Tight cycle spectrum gaps of cubic 3-connected toroidal graphs

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Abstract. This note complements results on cycle spectra of planar graphs by investigating the toroidal case. Let $k \geq 3$ be an integer and G a cubic graph polyhedrally embedded on the torus with circumference at least k. For $k = 3$, it follows from Euler's formula that G has some (facial) cycle of length in $[3, 6]$. For $k = 4$ or 5, we show that G contains a cycle whose length lies in the interval $[k, 12]$; and for $k > 5$, we show that the same holds for the interval $[k, 2k + 4]$. This is best possible for all $k \geq 4$. On the other hand, for any non-negative integer γ there exist 2-connected cubic graphs G of genus γ, arbitrarily large face-width, and with arbitrarily large gaps in their cycle spectrum. The behavior of cycle spectrum gaps of graphs polyhedrally embedded on surfaces of genus greater than 1 remains largely unknown.

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1 Introduction

For a graph G we denote by $\mathfrak{S}(G)$ the set of all cycle lengths of G, i.e. its *cycle spectrum.* For positive integers a and b such that $3 \leq a \leq b$, the interval $[a, b]$ is a *gap* in the cycle spectrum of G if $\mathfrak{S}(G) \cap [a, b] = \emptyset$. For a graph G that has a gap $[a, b]$ in $\mathfrak{S}(G)$, a cycle in G is *short* (*long*) if its length is smaller than a (greater than b).

All embeddings of graphs in this paper are assumed to be closed 2-cell embeddings (in orientable surfaces) given by an orientation of the edges around each vertex. In an embedded graph, a cycle is *contractible* if it bounds a disk on the surface, and a walk is *facial* if it constitutes the boundary of a face. An embedding in a surface is *polyhedral* if every facial walk is a cycle and any two facial cycles meet either not at all, in a vertex, or in an edge. Equivalently, we require that the graph is 3-connected and that the embedding has face-width at least three, where the *face-width* (also known as *representativity*) of an embedded graph G is the minimum integer r such that G has r facial walks whose union contains a cycle which is noncontractible on the surface. (In the case when there are no noncontractible cycles, we let the face-width be ∞.) For a positive integer k and a non-negative integer γ, if there exists some integer $k' \geq k$ such that every polyhedrally embedded graph G of genus γ and circumference at least k satisfies $\mathfrak{S}(G) \cap [k, k'] \neq \emptyset$, we define $\sigma_\gamma(k)$ to be the smallest k'; otherwise we define $\sigma_\gamma(k)$ to be ∞. Let $\sigma_\gamma^r(k)$ be defined as $\sigma_\gamma(k)$ restricted to r-regular graphs.

Recently, Merker [3] began the investigation of gaps occurring in the cycle spectra of 3-connected planar graphs—his article elucidates why this is an interesting problem and mentions numerous related results, which we shall not repeat here. He showed that $\sigma_0^3(k) \leq 2k+9$ for all $k \geq 2$ and that $\sigma_0^3(k) \geq 2k+2$ for all even $k \geq 4$. Merker conjectured that $\sigma_0^3(k) \leq 2k+2$ for all $k \geq 2$. It is easy to see that this holds for $k \in \{2,3,4,5\}$. However, in a short note the second author proved that $\sigma_0^3(k) \geq 2k+3$ for all even $k \geq 6$, see [5]. Cui and the first author [2] expanded on this and gave a full description of the values of $\sigma_0(k)$ and $\sigma_0^3(k)$ for all k; in particular, they showed that $\sigma_0(k) = \sigma_0^3(k) = 2k+3$ for all $k \geq 10$. The purpose of this note is to treat analogues of these results pertaining to the cubic toroidal case. We will prove that $\sigma_1^3(3) = 6$, $\sigma_1^3(4) = \sigma_1^3(5) = 12$, and $\sigma_1^3(k) = 2k+4$ for all integers $k \geq 6$.

For an abstract or embedded graph G we denote with $V(G)$ ($E(G)$) its vertex (edge) set. For an embedded graph G we denote by $F(G)$ the set of its faces, and the boundary of a face F will be denoted by $\mathrm{bd}(F)$. For an embedded graph G, its *Euler characteristic* and *genus* are defined as

$$\chi(G) := |V(G)| - |E(G)| + |F(G)| \quad \text{and} \quad \gamma(G) := \frac{2-\chi(G)}{2},$$

respectively. For an abstract graph, its *genus* is the smallest integer γ such that the graph has an embedding with genus γ. Abstract (embedded) graphs of genus 0 are called *planar* (*plane*). For further details regarding topological graph theory we refer the reader to [4].

For a contractible cycle C in an embedded graph with a positive genus, the *interior* $\mathrm{int}(C)$ of C shall be the region that is bounded by C and homeomorphic to an open disk. We also set $\mathrm{Int}(C)$ to be the closure of $\mathrm{int}(C)$ in the surface we consider. If H is a path or a cycle in G, then $\ell(H) := |E(H)|$ denotes the *length* of H.

2 The cubic toroidal case

Lemma. *Let $k \geq 3$ be an integer and G be a cubic graph which has a polyhedral embedding on the torus. Suppose G has circumference at least k and gap $[k, 2k]$ in $\mathfrak{S}(G)$. Then there exists a 2-connected subgraph G' of G such that*

(i) *G' is embedded on either the plane or the torus with face-width at least 2;*

(ii) *no two short facial cycles of G' intersect;*

(iii) *G' contains a long facial cycle;*

(iv) *every long facial cycle in G' is also a facial cycle in G (of the same length); and*

(v) *if the intersection of two long facial cycles of G' is not empty, then it is an edge (together with its end-vertices).*

Moreover, if $k = 5$, we have $G' = G$.

Proof. We first show that G contains a long facial cycle. Let C be a shortest long cycle in G. Let P be a path in G with end-vertices u, v lying in C but internally disjoint from C. Then C is the union of two edge-disjoint uv-paths P_1 and P_2. Observe that any cycle contained in the union of two short cycles is also short as G has gap $[k, 2k]$. This implies that at least one of $P_1 \cup P$ and $P_2 \cup P$ is long as well (as C is long); assume the former holds. Hence, as C is a shortest long cycle in G, $\ell(P) \geq \ell(P_2) \geq \min\{\ell(P_1), \ell(P_2)\}$. We call this property (†).

Now, choose $u, v \in V(C)$ such that C is the union of two edge-disjoint uv-paths P_1 and P_2 and $\min\{\ell(P_1), \ell(P_2)\} \geq k$. Let F_u and F_v be two faces in G incident to u and v, respectively.

By a result of Barnette [1] there is a contractible cycle K whose interior contains F_u and F_v when G is polyhedrally embedded on the torus. We may take such a cycle K with a minimum number of interior faces. We now show that the graph induced by K and its interior must be an outerplane graph. Suppose this is not the case, so there is some vertex w lying in $\mathrm{int}(K)$. Since G is 3-connected, there are three distinct vertices $w_1, w_2, w_3 \in V(K)$ and three paths joining w to w_1, w_2, w_3 such that any two of these paths only have w as their common vertex and each of them is internally disjoint from K. These three paths separate $\mathrm{Int}(K)$ into three regions and the union of the regions that contain F_u or F_v is also bounded by a contractible cycle but has fewer interior faces, contradicting our assumption. Therefore $\mathrm{Int}(K)$ is an outerplane graph and, in particular, $u, v \in V(K)$.

Let Q_1, Q_2 be two edge-disjoint subpaths of K with end-vertices u, v such that K is the union of Q_1 and Q_2. We claim that K is long. By symmetry and the condition $\min\{\ell(P_1), \ell(P_2)\} \geq k$, it suffices to show that $\ell(Q_1) \geq \min\{\ell(P_1), \ell(P_2)\}$. Let $v_0 = u, v_1, \ldots, v_t = v$ be the vertices in $V(Q_1) \cap V(C)$, listed in the order in which they occur, and Q_1^i be the subpath of Q_1 with end-vertices v_{i-1}, v_i ($i \in \{1, \ldots, t\}$), such that $Q_1^1, \ldots, Q_1^t$ are pairwise edge-disjoint and every Q_1^i is internally disjoint from C. So Q_1 is the union of $Q_1^1, \ldots, Q_1^t$. For every $i \in \{1, \ldots, t\}$, denote by P_1^i, P_2^i two edge-disjoint paths with end-vertices v_{i-1}, v_i such that $C = P_1^i \cup P_2^i$. By (†), we have $\ell(Q_1^i) \geq \min\{\ell(P_1^i), \ell(P_2^i)\}$. Note that $\sum_{i=1}^t \min\{\ell(P_1^i), \ell(P_2^i)\}$ denotes the length of a uv-walk in C, which is at least $\min\{\ell(P_1), \ell(P_2)\}$. Altogether, we have

$$\ell(Q_1) = \sum_{i=1}^{t} \ell(Q_1^i) \geq \sum_{i=1}^{t} \min\{\ell(P_1^i), \ell(P_2^i)\} \geq \min\{\ell(P_1), \ell(P_2)\}.$$

Thus, there exists a long contractible cycle in G. It was shown in the proof of [2, Lemma 3] that every long contractible cycle contains some long face in its interior, from which we conclude that G contains a long face. In the rest of this proof, we fix a long face in G and denote it by F.

We construct a sequence of subgraphs of G as follows. Initially, set $G_0 := G$. When G_i is already constructed, we set $G' := G_i$ if no two short facial cycles in G_i intersect. Otherwise, we choose two short facial cycles C_1, C_2 that have a non-empty intersection. We see G_i as embedded in a surface S and C_1, C_2 as Jordan curves on S. By deleting from S the curves C_1 and C_2, the surface S falls apart into components. We define G_{i+1}, as a subgraph of G, to be the closure of the component S_F that contains F. Note that, since G is cubic, G_i is subcubic and the boundary of S_F is a disjoint family of short cycles in G_i. This enables us to obtain an orientable embedding of G_{i+1} by patching the holes of S_F with disks (which will be new short faces in G_{i+1}). Clearly, this embedding of G_{i+1} has genus no larger than that of G_i.

It is obvious that every graph in the above sequence is 2-connected and contains the long face F. Hence, it is not hard to see that the properties (i) to (iv) hold. Suppose there are two long facial cycles F_1, F_2 of G' that intersect. By (iv), F_1 and F_2 are also in G. Since G is a polyhedrally embedded cubic graph, $F_1 \cap F_2$ must be an edge. This justifies property (v).

It remains to show that $G' = G_0$ in the above construction if $k = 5$. In this case all short faces have length either 3 or 4. It is easy to check that no two short faces of G can intersect as G is a polyhedrally embedded cubic graph with no cycle length in $[5, 10]$. Therefore, we must have $G' = G_0$, and the last statement follows. □

Theorem. *Let $k \geq 3$ be an integer and G be a cubic graph polyhedrally embedded on the torus. If G has circumference at least k, then*

$$\mathfrak{S}(G) \cap [k, 2k+4] \neq \emptyset.$$

Moreover, if $k = 5$, then

$$\mathfrak{S}(G) \cap [5, 12] \neq \emptyset.$$

Finally, $\sigma_1^3(3) = 6$, $\sigma_1^3(4) = \sigma_1^3(5) = 12$, and $\sigma_1^3(\ell) = 2\ell + 4$ for all integers $\ell \geq 6$.

Proof. Suppose to the contrary that the above intersection is empty. Let G' be the subgraph of G given by our lemma. Next we consider the graph H which is obtained from G' by suppressing vertices of degree two. Faces in G' correspond in an obvious way to faces in H. For any face F in G', denote by $\ell(F)$ and $\ell_H(F)$ the length of F in G' and the length of the corresponding face in H, respectively. We define $\mathcal{X}$ to be the set of short faces in G', and $\mathcal{Y}$ the set of long faces in G'. Since G' has no facial cycle of length in $[k, 2k+4]$, we have $\ell(F) \leq k-1$ for all $F \in \mathcal{X}$, and $\ell(F) \geq 2k+5$ for all $F \in \mathcal{Y}$. Put $n := |V(H)|$, $x := |\mathcal{X}|$, and $y := |\mathcal{Y}|$.

Since H is a cubic graph of genus $\gamma(H) \leq 1$, by Euler's formula we have

$$x + y = \frac{n}{2} + 2 - 2\gamma(H) \geq \frac{n}{2}.$$

As H is a simple graph—this can be shown exactly as in the proof of [3, Theorem 2] and is therefore omitted—and G' satisfies (ii) in our lemma, we have

$$n \geq \sum_{F \in \mathcal{X}} \ell_H(F) \geq 3x. \tag{1}$$

Furthermore

$$3n + (k-7)x \geq \sum_{F \in \mathcal{Y}} \ell(F) \geq (2k+5)y \geq (2k+5)\left(\frac{n}{2} - x\right),$$

where one derives the first inequality as in [2]. This implies that

$$n \leq \frac{(6k-4)x}{2k-1} < 3x,$$

which contradicts (1).

For the case $k = 5$, it follows from the lemma that $H = G' = G$. Suppose all long faces of G have length larger than 12. We have

$$3n - 3x \geq 3n - \sum_{F \in \mathcal{X}} \ell(F) = \sum_{F \in \mathcal{Y}} \ell(F) \geq 13y \geq 13\left(\frac{n}{2} - x\right).$$

This implies $n \leq \frac{20}{7}x$, which, again, contradicts (1).

We have $\sigma_1^3(3) = 6$ as the hexagonal tiling can be embedded on the torus and a cubic toroidal graph has girth at most 6. As $\sigma_1^3(5) \geq \sigma_1^3(4)$, it remains to show that $\sigma_1^3(\ell) \geq 2\ell + 4$ for all integers $\ell \geq 4$ other than 5. Consider a hexagonal tiling of the torus with large face-width. Replacing vertices as shown in Figure 1 and following the indications in the figure's caption—note that operation f (g) replaces every cubic vertex with $\ell - 1$ (ℓ) vertices—, the statement is proven. □

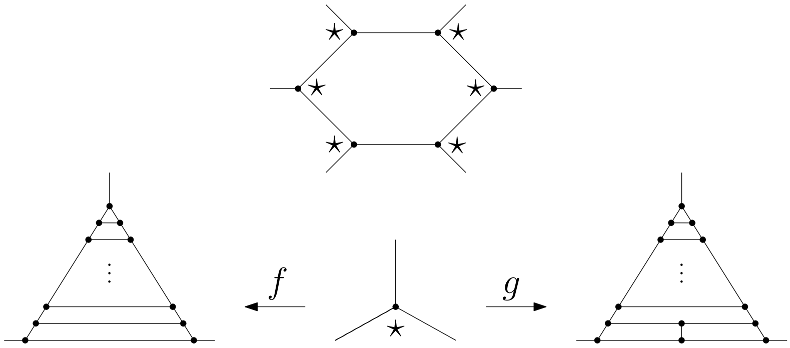

Figure 1: The top figure shows a hexagon in which each vertex has one of its three incident faces emphasised by a star ($\star$). The bottom figures depict how we replace each vertex according to the position of ($\star$), using operation f if k is even and operation g if k is odd.

The above result gives a clear picture of the behaviour of large gaps in cubic toroidal graphs—we did however impose 3-connectedness. Dropping this requirement changes the situation dramatically. In the proof of the next observation we use an idea similar to what was used by Merker in [3].

Proposition. *For any non-negative integer γ there exist 2-connected cubic graphs G of genus γ, of arbitrarily large face-width, with arbitrarily large gaps in their cycle spectrum.*

Proof. A *diamond* shall be a plane graph consisting of two triangles sharing exactly one edge. By considering two disjoint diamonds and joining two of their non-cubic vertices, we form a *diamond chain* consisting of two diamonds. In the same way, a diamond chain of arbitrary length can be constructed. We do so such that the resulting graph is plane. Consider a cubic closed-2-cell graph G embedded on a surface of genus γ and with sufficiently large face-width, a face F of G, and insert into the interior of F a diamond chain D, its two dangling edges connected to two new vertices obtained by splitting two distinct edges of the boundary cycle of F. In the resulting graph, cycles contained entirely in D must be 3- or 4-cycles, and any other cycle either does not visit D at all, or it visits at least 3/4 of its vertices. $\square$

We conclude this note with the following question. Are there 3-connected cubic graphs G with a closed 2-cell embedding on the torus which have gap $[k, 2k+4]$ in $\mathfrak{S}(G)$ for some integer $k \geq 3$?

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