On 2-factors splitting an embedded graph into two plane graphs

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Abstract

We investigate 2-planarizing 2-factors, i.e. 2-factors of embedded graphs so that cutting along the cycles of the 2-factor we get two plane graphs where the cycles of the 2-factors are a spanning set of face boundaries in each of the graphs. We will give necessary criteria for an abstract graph to have an embedding with a 2-planarizing 2-factor as well as necessary criteria for embedded graphs to have such a 2-factor. Along the way, we discuss to which degree classical results from planar hamiltonicity theory can be extended in our framework. In addition we present computational results on how common 2-planarizing 2-factors are in small cubic graphs.

Key words. 2-factor; hamiltonian cycle; embedded graph

MSC 2020. 05C10; 05C38; 05C45

1 Introduction

2-planarizing 2-factors are a generalization of hamiltonian cycles in plane graphs, where a graph is *plane* if it is planar and embedded in the Euclidean plane. A hamiltonian cycle in a plane graph G splits the graph into two parts: an interior and an exterior part. Both parts—equipped with the embedding induced by G—are plane and in both parts the hamiltonian cycle is the boundary of a face, only with the direction once clockwise and once counterclockwise. So a plane graph G has a hamiltonian cycle if and only if there are two plane connected graphs G_1, G_2 with the same vertices and the (except for the orientation, which is different) same spanning facial cycles f_1, f_2, so that G is the graph obtained by gluing G_1 to G_2 along the cycles f_1, f_2. (We recall that a cycle in a plane graph is *facial* if it bounds a face of the graph.) A hamiltonian cycle is a special case of a 2-*factor*, that is: a 2-regular spanning subgraph of the graph. For higher genera, where hamiltonian cycles do not necessarily separate the graph any more (and never into two plane graphs), we need 2-factors with more than one cycle to split the graph into two plane parts. In general, an embedded graph is defined to have a 2-planarizing 2-factor if there are plane connected graphs G_1 and G_2 on the same set of vertices and 2-factors $\mathcal{F}_1$ of G_1 and $\mathcal{F}_2$ of G_2, composed of (except for the orientation, which is different) the same facial cycles, so that G is the graph obtained by gluing G_1 to G_2 along identical faces.

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For a graph G we denote with $V(G)$ ($E(G)$) its vertex (edge) set, and for an embedded graph $F(G)$ shall be the set of its faces. In this article, edges $\{u, v\}$ in an embedded graph are interpreted as two oppositely directed edges (u, v) from u to v and $(u, v)^{-1} = (v, u)$ from v to u. An embedded graph is a connected graph G together with a cyclic ordering of the oriented edges starting at the same vertex, that we will interpret as clockwise. When we refer to an oriented edge e as being *between* an oriented edge e_1 and an oriented edge e_2 in the order, this means that starting at e_1 and proceeding in the rotational order, the edge e is reached before the edge e_2. Such combinatorial embeddings correspond to 2-cell embeddings on orientable surfaces [7, 9]. Each cycle c corresponds to two oppositely oriented directed cycles c^+, c^- of oriented edges. A face in an embedded graph is a cyclic sequence $e_0, \ldots, e_{n-1}$ of oriented edges, so that for $0 \leq i < n$ the edge $e_{(i+1) \pmod n}$ is the next edge of e_i^{-1} in the ordering around the end vertex of e_i. We call such a pair $(e_i, e_{(i+1) \pmod n})$ an *angle of the face*. When drawing an embedded graph and interpreting a face as the region bounded by edges, due to the clockwise interpretation of the ordering around the vertices, a face is the sequence of oriented edges with the face on the left one gets by a counterclockwise traversal of the oriented boundary edges. With this concept we can make the definition of a 2-planarizing 2-factor exact:

Let G be an embedded graph with a 2-factor $\mathcal{F}$ consisting of cycles $c_1, \ldots, c_k$. We say that $\mathcal{F}$ is a *2-planarizing 2-factor* if there are plane graphs G_1 and G_2 with $V(G_1) = V(G_2) = V(G)$—where G_1 and G_2 shall be called the *decomposition graphs*—with the following properties:

- $E(G_1) \cup E(G_2) = E(G)$;
- $E(G_1) \cap E(G_2) = E(\mathcal{F})$;
- for each cycle c_i in $\mathcal{F}$, one of the two oppositely oriented directed cycles occurs as a face in G_1 and the other in G_2; and
- the orientation of edges around the vertices in G_1 and G_2 is the one induced by G, that is: the rotational order in G with the edges not in G_1, resp. G_2 removed.

The faces $c_i^{+/-}$ in G_1 and G_2 are called *external faces* and the others *internal faces*. We say that an abstract graph G *has a 2-planarizing 2-factor* if G has an embedding with a 2-planarizing 2-factor. Examples of graphs having and graphs not having a 2-planarizing 2-factor are given in [5]; Petersen's graph is among the former, Heawood's graph among the latter.

Counting the number of vertices, edges, and faces we may conclude that if a 2-planarizing 2-factor consists of k cycles, the graph G is embedded in an orientable surface of genus $k - 1$. For $k = 1$ we have that G is a plane hamiltonian graph.

For an embedded graph G, the *Euler characteristic* is defined as $\chi(G) := |V(G)| - |E(G)| + |F(G)|$ and we write $\gamma(G) = \frac{2-\chi(G)}{2}$ for its *genus*. For an abstract graph G, its *genus* $\text{mingen}(G)$ is the smallest integer γ such that the graph has an embedding with genus γ.

Let $G_1 = (V, E_1), G_2 = (V, E_2)$ be two embedded graphs that have the same set of vertices and where the set $E_1 \cap E_2$ of common edges forms a facial 2-factor—that is: a 2-factor whose cycles are face boundaries—in both graphs, only that the directed cycles forming the faces are oppositely directed. Then we can form a graph $G_{1,2}$ by taking a copy of G_1 and inserting the edges of $E_2 \setminus E_1$ in the following way. For each vertex v there are exactly two oriented edges $e_{v,1}, e_{v,2}$ starting at v that occur in both graphs. The opposite orientation of the faces implies that in the copy of G_1 the edge $e_{v,2}$ follows $e_{v,1}$ in the rotational order and in G_2 it is the other way around—that is: all other edges are between $e_{v,1}$ and $e_{v,2}$. To this end we can insert all oriented edges of $E_2 \setminus E_1$ starting at v into the copy of G_1 between $e_{v,1}$ and $e_{v,2}$. For these edges we keep the order given by G_2. We call this *identification of 2-factors*. Starting with two connected plane graphs G_1, G_2 with these properties, we can construct a graph with a 2-planarizing 2-factor.

Note that a set of pairwise disjoint cycles (which need not form a 2-factor) in a (connected) embedded graph is called *planarizing* by Mohar and Thomassen [9, p. 177] if cutting along these cycles yields one plane connected graph; see also [1]. In [5] and [11] the same term is used if by cutting along these cycles a disjoint union of plane graphs is obtained. In this article we have chosen to call the second concept "2-planarizing" in order to be able to distinguish these two concepts easily in the future.

2 Results

The topic of [5] were generalizations of Grinberg's classical formula [6], but one of the main results is about 2-planarizing 2-factors, so we will repeat it here. We denote the size of a face f by $s(f)$.

Theorem 1 ([5]). *Let G be an embedded graph with a 2-planarizing 2-factor, so that the corresponding plane graphs are G_1 and G_2. Then*

$$\sum_{f \in F(G_1)} (s(f) - 2) = \sum_{f \in F(G_2)} (s(f) - 2).$$

Note that it makes no difference whether the sums are over all faces of G_1 and G_2 or just the internal faces, as there are equally many external faces in both graphs and they have the same sizes.

Tutte proved that plane 4-connected graphs are hamiltonian [10] and thus have a 2-planarizing 2-factor. Unfortunately, already for the torus there exist 4-connected embedded graphs not admitting a 2-planarizing 2-factor: consider the canonical toroidal embedding G of the (4-connected) Cartesian product of two odd-length cycles. As $F(G)$ consists exclusively of 4-faces and $|F(G)|$ is odd, by applying Theorem 1 we obtain that G has no 2-planarizing 2-factor. However, it is not difficult to see that the Cartesian product of two 3-cycles does have another embedding on the torus with a 2-planarizing 2-factor. Later we will see in Corollary 2 that for any $n \in \{5\} \cup \{k \in \mathbb{N} : k \geq 8\}$, the complete graph K_n is a 4-connected graph of genus $\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ for which *no* embedding admits a 2-planarizing 2-factor.

A hamiltonian graph has neither a cut-vertex nor a bridge—is this true for non-hamiltonian 2-planarizing 2-factors, as well? For the former it is not, as shown in Figure 1: there exist embedded graphs with a 2-planarizing 2-factor and a cut-vertex. For bridges, the situation is different.

Theorem 2. *If a graph G has an embedding with a 2-planarizing 2-factor, then G is bridgeless.*

Proof. Let G be an embedded graph with a 2-planarizing 2-factor $\mathcal{F}$. Suppose that G contains a bridge b. As b is a bridge, it cannot lie on a cycle—in particular, b is not an edge of $\mathcal{F}$. In G, there exist exactly two graphs G_1 and G_2 under decomposition by $\mathcal{F}$, and these graphs must be connected. Only edges of $\mathcal{F}$ can occur in both G_1 and G_2, so b lies either in G_1 or in G_2. Without loss of generality assume the former. Thus, b is not in G_2. As G_2 spans G, the graph $G - b$ is connected, a contradiction. □

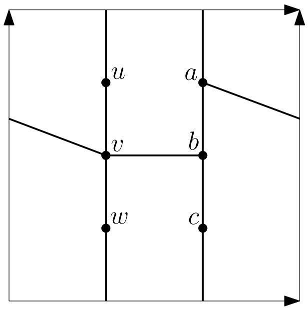

Figure 1: A graph embedded on the torus with a 2-planarizing 2-factor composed of cycles uvw and abc, and cut-vertex v. Thus, there exist graphs of connectivity 1 with an embedding admitting a 2-planarizing 2-factor.

2.1 Bounding the size

The genus of a graph gives a lower bound on the number of cycles in a 2-planarizing 2-factor, so that sometimes there are simply not enough vertices for this number of cycles. If the genus of an abstract graph is not known, we can still give a ratio between the number of vertices and the number of edges, above which no 2-planarizing 2-factor can exist. We make use of the *girth* of a graph, i.e. the length of a shortest cycle in the graph. We will assume here G to always contain at least one cycle—otherwise we know that it cannot have a 2-planarizing 2-factor—so for the girth g of G we have $3 \leq g < \infty$. If g is the girth of an abstract graph G, then $|V(G)| \geq g(\gamma(G)+1)$ is a necessary condition for an embedding of G to have a 2-planarizing 2-factor. We also point out that, as a consequence of both decomposition graphs having to be connected, we have that if an embedded graph G has a 2-planarizing 2-factor, then $|E(G)| \geq |V(G)| + 2\gamma(G)$.

Lemma 1. *Let G be a graph of girth at least g. If G has a 2-planarizing 2-factor, then*

$$|E(G)| \leq \frac{(g+2)|V(G)| - 4g}{g-2}$$

and this inequality is strict if $|V(G)|$ is not divisible by g. As a consequence we have that

$$|E(G)| \leq 5|V(G)| - 12$$

and if $|V(G)|$ is not divisible by 3, this inequality is also strict.

Proof. Assume that G is embedded in a surface of genus $\gamma(G)$ with a 2-planarizing 2-factor, which then has $\gamma(G)+1$ cycles. Since every such cycle contains at least g vertices, we have $|V(G)| \geq g(\gamma(G)+1)$ and $|V(G)| > g(\gamma(G)+1)$ if $|V(G)|$ is not divisible by g. For G we have

$$\gamma(G) = \frac{2-\chi(G)}{2} = \frac{2-|V(G)|+|E(G)|-|F(G)|}{2}$$

with $|F(G)| \leq \frac{2|E(G)|}{g}$, as the boundary of a face must contain a cycle. We get

$$|V(G)| \geq g(\gamma(G)+1) = g\left(\frac{4-|V(G)|+|E(G)|-|F(G)|}{2}\right)$$
$$\geq \frac{4g-|V(G)|g+|E(G)|g-2|E(G)|}{2},$$

from which the first statement follows. The second statement is obtained by inserting $g=3$ for which the maximum is attained and both inequalities are strict if not all cycles in the 2-factor can have length g, resp. 3. □

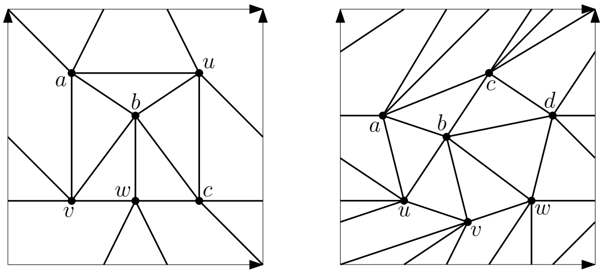

Figure 2: An embedding of K_6 with a 2-planarizing 2-factor composed of cycles abc, uvw, and an embedding of K_7 with a 2-planarizing 2-factor composed of cycles $abcd$, uvw.

We present three consequences of Lemma 1:

Corollary 1. *For $r \geq 10$ no r-regular graphs with a 2-planarizing 2-factor exist.*

Corollary 2. *The complete graph K_n admits an embedding containing a 2-planarizing 2-factor if and only if $n \in \{3, 4, 6, 7\}$.*

Proof. Lemma 1 implies $n \leq 7$. The graphs K_1 and K_2 are acyclic. The graphs K_3 and K_4 are hamiltonian planar graphs and therefore have a 2-planarizing 2-factor. For K_5, which is non-planar, there is no embedding with a 2-planarizing 2-factor, as there are not enough vertices for at least two disjoint cycles. In Figure 2, we give toroidal embeddings of K_6 and K_7 possessing 2-planarizing 2-factors. □

Corollary 3. *The complete bipartite graph $K_{s,t}$ has an embedding with a 2-planarizing 2-factor if and only if $s = t \in \{2, 4\}$.*

Proof. For $s \neq t$, $K_{s,t}$ does not admit any 2-factor, so $s = t$ is a necessary condition. Using $g = 4$, Lemma 1 gives $t < 5$. The graph $K_{1,1}$ is acyclic and $K_{3,3}$ is a toroidal 6-vertex graph of girth 4, so it has no non-hamiltonian 2-factor. $K_{2,2}$ is a 4-cycle, so it is plane and hamiltonian, and $K_{4,4}$ has a 2-planarizing 2-factor as depicted in Figure 3. □

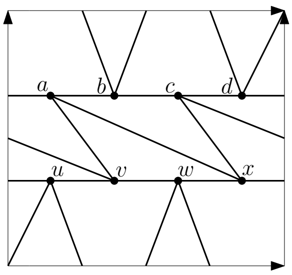

Figure 3: An embedding of $K_{4,4}$ with a 2-planarizing 2-factor composed of cycles $abcd$, $uvwx$.

If G is k-regular, then $|E(G)| = \frac{k|V(G)|}{2}$. Combining this with Lemma 1 we obtain for k-regular graphs of girth g

$$\frac{k|V(G)|}{2} \leq \frac{g|V(G)| + 2|V(G)| - 4g}{g - 2}$$

which gives

$$|V(G)| - \frac{kg - 2g - 2k}{4}|V(G)| \geq 2g.$$

We can extract the following result.

Theorem 3. *Let G be a k-regular graph of girth g. If $kg - 2g - 2k \geq 4$, then there exists no embedding of G with a 2-planarizing 2-factor. If $kg - 2g - 2k < 4$ and an embedding of G admits a 2-planarizing 2-factor, then*

$$|V(G)| \geq \left\lceil \frac{8g}{4 - kg + 2g + 2k} \right\rceil.$$

Barnette conjectured in 1969 that every 3-connected bipartite cubic plane graph is hamiltonian [2]. This conjecture remains open. But its analogue for higher genera and 2-planarizing 2-factors is not true—as can be seen at examples like $K_{3,3}$, see Corollary 3, and the Heawood graph [5].

For $k \geq 3$ and $g \geq 3$, let $n_{k,g}$ be the minimum order of a k-regular graph of girth g admitting an embedding with a 2-planarizing 2-factor. If for the given values of k and g no such graph exists, put $n_{k,g} := \infty$.

Lemma 2. *The following table presents bounds and exact values of $n_{k,g}$. For values of $n_{k,g}$ not mentioned in the table we have $n_{k,g} = \infty$ by Theorem 3. Bounds inferred from Theorem 3 are given between round brackets.*

$k \backslash g$	3	4	5	6	7	8	9
3	4 (≥ 4)	8 (≥ 6)	10 (≥ 8)	16 (≥ 12)	30 (≥ 19)	40 (≥ 32)	(≥ 72)
4	6 (≥ 4)	8 (≥ 8)	25 (≥ 20)				
5	6 (≥ 5)	16 (≥ 16)					
6	7 (≥ 6)						
7	10 (≥ 8)						
8	12 (≥ 12)						
9	24 (≥ 24)						

Proof. Case $k = 3$: Due to K_4, we have $n_{3,3} = 4$. The only cubic graph of girth 4 and order 6 is $K_{3,3}$, which by Corollary 3 does not admit an embedding with a 2-planarizing 2-factor. A k-regular graph with k odd must have an even number of vertices, so $n_{3,4} \geq 8$. The cube is a plane hamiltonian cubic graph of girth 4, so $n_{3,4} = 8$. The $(3,5)$-cage, $(3,6)$-cage, and $(3,7)$-cage have order 10, 14, and 24, resp. The $(3,5)$-cage is the Petersen graph which admits an embedding with a 2-planarizing 2-factor [5], so $n_{3,5} = 10$. For $n_{3,6}$, $n_{3,7}$, and $n_{3,8}$ we used the computer program described in Section 3 applied to lists that can be obtained from [4] to determine $n_{3,\cdot}$. Graphs realizing the minimal values can be found in the database *HoG – House of Graphs* [4] when searching for the string `pl_2_fac`. The result is $n_{3,6} = 16$, $n_{3,7} = 30$, and $n_{3,8} = 40$.

Case $k = 4$: By Corollary 2, no embedding of K_5 has a 2-planarizing 2-factor, and as the octahedron is a plane hamiltonian 4-regular graph of girth 3, we have $n_{4,3} = 6$. The graph $K_{4,4}$ has an embedding with a 2-planarizing 2-factor, see Figure 3, so $n_{4,4} = 8$. For $n_{4,5}$ the program *genreg* described in [8] was used to generate all graphs that were then tested by the program from Section 3. The result is $n_{4,5} = 25$ and an example graph can be found again in *HoG* when searching for the string `pl_2_fac`.

Case $k = 5$: By Corollary 2, from K_6 we can infer that $n_{5,3} = 6$. With a computer we tested all 388 graphs on 16 vertices that are 5-regular and have girth 4. The result is that 32 of them have a planarizing 2-factor, so that $n_{5,4} = 16$. An example graph can be found again in *HoG*.

Case $k \in \{6, 7, 8, 9\}$: By Corollary 2, from K_7 we can infer that $n_{6,3} = 7$, and since K_8 admits no embedding with a 2-planarizing 2-factor, $n_{7,3} \geq 9$. As 7 is odd, $n_{7,3} \geq 10$. In Figure 4 a graph decomposed into two plane graphs is given, showing that $n_{7,3} = 10$. Graphs showing that $n_{8,3} = 12$ and $n_{9,3} = 24$ are given in Figure 5, resp. Figure 6. □

It remains an open problem to determine $n_{3,9}$.

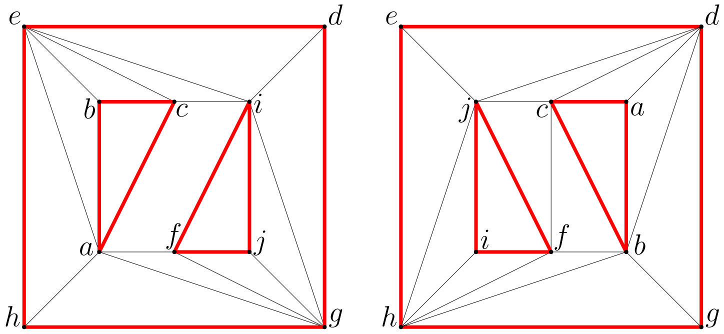

Figure 4: Graphs required to prove that $n_{7,3} = 10$.

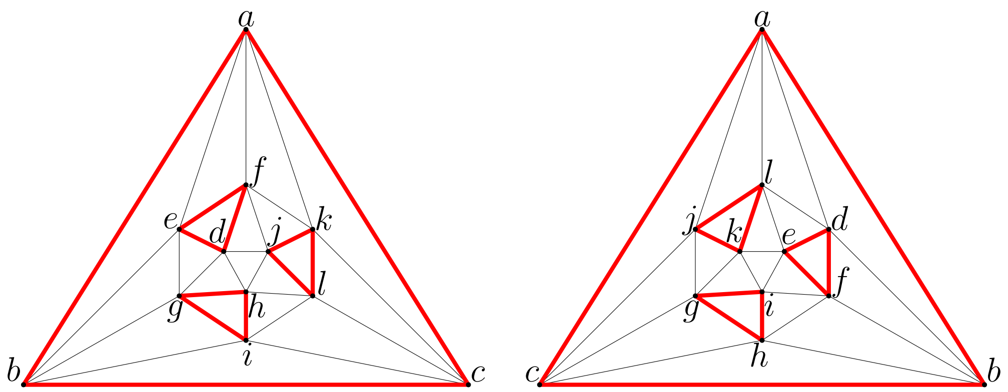

Figure 5: Two icosahedra with 2-factors in red. Identifying these yields $n_{8,3} \leq 12$.

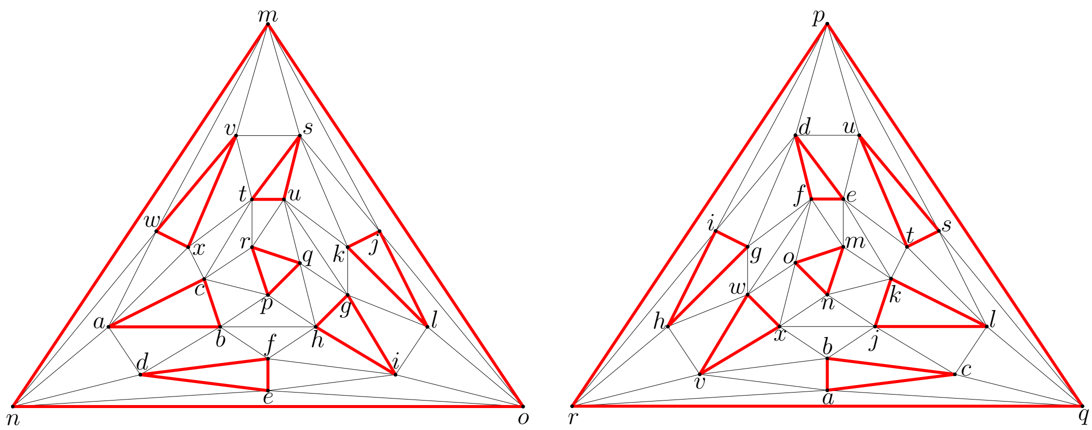

Figure 6: Graphs showing that $n_{9,3} \leq 24$.

2.2 The genus of embeddings with a 2-planarizing 2-factor

Lemma 3. *If G is an embedding with genus γ of a graph with girth at least g that has a 2-planarizing 2-factor, then*

$$\gamma \le \min\left\{\frac{|V(G)|}{g} - 1, \frac{|E(G)| - |V(G)|}{2}\right\}.$$

For r-regular graphs ($r \ge 2$), this means that $\gamma = 0$ if $r = 2$, $\gamma \le \frac{|V(G)|}{4}$ if $r = 3$, and $\gamma \le \frac{|V(G)|}{3} - 1$ otherwise.

Proof. A 2-factor in G can have at most $\frac{|V(G)|}{g}$ cycles, so the bound $\frac{|V(G)|}{g} - 1$ for the genus is immediate.

As a 2-planarizing 2-factor has $|V(G)|$ edges, one of the two planar parts into which the graph is decomposed has at most $\frac{|E(G)|-|V(G)|}{2}$ edges not belonging to the 2-factor and as the graph must be connected, there are at most $\frac{|E(G)|-|V(G)|}{2} + 1$ cycles, giving a bound of $\frac{|E(G)|-|V(G)|}{2}$ for the genus. □

The following lemma will be useful when showing that certain graphs have embeddings on surfaces of many different genera, all of which admit a 2-planarizing 2-factor. If two vertices lie in the same face of an embedded graph, then there is only one cyclic order, so this cyclic order occurs in the facial walk in the graph as well as in the corresponding facial walk of the mirror image. A given ordering of three vertices of the same face occurs either in the graph or in its mirror image, but not in both—at least if the facial walk is simple. For four or more vertices there can be cyclic orders that occur neither in the graph nor in its mirror image. These facts are the basis of the following lemma.

Lemma 4. *Let G and G' be graphs with disjoint vertex sets, 2-planarizing 2-factors $\mathcal{F}$, resp. $\mathcal{F}'$, and (embedded) decomposition graphs G_1 and G_2 (for G), resp. G'_1 and G'_2 (for G'). For $1 \le i \le 2$ and $1 \le j \le 3$ let $x_1, \dots, x_i$ be vertices in a face of G_1 that is not in $\mathcal{F}$, $x'_1, \dots, x'_i$ be vertices in a face of G'_1 that is not in $\mathcal{F}'$, $y_1, \dots, y_j$ be vertices in a face of G_2 that is not in $\mathcal{F}$, and $y'_1, \dots, y'_j$ be vertices in a face of G'_2 that is not in $\mathcal{F}'$.*

Then the graph

$$G^{1,2} = (V(G) \cup V(G'), E(G) \cup E(G') \cup \{\{x_1, x'_1\}, \dots, \{x_i, x'_i\}, \{y_1, y'_1\}, \dots, \{y_j, y'_j\}\})$$

has the 2-planarizing 2-factor $\mathcal{F} \cup \mathcal{F}'$.

Proof. If $j = 3$ then—by possibly taking the mirror images of G'_1 and G'_2—we can guarantee that in the face of G_2 the cyclic order is y_1, y_2, y_3 and in the face of G'_2 the order is y'_3, y'_2, y'_1.

It is then easy to see that there are plane decomposition graphs $G_1^{1,2}$ and $G_2^{1,2}$ with facial 2-factor $\mathcal{F} \cup \mathcal{F}'$ by placing G'_1 (G'_2) in a face of G_1 (G_2) that is not in $\mathcal{F}$. The case with three vertices is depicted in Figure 7. □

We note that if, in the above lemma, $i > 2$ or $j > 3$, we cannot guarantee that the edges $\{x_1, x'_1\}, \dots, \{x_i, x'_i\}, \{y_1, y'_1\}, \dots, \{y_j, y'_j\}$ do not cross when constructing the decomposition graphs of the larger graph.

As for $g = 3$ the formula in Lemma 3 gives the maximum value of γ for all possible g, let $\gamma_{max}(G) = \min\left\{\left\lfloor \frac{|V(G)|}{3} - 1 \right\rfloor, \left\lfloor \frac{|E(G)|-|V(G)|}{2} \right\rfloor\right\}$.

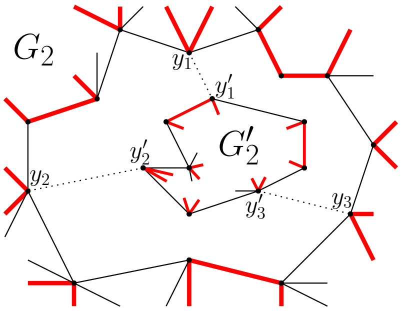

Figure 7: A decomposition graph of the graph $G^{1,2}$.

Lemma 5. *For $r \in \{2,3,4,5\}$ there are infinitely many r-regular graphs not only with embeddings with a 2-planarizing 2-factor of genus $\gamma_{max}(G)$, but even with embeddings with 2-planarizing 2-factor in each genus from 0 to $\gamma_{max}(G)$.*

Proof. For $r = 2$ the result is trivial, as the upper bound is 0, so each cycle has this property. So let $r \in \{3,4,5\}$. In order to prove the lemma, we will give an r-regular planar hamiltonian graph and a 2-factor to which Lemma 4 can be applied iteratively. We will choose for the cases, which are in a certain sense maximal, that is where the rounding in $\left\lfloor \frac{|V(G)|}{3} - 1 \right\rfloor$, resp. $\left\lfloor \frac{|E(G)|-|V(G)|}{2} \right\rfloor$ has no effect.

In Figures 8 to 11 some 3-, 4- and 5-regular graphs built from repeated building blocks are presented, together with 2-factors of the blocks and cycles through several blocks that show that 2-planarizing 2-factors with any number of cycles from 1 to $\gamma_{max}(G) + 1$ exist. The fact that they are 2-planarizing 2-factors can either be seen directly or by applying Lemma 4. $\square$

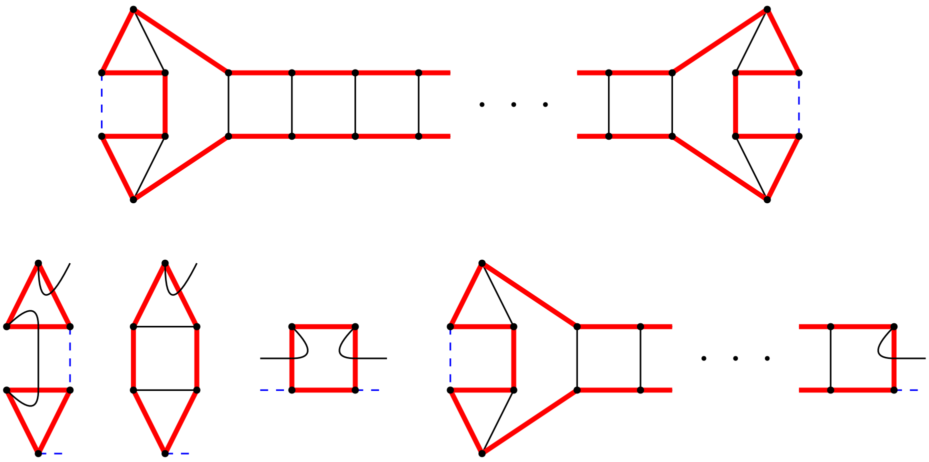

Figure 8: A 3-regular graph with $12 + 4n$ vertices for an arbitrary $n \geq 0$ and with a 2-planarizing 2-factor in bold red and building blocks to build this graph with 2-planarizing 2-factors with any number of cycles from 1 to $\gamma_{max}(G) + 1$. In this figure—just like in the following ones—the 2-planarizing 2-factor is drawn in red and the decomposition graphs are the graphs with once the red and black and once the red and dashed blue edges. The dashed blue edges are always drawn outside the red cycles and the black edges inside the red cycles, but both graphs with the given orientations around the vertices are plane graphs.

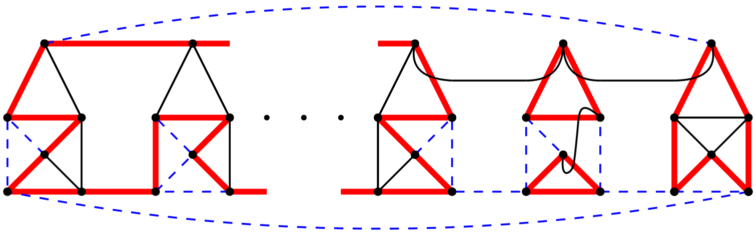

Figure 9: A 4-regular graph with $6n$ vertices for an arbitrary $n > 0$ and with a 2-planarizing 2-factor with a cycle spanning some building blocks and smaller cycles in the last two blocks. Combining a cycle through some blocks with blocks with one or two cycles 2-planarizing 2-factors with any number of cycles from 1 to $\gamma_{max}(G) + 1$ can be formed.

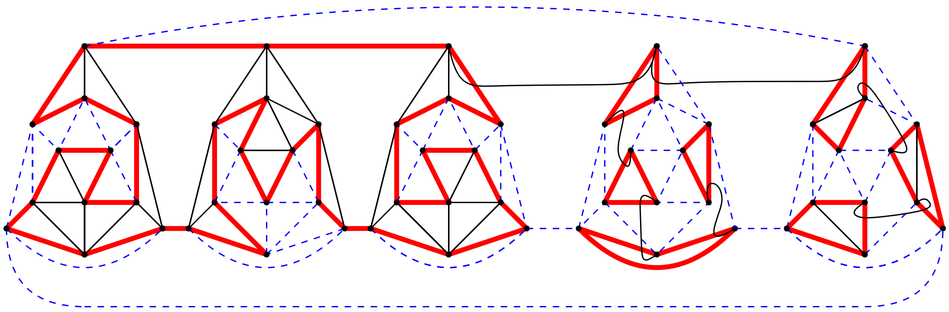

Figure 10: An example for $n = 5$ illustrating a construction of 5-regular graphs with $12n$ vertices for an arbitrary $n > 0$. In this case we have a 2-planarizing 2-factor with a cycle spanning three building blocks and smaller cycles in the last two blocks. Combining a cycle through some blocks with blocks with one to four cycles (see also Figure 11) 2-planarizing 2-factors with any number of cycles from 1 to $\gamma_{max}(G) + 1$ can be formed.

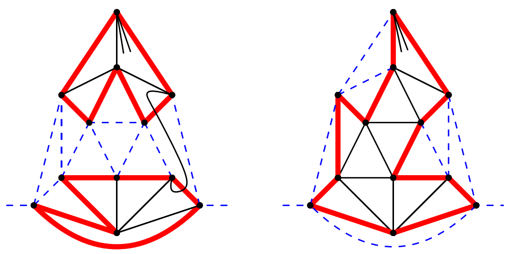

Figure 11: Building blocks of the 5-regular plane graph with one and two cycles. Blocks with three and four cycles are given in Figure 10. Connections to the other blocks are drawn as dangling edges. If the block is at the end of the row of blocks given in Figure 10, one of the dangling edges at the top must be a dashed blue edge on the outside.

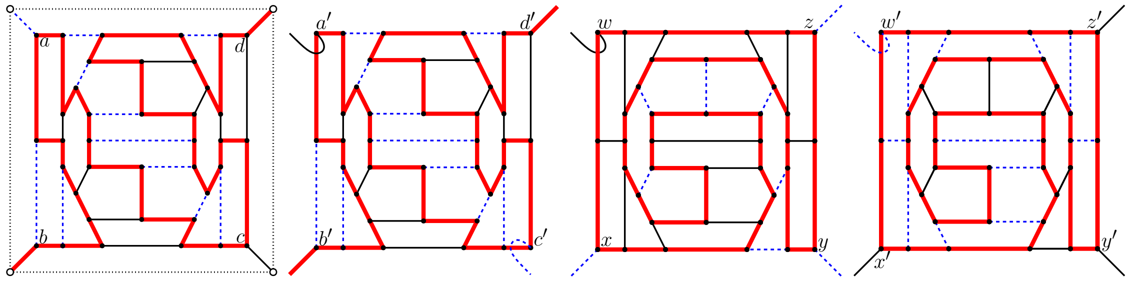

Figure 12: The plane 3-connected 3-regular cyclically 4-edge-connected non-hamiltonian graph J_1 on 42 vertices due to Grinberg [6]. Removing the vertices of the unique 4-gon, we obtain a hamiltonian graph J_1'.

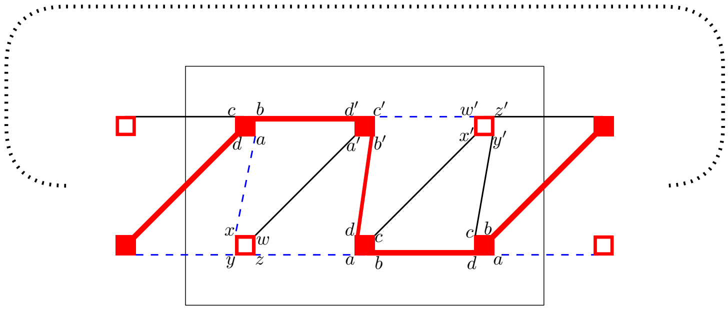

Figure 13: Taking m copies of the central region of this figure and arranging them in a circular fashion we get a plane cyclically 4-connected cubic graph with a smallest 2-factor of size $2m+1$. The squares stand for copies of the graph J_1'—solid squares for copies with a spanning path that is part of a cycle in the 2-factor and hollow squares for copies with a hamiltonian cycle that is part of the 2-factor.

The next result will show that there is no bound on the difference between the genus of a graph G and the smallest genus for which an embedding of G with a 2-planarizing 2-factor exists.

Theorem 4. *For every non-negative integer g there exists a planar 3-connected, cyclically 4-connected 3-regular graph G where a smallest 2-factor contains $k+1 \geq g+1$ cycles and such a 2-factor can be chosen in a way that G has an embedding on a surface of genus k with this 2-planarizing 2-factor.*

Proof. For $g = 0$ e.g. the cube can serve as an example, so assume $g > 0$. The graph J_1 in Figure 12 is plane and non-hamiltonian. The graph J_1' obtained from J_1 by removing the vertices of the unique 4-gon has no hamiltonian path between two vertices adjacent to neighboring vertices of the removed 4-gon (e.g. no path between a and b) and also no two spanning paths between vertices adjacent to neighboring vertices of the removed 4-gon—as in both cases the paths could be extended to a hamiltonian cycle of J_1. There can also be no disjoint paths between a and c as well as b and d, as these paths would have to cross in the plane graph J_1'. This implies that if J_1' is an induced subgraph of a larger graph G, then any 2-factor of G either contains a cycle completely in this copy of J_1' or a path spanning the copy and starting and ending at diagonally opposite vertices—e.g. a and c.

For even genus g we consider the graph G sketched in Figure 13. Calling the faces with three copies of J_1' as cornerpoints *triangles*, it is easy to see that due to the fact that a

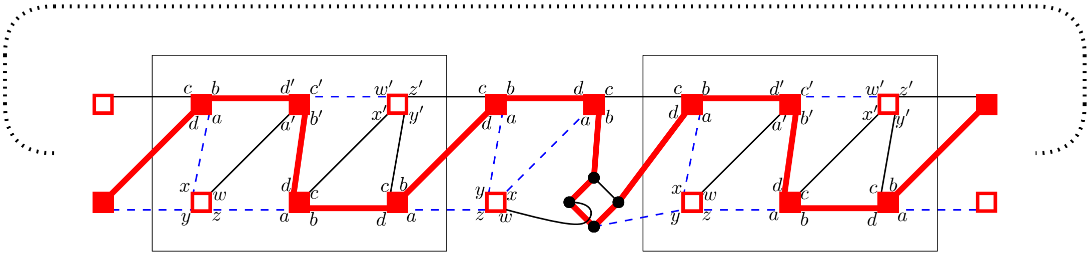

Figure 14: The construction for odd genus. The fat vertices are vertices of the graph and do not represent nontrivial subgraphs.

copy can only be traversed diagonally by a 2-factor, for any 2-factor each triangle contains at least one copy of J_1' that contains a cycle of the 2-factor in its interior. As for m copies of the central region there are $6m$ triangles and as each copy lies in only three triangles, a 2-factor contains at least $2m$ cycles that lie completely inside one copy and—unless all cycles lie inside such a copy in which case the 2-factor has $6m$ cycles—also at least one cycle not inside a copy. So any 2-factor of G contains at least $2m + 1$ cycles. Taking the hamiltonian cycle and paths of J_1' shown in Figure 12 and combining them as shown in Figure 13, one gets a 2-factor with exactly $2m + 1$ cycles.

Taking the red squares as copies of J_1' with the rotational order of the vertices as given by the vertex labels, the red and black edges, resp. the red and blue edges give the two decomposition graphs showing that this is a 2-planarizing 2-factor.

For odd genus, the construction is given in Figure 14. One copy of J_1' is replaced by a 4-gon—embedded and traversed by the cycle as described in the figure. □

3 A computer program

In order to determine some values for $n_{k,g}$ and to get an intuition on how common 2-planarizing 2-factors are, we developed simple programs to find embeddings of graphs that have a 2-planarizing 2-factor. On one hand (a) we wrote a program generating all possible embeddings of a given graph and combined it with a program searching for 2-planarizing 2-factors in the embedded graphs and on the other hand (b) we developed a program first searching for 2-factors and then trying to distribute the remaining edges to two copies of the 2-factors by at the same time keeping the cycles of the 2-factors facial cycles, the graphs plane, and finally having all cycles connected. Due to the enormous number of possible embeddings already for small vertex numbers—especially in case of large degrees—program (a) is much slower than program (b) and was only used as a completely independent check.

With more theoretical background—focused on results that can be exploited by algorithms—faster programs should be possible, but already these fairly simple programs applied to large lists of regular graphs produced by the programs described in [3] and [8] allow some insights. Results of program (b) are given in Table 1. In order to test the implementation, we compared the number of graphs with a 2-planarizing 2-factor computed by the two programs for all 3-regular graphs on up to 20 vertices, all 4-regular graphs on up to 11 vertices, and all graphs with degree sequences $(0, 2, 2, 2, 2)$, $(0, 2, 3, 2, 3)$, $(0, 2, 2, 2, 2, 1)$, and $(0, 0, 2, 3, 2, 2)$. We had complete agreement.

order	3-regular	3-regular girth 4	4-regular	5-regular	6-regular
4	100 %				
5			0 %		
6	50 %	0 %	100 %	100 %	
7			100 %		100 %
8	80 %	100 %	100 %	100 %	100 %
9			100 %		100 %
10	88.89 %	83.33 %	98.31 %	98.33 %	100 %
11			98.49 %		92.86 %
12	92.59 %	90.91 %	99.94 %	99.60 %	99.46 %
13			99.96 %		99.97 %
14	93.75 %	86.24 %	99.99 %	99.99 %	
15			99.98 %		
16	93.91 %	86.29 %	99.90 %		
17			99.96 %		
18	94.01 %	85.34 %			
20	95.51 %	91.12 %			
22	96.73 %	94.66 %			
24	96.98 %	93.82 %			
26	95.94 %	88.82 %			

Table 1: The percentage of embedded graphs with a 2-planarizing 2-factor among the bridgeless regular graphs with given vertex degree; rounded to two digits after the comma.

4 Declarations

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Conflicts of interest: The authors have no conflicts of interest / competing interests.

Availability of data and material: All generated data can be obtained from the first author by e-mail.

Code availability: The code for the programs used in this paper can be obtained from the first author by e-mail.

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