $n-k$ hamiltonian vertex-deleted subgraphs if and only if $k \geq 5$.*

Proof. Let us call a triangulation that is a k-graph a *k-triangulation.* We first prove that k-triangulations exist for each $k \geq 5$. Consider the graphs from Fig. 2. Each contains k black and $k+1$ white vertices. By removing all k black vertices, we obtain $k+1$ components, so these triangulations are non-hamiltonian by a standard toughness argument. After removing a black vertex, a similar reasoning yields that we also obtain a non-hamiltonian graph. To verify that the removal of any white vertex yields a hamiltonian graph is, for this particular construction, a straightforward case analysis left to the reader (the crucial observation being that any such hamiltonian cycle alternates between black and white vertices). A prominent example of a 5-triangulation is the Goldner-Harary graph, see the end of this section.

Now assume there exists a k-triangulation T with $k < 5$ (reductio ad absurdum). In the following arguments we, sometimes tacitly, use that T contains at least four separating triangles by the theorem of Jackson and Yu stating that triangulations with at most three separating triangles are hamiltonian [15], and that $|V(T)| \geq 11$ holds, since all polyhedral graphs up to 10 vertices are hamiltonian [3]. Assume T contains a separating triangle $\Delta = xyz$ such that x and y are non-exceptional. Consider, for the remainder of this proof, T to be embedded in the plane; by a fundamental result of Whitney this embedding is unique. Since $T-x$ is hamiltonian, there exists a hamiltonian yz-path outside of Δ, and as $T-y$ is hamiltonian, there is a hamiltonian xz-path inside of Δ. Joining these paths and adding the edge xy, we

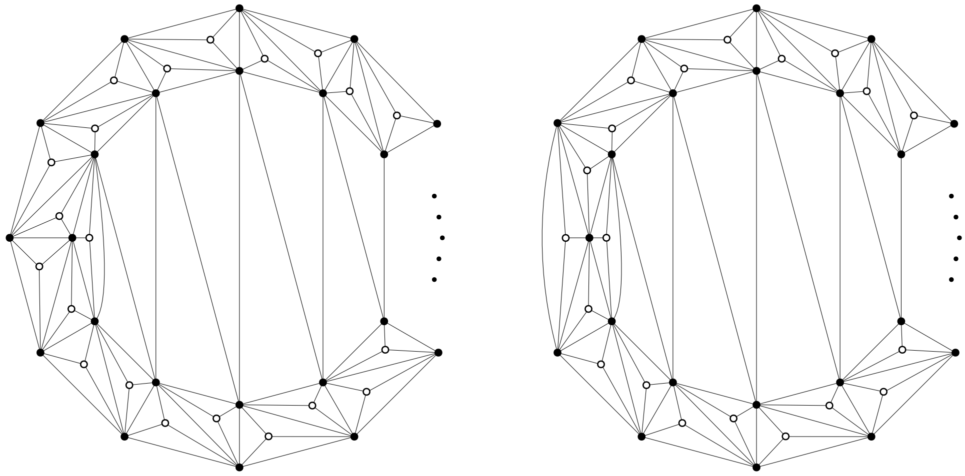

Fig. 2: A k-triangulation for even $k \geq 6$, left-hand side,
and odd $k \geq 5$, right-hand side.
(Edges subdividing the unbounded face are omitted and irrelevant for the proof.)

obtain a hamiltonian cycle in T, a contradiction. Therefore, any separating triangle of T contains at most one non-exceptional vertex; we abbreviate this fact by (†).

Suppose T contains a separating triangle $\Delta' = xyz$ with exactly one non-exceptional vertex z among $V(\Delta')$, such that every separating triangle in the (without loss of generality) interior of Δ' has xy as one of its edges. We consider Δ' and its interior as a sub-triangulation T' of T. The decomposition tree of T' is a path, so we can apply Lemma 2 of Jackson and Yu to T' and a piece therein having Δ' as facial triangle, and obtain that there exists a hamiltonian cycle of T' containing xy and yz, and thus a hamiltonian xz-path $\mathfrak{p}'$ in $T' - y$. Since z is non-exceptional in T, there is a hamiltonian xy-path $\mathfrak{p}$ in $T[(V(T) \setminus V(T')) \cup \{x, y\}]$. Combining, in T, the paths $\mathfrak{p}'$ and $\mathfrak{p}$ with the edge zy, we obtain a hamiltonian cycle in T, a contradiction. We refer to this argument as ($\star$). Note that we do not require Δ' to contain any separating triangles in its interior. Moreover, ($\star$) holds if $V(\Delta')$ forms a trivial 3-cut.

By Theorem 3, T must contain a cubic vertex v. Each neighbour of v is exceptional—otherwise we would obtain, by obvious rerouting, a hamiltonian cycle in T—, so T contains at least three exceptional vertices. By (†), there can be no separating triangle vertex-disjoint from the triangle $T[N(v)]$. If the set of exceptional vertices of T is either $N(v)$ or $N[v]$, then we obtain a contradiction by ($\star$). So T contains exactly one exceptional vertex w other than the vertices in $N[v]$. Recall that T contains at least three separating triangles different from $T[N(v)]$.

Case 1. *Every vertex of each separating triangle is exceptional.* In this case, there are exactly four separating triangles (since T contains exactly four exceptional vertices) whose vertices induce a K_4. Suppose there exists a separating triangle Δ_{nt} forming a non-trivial 3-cut in T, then Δ_{nt} together with its (without loss of generality) interior I yields a 4-connected triangulation. In I we have at least two vertices, each of which is non-exceptional, which guarantees that there exists, in $T - I$, a path P between two vertices of Δ_{nt} that visits all or all but one vertices of $T - I$ (if a vertex is omitted, it must be the third vertex of Δ_{nt}). We apply [15, Theorem 4.1] to obtain a suitable path in the aforementioned 4-connected triangulation which together with P yields a hamiltonian cycle in T, a contradiction. Therefore, the vertices of each separating triangle form a trivial 3-cut in T, but then the resulting triangulation has order 8, so it is hamiltonian, a contradiction.

CASE 2. *There exists a separating triangle Δ'' with a non-exceptional vertex u.* By (†), the other two vertices of Δ'' are exceptional. If w is among them, then either the interior or exterior of Δ'' contains no exceptional vertex, and we obtain a contradiction by (†) or ($\star$). Otherwise, either w lies in the exterior of Δ'', in which case the interior of Δ'' contains no exceptional vertex and we may argue as before, or (now w lies in the interior of Δ'') there exists a 4-cycle C formed by $N(v)$ and u which does not separate v and w. Then outside of C either at least one separating triangle exists which has two exceptional and one non-exceptional vertex, namely u, and which cannot contain any exceptional vertices in its interior (and we are done, as earlier), or no such triangle exists and we can apply Lemma 2 to the triangulation T'' formed by Δ'' minus its interior, and a piece thereof having Δ'' as facial triangle. This yields the existence of a hamiltonian cycle of T'' using, among edges of Δ'', the one not incident with u and one of the edges incident with u. Removing the former together with the fact that u is non-exceptional we obtain the hamiltonicity of T, a contradiction. □

Proposition 1. *If G is a bipartite k-graph with bipartition (A, B) with $|A| \geq |B|$, then $|A| - 1 = |B| \leq k$, $B \subset \text{exc}(G)$, and for all $vw \in E(\overline{G})$ such that not both v and w lie in A, we have that $G + vw$ is an ℓ-graph with $\ell \leq k$. If $|B| = k$, then $\text{exc}(G) = B$ and $G + vw$ is a k-graph. Furthermore, no hypohamiltonian or almost hypohamiltonian graph is bipartite, but bipartite k-graphs exist for every $k \geq 2$.*

Proof. Let G have order n. If $|A| = |B|$ or $|A| > |B| + 1$, the circumference of G cannot be $n - 1$, whence $|A| - 1 = |B|$. Since the removal of an arbitrary vertex residing in B yields a non-hamiltonian graph, every vertex in B is exceptional, so $k \geq |B|$. The graph $G' = G + vw$ is non-hamiltonian if not both v and w lie in A (when both v and w lie in A, the graph G' may be hamiltonian and thus not a k-graph) and since adding an edge does not destroy but might have created new $(n-1)$-cycles, we have that G' is an ℓ-graph with $\ell \leq k$. If $|B| = k$ then $\text{exc}(G)$ coincides with B and for any vw as introduced above the number of exceptional vertices can neither increase nor decrease, so $G + vw$ is a k-graph. For hypohamiltonian and almost hypohamiltonian graphs we have $|B| \leq k \in \{0, 1\}$, which gives as sole possibility $K_{1,2}$, which is neither hypohamiltonian nor almost hypohamiltonian. Finally, $K_{k,k+1}$ is a bipartite k-graph for any $k \geq 2$. □

Corollary 2. *Herschel's graph and the Goldner-Harary graph are 5-graphs.*

Proof. Herschel's graph is bipartite and it is well-known that it is the smallest non-hamiltonian polyhedral graph. Its vertex set admits a bipartition into sets of size 5 and 6. Each vertex in the smaller set must be exceptional, and it is easy to verify that each vertex in the bigger set is non-exceptional. Thus, Herschel's graph is 5-hypohamiltonian. We can obtain the Goldner-Harary graph by adding edges to Herschel's graph in such a manner that in each step the condition from Proposition 1 is satisfied. □

4 Bondy's Meta-conjecture

In 1971, Bondy [5] proposed his famous *meta-conjecture*: "Almost any non-trivial condition on a graph which implies that the graph is hamiltonian also implies that the graph is pancyclic. (There may be a simple family of exceptional graphs.)"

Considering Thomassen's Theorem 1, is it true that a planar graph with minimum degree 4, in which every vertex-deleted subgraph is hamiltonian, must be pancyclic? Certainly not in general, since even in the subclass of planar 4-connected graphs we have the line-graph of the (1-skeleton of the) dodecahedron which contains no 4-cycle. However, no other exceptions than 4-cycles are known, which relates to a well-known question of Malkevitch [18]. As we shall see, even excluding 4-cycles the answer is *no*, and the (infinitely many) exceptions do not seem particularly simple.

Theorem 7. *For any $k \geq 1$ there exists a planar 4-regular graph in which all vertex-deleted subgraphs are hamiltonian such that its cycle spectrum has a contiguous gap of size at least k.*

Proof. Consider the line-graph L of the (1-skeleton of the) dodecahedron G, which is a planar cyclically 4-edge-connected cubic graph of girth 5. Thus L is 4-connected and planar, so by Nelson's observation [19] stating that planar 4-connected graphs are 1-hamiltonian, every vertex-deleted subgraph of L is hamiltonian. However, L does not contain a 4-cycle.

Let uvw be a triangle in L. For any integer $k \geq 1$, we say that we *k-expand* uvw if we replace the edge uv with the path $uu_1 \dots u_{4k}v$, the edge vw with the path $vv_1 \dots v_{4k}w$, and the edge wu with the path $ww_1 \dots w_{4k}u$. Furthermore, we add new vertices x_i, y_i, z_i, join x_i to $u_{2i-1}, u_{2i}, w_{4k-2i+1}, w_{4k-2i+2}$, and add the edges $u_{2i-1}w_{4k-2i+2}, u_{2i}w_{4k-2i+1}$ for $1 \leq i \leq k$. The analogous procedure is performed for y_i and z_i. See Fig. 3. We denote by L_k the graph we obtain if every triangle of L is k-expanded.

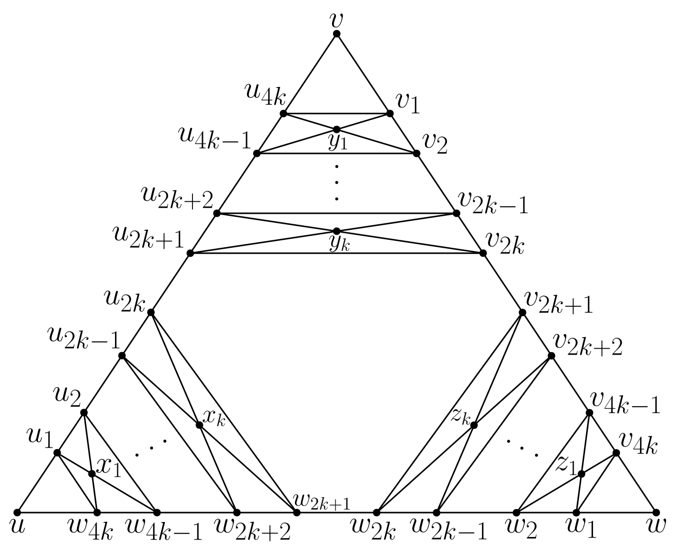

Fig. 3: The k-expansion of the triangle uvw.

It is clear that L_k is planar and 4-regular. If a cycle is fully contained in the k-expansion of a triangle uvw, then it has length at most $15k + 3$. Otherwise—now seeing u, v, w as lying in L_k—, since $\{u, v, w\}$ is a 3-cut in L_k and L contains no 4-cycles, its length is at least $20k + 5$.

It remains to show that for every $a \in V(L_k)$ the graph $L_k - a$ is hamiltonian. We shall only treat the cases when $a \in \{v, v_1, y_1, v_{2k}\}$, as all other cases can be resolved in a very similar manner, and recommend using Fig. 3 in the following arguments. In the k-expansion Δ of the triangle uvw, consider the path

$$uu_1x_1w_{4k}w_{4k-1}u_2u_3x_2w_{4k-2}w_{4k-3}u_4 \ldots u_{2k}u_{2k+1}y_ku_{2k+2} \ldots u_{4k-1}y_1u_{4k}vv_1v_2v_3 \ldots$$
$$\ldots v_{2k+1}z_kw_{2k}w_{2k-1}v_{2k+2} \ldots v_{4k-1}z_1w_2w_1v_{4k}w.$$

This path, and the path obtained when replacing in the above path $u_{4k}vv_1$ by $u_{4k}v_1$, show that for any distinct $b, c \in \{u, v, w\}$ there exists a hamiltonian bc-path in Δ and $\Delta - d$, where $d \in \{u, v, w\} \setminus \{b, c\}$, a fact we abbreviate by $(\star)$. For any edge e in G there exists a hamiltonian cycle $\mathfrak{h}_e$ in $G - e$. Denote by $e(v)$ the edge of G corresponding to the vertex v in L_k.

Case $a = v$. Consider a hamiltonian cycle $\mathfrak{h}_{e(v)}$ in $G - e(v)$. Using $(\star)$ we can transform $\mathfrak{h}_{e(v)}$ into the disjoint union $\mathfrak{P}$ of two paths in L_k, which visits every vertex of every expanded triangle except for Δ and Δ', the two expanded triangles sharing v. Applying $(\star)$ to Δ and Δ', in each avoiding only the vertex v, we obtain paths $\mathfrak{p}$ and $\mathfrak{p}'$, respectively. Then $\mathfrak{P} \cup \mathfrak{p} \cup \mathfrak{p}'$ is a hamiltonian cycle in $L_k - v$.

Case $a = v_1$. We slightly modify the hamiltonian cycle just obtained: in Δ' visit all vertices instead of avoiding v (possible by $(\star)$), and in $\mathfrak{p}$, replace the subpath $u_{4k-1}y_1u_{4k}v_1v_2$ by $u_{4k-1}u_{4k}y_1v_2$. We call $\mathfrak{h}$ this hamiltonian cycle of $L_k - v_1$.

Case $a = y_1$. Replace in $\mathfrak{h}$ the subpath $u_{4k-1}u_{4k}y_1v_2$ by $u_{4k-1}u_{4k}v_1v_2$.

Case $a = v_{2k}$. Consider the hamiltonian cycle $\mathfrak{h}_{e(u)}$ in $G - e(u)$. In the light of $(\star)$, it remains to show that there exists a hamiltonian wv-path in $\Delta - \{u, v_{2k}\}$. As in previous arguments it can be shown that there exists a path between w and u_{2k} visiting all vertices in $\Delta - \{u, u_{2k+1}, \ldots, u_{4k}, v_1, \ldots, v_{2k}, y_1, \ldots, y_k\}$. Add to this path the path $u_{2k+1}y_kv_{2k-1}u_{2k+2} \ldots u_{4k-1}y_1v_2v_1u_{4k}v$ and we obtain a hamiltonian cycle in $L_k - v_{2k}$ and are thus done. $\square$

5 Thomassen's Theorem for Paths

Among the most widely studied variations of hamiltonicity is traceability—a graph is *traceable* if it contains a hamiltonian path. We replace in the definition of hypohamiltonicity "cycle" by "path" and obtain what a *hypotraceable* graph is (see for instance [30]). In applications concerning longest paths and longest cycles, instead of 3-fragments of hypohamiltonian graphs we may also consider 3-fragments obtained from an almost hypohamiltonian graph with a cubic exceptional vertex, from which we remove the exceptional vertex—along the same lines we aim at better understanding 3-fragments of 3-connected hypotraceable and almost hypotraceable graphs (see [37] on recent progress concerning the latter, and their connection to Gallai's problem on the intersection of longest paths) and the occurrence of cubic vertices in such graphs. We now prove a path-analogue of Theorem 1, but with an annoying (and perhaps unnecessary) restriction.

Theorem 8. *If in a planar graph G of order n, minimum degree at least 4, and not of connectivity 2 at least $n-5$ vertex-deleted subgraphs are traceable, then G is traceable.*

Proof. By Tutte's theorem [32] planar 4-connected graphs are traceable, so we may restrict ourselves to the situation that G has connectivity 1 or 3. We first treat the former case. Let w be a cut-vertex of G and F, F' the 1-fragments with attachment $\{w\}$ (it is easy to see that there must be exactly two such fragments). As K_5 is non-planar, F and F' each have order at least six, and each contains a non-exceptional vertex v and v', respectively, neither of which can be w. Let $\mathfrak{p}$ ($\mathfrak{p}'$) be a hamiltonian path in $G - v$ ($G - v'$). Then $(\mathfrak{p}' \cap F) \cup (\mathfrak{p} \cap F')$ is a hamiltonian path in G.

Let G have connectivity 3. We follow a similar strategy as in the proof of Theorem 3—let $X = \{x, y, z\}$, F, F', and F_Δ be defined as in that proof. We shall abbreviate by $(\star)$ the fact that F contains a hamiltonian vw-path for any distinct $v, w \in X$ as shown in the proof of Theorem 3. We know that F contains a non-exceptional vertex v. Since X is a 3-cut, the traceability of $G - v$ yields, ignoring analogous cases, the following situations. We shall make use of the fact that, by Sanders' theorem stating that in a planar 4-connected graph there exists a hamiltonian cycle through any two edges [24], there exists a hamiltonian xy-path $\mathfrak{p}$ in $F - z$.

If there is a hamiltonian x-path $\mathfrak{p}'$ in F', then remove from $\mathfrak{p}$ the edge incident with y to obtain a path $\mathfrak{p}^-$, and we find in $\mathfrak{p}' \cup \mathfrak{p}^-$ a hamiltonian path in G. If there is a hamiltonian x-path $\mathfrak{p}'$ in $F' - y$, then $\mathfrak{p}' \cup \mathfrak{p}$ gives a hamiltonian path in G. In case there is a hamiltonian x-path $\mathfrak{p}'$ in $F' - y - z$, due to $(\star)$ there exists a hamiltonian xz-path in F, leading to the desired conclusion. If there is an x-path $\mathfrak{p}'_x$ and a y-path $\mathfrak{p}'_y$ which together span $F' - z$, then again we make use of $(\star)$ which implies that there exists a hamiltonian xy-path $\mathfrak{p}_{xy}$ in F. Then $\mathfrak{p}'_x \cup \mathfrak{p}'_y \cup \mathfrak{p}'_{xy}$ is a hamiltonian path in G. Finally, if there is an x-path $\mathfrak{p}'_x$ and a y-path $\mathfrak{p}'_y$ which together span F', then $\mathfrak{p}'_x \cup \mathfrak{p}'_y \cup \mathfrak{p}$ gives a hamiltonian path in G. ☐

The connectivity 2 case remains open. We do have the following result.

Proposition 2 [39]**.** *Each 2-fragment of edge-connectivity 2 in a planar hypotraceable graph contains a cubic vertex.*

6 Discussion

1. If we allow a modest set of exceptions $\mathcal{E}$ (of which there is at least one, namely $K_2 + 3K_1$), is it true that if in a planar n-vertex graph $G \notin \mathcal{E}$ containing no cubic vertex, $n - 2$ vertex-deleted subgraphs are hamiltonian, then G itself must be hamiltonian? By Tutte's theorem [32] and Theorem 3, only the connectivity 2 case remains open.

2. We have discussed in this article what can be inferred, in terms of hamiltonian properties, from the vertex-deleted subgraphs. A natural question is to look at the set of all subgraphs obtained when deleting any two vertices. If every such subgraph is hamiltonian, then the graph must be 4-connected. If it is also planar, then by Tutte's theorem [32], the graph must be hamiltonian. But what if G is non-planar? Is there a non-hamiltonian graph in which the removal of any pair of vertices yields a hamiltonian graph? This is a question Katona, Kostochka, Pach, and Stechkin asked in [17] (and equivalent to a recent question of van Aardt, Burger, Frick, Llano, and Zuazua [1]). Grünbaum conjectured that there exists no n-vertex graph of circumference $n-2$ such that any pair of vertices is avoided by a longest cycle [13]. Both remain open, but an important special cases of the question of Katona, Kostochka, Pach, and Stechkin is being studied in [40]. We do know that there exist planar cubic 3-connected graphs in which any pair of vertices is avoided by a longest cycle [25].

3. As mentioned in the introduction, planar graphs of minimum degree at least 4 in which every vertex-deleted subgraph is hamiltonian are a superclass of planar 4-connected graphs. A very general question arises: which results on the latter family can be extended to the former? For instance, how many hamiltonian cycles must there be in a planar graph of minimum degree at least 4 in which every vertex-deleted subgraph is hamiltonian? Moreover, inspired by conjectures of Malkevitch [18, Conjecture 6.1] and Chen, Fan, and Yu [10], we would be interested in determining such a graph that contains no k-cycle, for k as large as possible.

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