Planar Hypohamiltonian Oriented Graphs

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Abstract. In 1978 Thomassen asked whether planar hypohamiltonian oriented graphs exist. Infinite families of such graphs have since been described but for infinitely many n it remained an open question whether planar hypohamiltonian oriented graphs of order n exist. In this paper we develop new methods for constructing hypohamiltonian digraphs, which, combined with efficient graph generation algorithms, enable us to fully characterise the orders for which planar hypohamiltonian oriented graphs exist. Our novel methods also led us to discover the planar hypohamiltonian oriented graph of smallest order and size, as well as infinitely many hypohamiltonian orientations of maximal planar graphs. Furthermore, we answer a question related to a problem of Schiermeyer on vertex degrees in hypohamiltonian oriented graphs, and characterise all the orders for which planar hypotraceable oriented graphs exist.

Key words. Oriented graph, planar, hypohamiltonian, hypotraceable

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1 Introduction

An *oriented graph* is obtained from a graph by assigning a direction to each edge. We regard an oriented graph simply as a digraph without 2-cycles. We identify a graph G with the symmetric digraph $\overleftrightarrow{G}$ (the digraph obtained from G by replacing each edge with two oppositely directed arcs), so that definitions that we state for digraphs apply also to graphs. A digraph G is *hypohamiltonian* if every vertex-deleted subdigraph of G has a hamiltonian cycle but G does not.

In 1978, answering a question of Murty, Thomassen [15] presented an infinite family of hypohamiltonian oriented graphs. Every member of that family is nonplanar and 2-diregular (i.e., each vertex has in-degree 2 and out-degree 2). In the same paper Thomassen proved that there exists a planar hypohamiltonian *di*graph of order n for every $n \geq 6$, albeit containing many 2-cycles (half of the arcs are in 2-cycles) and he asked whether planar

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hypohamiltonian oriented graphs exist. This question remained open until 35 years later, when van Aardt, Burger and Frick [1] showed that for every integer $k \geq 0$ there exists a planar hypohamiltonian oriented graph of order $9 + 12k$. In [3] the same authors together with Kemnitz and Schiermeyer [3] extended the result to cover orders $9+6k$, $k \geq 0$. It was also shown in [1] that the smallest possible order of a planar hypohamiltonian oriented graph is 9, but the question of smallest size was left open.

In Section 2 we develop new methods for constructing hypohamiltonian oriented graphs, which we then use in Section 3 in combination with results obtained by computer to answer several questions concerning planar hypohamiltonian oriented graphs.

In Section 3.1 we prove that there exists a planar hypohamiltonian oriented graph of order n if and only if $n = 9$ or $n \geq 11$. For undirected graphs such a characterisation is still an open problem, despite significant recent efforts—it is known that a planar hypohamiltonian graph of order n exists if $n = 40$ or $n \geq 42$, see [12], and that no such graph exists on fewer than 23 vertices [9]. The situation for $23 \leq n \leq 39$ and $n = 41$ is unknown, and relates to a problem raised by Holton [11].

In Section 3.2 we show that there exists a planar hypohamiltonian oriented graph of order n whose underlying undirected graph is a maximal planar graph if and only if $n = 9$ or $n \geq 11$. This is in stark contrast to the fact that no maximal planar graph is hypohamiltonian: by Whitney's theorem that 4-connected triangulations of the plane are hamiltonian, any planar hypohamiltonian graph has a 3-vertex-cut. For triangulations, this cut forms a separating triangle. A short argument involving the fact that removing any vertex of this triangle yields a hamiltonian graph implies that the triangulation itself is hamiltonian, a contradiction.

In an undirected hypohamiltonian graph every vertex has degree at least 3, so we say that an undirected hypohamiltonian graph is *edge-minimal* if it is cubic (3-regular). It is well-known that infinite families of edge-minimal planar hypohamiltonian graphs exist. In a hypohamiltonian digraph, every vertex has in-degree at least 2 and out-degree at least 2, so we call a hypohamiltonian digraph *arc-minimal* if it is 2-diregular. The nonplanar hypohamiltonian oriented graphs constructed by Thomassen in [15] are all arc-minimal, but no planar arc-minimal hypohamiltonian oriented graph has yet appeared in the literature. In Section 3.2 we construct a 9-vertex arc-minimal planar hypohamiltonian oriented graph. We have not found any other arc-minimal planar hypohamiltonian oriented graph, and our computational results show that there are none of order n for $9 < n < 25$ (and none on fewer than 9 vertices). However, we show that for infinitely many n there exist planar hypohamiltonian oriented graphs of order n with only $2n + 1$ arcs (we call these *almost arc-minimal*).

During a talk given at the fourth Ilmenau-Košice DAAD Research Workshop held in Heyda, Germany in March 2018, Schiermeyer raised the question whether every hypohamiltonian oriented graph contains a vertex with in-degree as well as out-degree 2. We call such a vertex *quartic* since its total degree is 4. One can see such vertices as analogous to cubic vertices (vertices of degree 3) in undirected hypohamiltonian graphs. But while Thomassen's well-known question whether undirected hypohamiltonian graphs without cubic vertices exist [15] remains open, Schiermeyer's question admits a negative answer by an infinite family of so-called 2-hypohamiltonian oriented graphs (which have in-degree and out-degree at least 3) constructed in [4]. The members of that family also happen to be hypohamiltonian but they are nonplanar. Thomassen [15] proved that every undirected *planar* hypohamiltonian graph has a cubic vertex. It still remained an open question whether every planar hypohamiltonian oriented graph has a quartic vertex. We answer that question in Section 3.3.

A digraph G is *hypotraceable* if all of its vertex-deleted subgraphs are *traceable* (i.e., contain a hamiltonian path), but G does not. Grötschel, Thomassen and Wakabayashi [10] observed that if G is any hypohamiltonian digraph, then splitting an arbitrary vertex of G into a *source* (a vertex with no incoming arcs) and a *sink* (a vertex with no outgoing arcs) yields a hypotraceable digraph. However, the existence of planar hypohamiltonian oriented graphs does not immediately imply the existence of planar hypotraceable oriented graphs, since the vertex splitting operation does not necessarily retain planarity. Nevertheless, van Aardt, Burger and Frick [2] proved that there exists a planar hypotraceable oriented graph of order n for every even $n \geq 10$, with the possible exception of 14. Whether planar hypotraceable oriented graphs of odd order or of order 14 exists is stated as an open problem in [2]. In Section 4 we establish that there exists a planar hypotraceable oriented graph of order n if and only if $n \geq 10$.

For digraph notation and terminology we follow [5]. In particular, we denote the vertex set and arc set of a digraph G by $V(G)$ and $A(G)$, respectively, and their respective cardinalities are called the *order* and *size* of G. The *converse* G^{-1} of a digraph G is obtained from G by reversing all orientations, i.e., each arc is replaced by an oppositely directed arc. If G is hypohamiltonian, then so is G^{-1}, but note that G and G^{-1} may be isomorphic.

2 New construction methods

This section provides new techniques for constructing hypohamiltonian digraphs and, in particular, planar hypohamiltonian oriented graphs.

2.1 The θ-replacement method

Throughout this section, k will denote an arbitrary positive integer.

A θ-graph is a graph isomorphic to $K_4 - e$ (the complete graph of order 4 with one edge removed). We denote by θ the orientation of a θ-graph shown in Figure 1, i.e.,

$$\theta = (\{v_1, v_2, v_3, v_4\}, \{v_1v_2, v_2v_3, v_3v_1, v_4v_2, v_3v_4\}).$$

Let T_k denote the oriented graph with vertex set $\{a_i, b_i, c_i, d_i, e_i, f_i\}_{i=1}^{k}$ consisting of the path $a_1b_1c_1d_1e_1f_1 \ldots a_kb_kc_kd_ke_kf_k$, together with all the arcs on the paths $f_kd_kb_k \ldots f_1d_1b_1$ and $e_kc_ka_k \ldots e_1c_1a_1$.

Let $T_k(\theta)$ be the disjoint union of $\theta - v_2v_3$ and T_k, together with all the arcs on the paths $b_1v_2a_1v_1$ and $v_4f_kv_3e_k$, as shown in Figure 1 on the right.

Now let G be a digraph of order n containing the oriented graph θ as induced subdigraph (we call this a *θ-subdigraph*), and let $T_k(G)$ be the digraph of order $n + 6k$ obtained from G by replacing θ with $T_k(\theta)$.

If G is planar and $v_1v_2v_3v_1$ and $v_2v_3v_4v_2$ are facial triangles in the planar embedding of G, then $T_k(G)$ is also planar, and if G is an oriented graph, then so is $T_k(G)$.

Our first result in this section provides conditions under which $T_k(G)$ is hypohamiltonian.

Lemma 1. *Let G be a hypohamiltonian digraph of order n containing the oriented graph θ (labelled as in Figure 1) as induced subdigraph, such that* *(i) for every $v \in V(G)$ there exists a hamiltonian cycle in $G - v$ that contains at least one of the arcs on the path $v_4v_2v_3v_1$, and*

(ii) $G - v_1$ or $G - v_4$ contains a hamiltonian cycle using v_4v_2 or v_3v_1 respectively, but not using v_2v_3.

Then $T_k(G)$ is a hypohamiltonian digraph of order $n + 6k$.

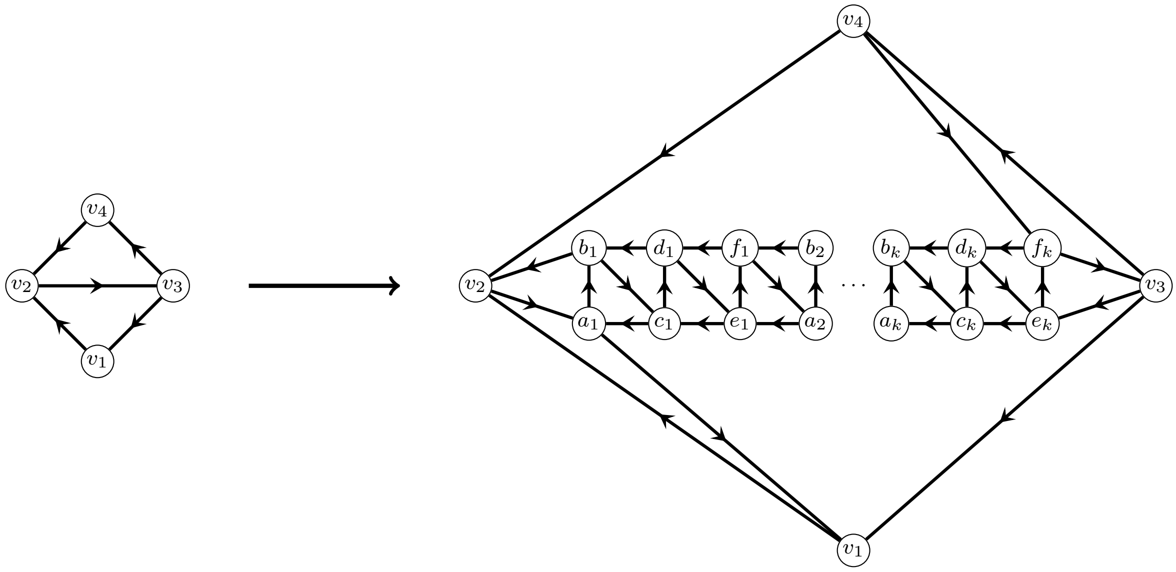

Figure 1: Replacing θ with $T_k(\theta)$.

Proof. For ease of notation we put

$$(af)_i = a_ib_ic_id_ie_if_i, \quad (fb)_i = f_id_ie_ic_ia_ib_i, \quad \text{and} \quad (ea)_i = e_if_id_ib_ic_ia_i.$$

We first prove that $T_k(G)$ is non-hamiltonian. Suppose, to the contrary, that there is a hamiltonian cycle $\mathfrak{h}$ in $T_k(G)$. Seeing T_k as a subdigraph of $T_k(G)$, we note that T_k is attached to $T_k(G) - T_k$ only by the in-going arcs v_2a_1, v_3e_k, v_4f_k and the out-going arcs f_kv_3, a_1v_1, b_1v_2. Therefore, $\mathfrak{h}$ contains either one of the four paths

$$Q_1 = v_2(af)_1 \dots (af)_kv_3, \quad Q_2 = v_3(ea)_k \dots (ea)_1v_1, \quad Q_3 = v_4(fb)_k \dots (fb)_1v_2,$$

$$Q_4 = v_4f_kv_3e_kc_kd_kb_kf_{k-1}a_k \dots e_1c_1d_1b_1v_2a_1v_1,$$

or the disjoint union of two paths

$$Q_5 = v_3e_kc_ka_k \dots e_1c_1a_1v_1 \cup v_4f_kd_kb_kf_1d_1b_1v_2.$$

In each case we obtain a corresponding hamiltonian cycle in G by performing one of the following replacements.

$$Q_1 \to v_2v_3, \quad Q_2 \to v_3v_1, \quad Q_3 \to v_4v_2, \quad Q_4 \to v_4v_2v_3v_1, \quad Q_5 \to v_3v_1 \cup v_4v_2.$$

But G is non-hamiltonian, so this proves that $T_k(G)$ is also non-hamiltonian.

For each vertex v in G, we let $\mathfrak{h}_v$ be a hamiltonian cycle in $G-v$ that satisfies (i). We now show that $T_k(G)-v$ is hamiltonian for every vertex v in $T_k(G)$. We first consider $v \notin V(T_k)$. Seeing v as a vertex in G, by (i) $\mathfrak{h}_v$ uses at least one of the arcs on the path $v_4v_2v_3v_1$. If $\mathfrak{h}_v$ uses v_2v_3, we replace v_2v_3 with Q_1, and if $\mathfrak{h}_v$ does not use v_2v_3, we either replace v_3v_1 with Q_2 or we replace v_4v_2 with Q_3. In either case we obtain a hamiltonian cycle of $T_k(G)-v$.

Next, we consider the following cases.

Case $v \in \{a_1, d_1\}$: By (i), $v_4v_2 \in A(\mathfrak{h}_{v_3})$. We replace this arc in $\mathfrak{h}_{v_3}$ by the v_4v_2-path $Q_4 - a_1 - v_1$ and obtain a hamiltonian cycle $\mathfrak{h}'_{a_1}$ of $T_k(G)-a_1$. By replacing the path $c_1d_1b_1$ in $\mathfrak{h}'_{a_1}$ by $c_1a_1b_1$, we also obtain a hamiltonian cycle of $T_k(G)-d_1$.

Case $v \in \{b_1, e_1\}$: By (ii), we have a hamiltonian cycle in $G-v_1$ that uses v_4v_2 but not v_2v_3, or a hamiltonian cycle in $G-v_4$ that uses v_3v_1 but not v_2v_3. But v_4v_2 can be replaced by the path

$$v_4(fb)_k \dots (fb)_2 f_1 d_1 e_1 c_1 a_1 v_1 v_2,$$

and v_3v_1 can be replaced by

$$v_3 v_4 (fb)_k \dots (fb)_2 f_1 d_1 e_1 c_1 a_1 v_1,$$

so in either case we obtain a hamiltonian cycle $\mathfrak{h}'_{b_1}$ of $T_k(G)-b_1$. By replacing the path $d_1e_1c_1$ in $\mathfrak{h}'_{b_1}$ by the path $d_1b_1c_1$, we also obtain a hamiltonian cycle of $T_k(G)-e_1$.

Case $v \in \{c_1, f_1\}$: By (i), $v_3v_1 \in A(\mathfrak{h}_{v_2})$. Replacing the arc v_3v_1 with the path

$$v_3(ea)_k \dots (ea)_2 e_1 f_1 d_1 b_1 v_2 a_1 v_1,$$

we obtain a hamiltonian cycle $\mathfrak{h}'_{c_1}$ of $G-c_1$, which can be modified to a hamiltonian cycle of $T_k(G)-f_1$ by replacing in $\mathfrak{h}'_{c_1}$ the path $e_1f_1d_1$ by $e_1c_1d_1$.

The generalisations to $a_i, b_i, c_i, d_i, e_i, f_i$ for $i>1$ are straightforward due to the periodicity of our construction. We exhibit the case a_i and leave the remainder to the reader. A hamiltonian cycle of $T_k(G)-a_i$ is obtained from the hamiltonian cycle $\mathfrak{h}'_{a_1}$ defined earlier, by replacing the v_4v_2-subpath of $\mathfrak{h}'_{a_1}$ by the path

$$v_4 f_k v_3 e_k c_k d_k b_k f_{k-1} a_k \dots e_{i+1} c_{i+1} d_{i+1} b_{i+1} f_i a_{i+1} e_i c_i d_i b_i (fb)_{i-1} \dots (fb)_1 v_2.$$

We conclude that every vertex-deleted subdigraph of $T_k(G)$ is hamiltonian. This completes the proof. □

The operation from Lemma 1 increases the degrees of the vertices v_1, v_2, v_3, v_4. This can be an obstacle if we wish to construct hypohamiltonian oriented graphs of small size. But the only reason why Lemma 1 requires that $\{v_4v_2, v_3v_1\} \subset A(T_k(G))$, is that a hamiltonian cycle in $G-v$ that contains the path $v_4v_2v_3$ or the path $v_2v_3v_1$ converts to a hamiltonian cycle in $T_k(G)$ that necessarily contains an arc in $\{v_4v_2, v_3v_1\}$. Thus we have the following variant of Lemma 1, which we shall use in Section 3.2 to construct planar hypohamiltonian oriented graphs of small size.

Lemma 2. *Let G be a hypohamiltonian digraph of order n satisfying the conditions of Lemma 1 plus the following condition (which is stronger than Lemma 1 (i)).*

(i)$^+$ For any $v \in V(G)$ there is a hamiltonian cycle in $G-v$ that contains at least one of the arcs on the path $v_4v_2v_3v_1$ but contains neither the path $v_4v_2v_3$ nor the path $v_2v_3v_1$.

Then $T_k(G) - v_4v_2 - v_3v_1$ is a hypohamiltonian oriented graph. Moreover, if G is arc-minimal, then so is $T_k(G) - v_4v_2 - v_3v_1$.

For the construction of planar hypohamiltonian oriented graphs of larger size we shall use the following lemma.

Lemma 3. *Let G be a planar hypohamiltonian digraph that satisfies the conditions of Lemma 1 plus the condition*

(iii) $v_1v_2v_3v_1$ and $v_2v_3v_4v_2$ are facial triangles in the planar embedding of G

and let

$$\mathcal{A} = \{v_4b_i, v_4d_i, v_4f_i, a_iv_1, c_iv_1, e_iv_1\}_{i=1}^{k} \setminus \{v_4f_k, a_1v_1\}.$$

Then $T_k(G) \cup A$ is a planar hypohamiltonian oriented graph for every subset A of $\mathcal{A}$. If G is an orientation of a maximal planar graph, then so is $T_k(G) \cup \mathcal{A}$.

Proof. Condition (iii) implies that $T_k(G)\cup\mathcal{A}$ is planar. By Lemma 1, $T_k(G)$ is hypohamiltonian, so it follows that for every $A \subset \mathcal{A}$, every vertex-deleted subdigraph of $T_k(G) \cup A$ is hamiltonian. Thus, to complete the proof, we only need to show that $T_k(G) \cup \mathcal{A}$ is non-hamiltonian.

Suppose, to the contrary, that $T_k(G)\cup\mathcal{A}$ contains a hamiltonian cycle $\mathfrak{h}$. By construction, at most two arcs in $\mathcal{A} \cup \{v_4v_2, v_3v_1\}$ lie in $\mathfrak{h}$. Since $T_k(G)$ is non-hamiltonian, $\mathfrak{h}$ contains at least one arc in $\mathcal{A}$. We treat the case where this arc is c_iv_1, $i \in \{2, \dots, k-1\}$. The proof for the other arcs in $\mathcal{A}$ is essentially the same. The predecessor of c_i on the cycle $\mathfrak{h}$ is either b_i or e_i, so we consider two cases.

Case $b_ic_i \in A(\mathfrak{h})$. Inspecting the out-neighbours of a_i we see that either (1) $a_ib_i \in A(\mathfrak{h})$ or (2) $a_ie_{i-1} \in A(\mathfrak{h})$. In case (1), since the out-neighbours of d_i are b_i and e_i and we have already visited b_i, we have

$$\mathfrak{h} \cap T_k(\theta) = v_2(af)_1 \dots (af)_{i-1}a_ib_ic_iv_1 \cup v_4d_ie_if_i(af)_{i+1} \dots (af)_kv_3,$$

which contradicts the planarity of $T_k(G)$. In case (2), as $f_{i-1}a_i \in A(\mathfrak{h})$ we necessarily have $v_4f_{i-1}a_ie_{i-1}c_{i-1} \subset A(\mathfrak{h})$, but then $\mathfrak{h}$ cannot contain all the vertices in $\{a_j, b_j, c_j, d_j, e_j, f_j\}_{j=1}^{i-1}$, a contradiction.

Case $e_ic_i \in A(\mathfrak{h})$. Then $f_{i-1}a_i \in A(\mathfrak{h})$. Since $a_ib_i \in A(\mathfrak{h})$ directly leads to a contradiction (as the only remaining available out-neighbour of b_i is f_{i-1}), it follows that $a_ie_{i-1} \in A(\mathfrak{h})$. Similarly, $b_if_{i-1} \in A(\mathfrak{h})$. Therefore $e_{i-1}c_{i-1} \in A(\mathfrak{h})$. We observe that if $c_{i-1}d_{i-1} \notin A(\mathfrak{h})$, then necessarily $v_4d_{i-1} \in A(\mathfrak{h})$, which leads to a contradiction, so $c_{i-1}d_{i-1} \in A(\mathfrak{h})$. But then we again have a contradiction, since not all vertices from $\{a_j, b_j, c_j, d_j, e_j, f_j\}_{j=1}^{i-1}$ can lie in $\mathfrak{h}$. □

2.2 Extending Thomassen's gluing method

Suppose G is a digraph with a vertex-cut $X = \{x_0, x_1, x_2\}$ such that $G-X$ consists of two components, G_1 and G_2. Let $F_i = G[V(G_i) \cup X]$, $i \in \{1, 2\}$. Then F_1 and F_2 are called 3-*fragments* of G with *attachments* x_0, x_1, x_2, or simply X-*fragments*. A 3-fragment of a hypohamiltonian digraph is called *trivial* if its underlying graph is isomorphic to $K_{1,3}$.

Thomassen [15] showed that 3-fragments of undirected hypohamiltonian graphs can be glued together, forming a hypohamiltonian graph. More precisely, he proved the following.

Theorem 4. [15] *If F_i is an $\{x_{i0}, x_{i1}, x_{i2}\}$-fragment of a hypohamiltonian graph G_i, $i \in \{1, 2\}$, with F_1 and F_2 not both trivial, then identifying x_{1j} with x_{2j} for $j \in \{0, 1, 2\}$, we obtain a hypohamiltonian graph.*

The proof of Theorem 4 follows from the following lemma.

Lemma 5. [15] *If F is a non-trivial $\{x_0, x_1, x_2\}$-fragment of a hypohamiltonian graph, then F satisfies the following two conditions.*

(*a*) *F does not have a hamiltonian $x_j x_k$-path for any $j, k \in \{0, 1, 2\}$ with $j \neq k$.*

(*b*) *For each $v \in V(F)$, there exist $j, k \in \{0, 1, 2\}$ with $j \neq k$ such that $F - v$ has a hamiltonian $x_j x_k$-path.*

Conversely, if F is a graph containing an independent set $\{x_0, x_1, x_2\}$ and F satisfies (a) and (b), then F is an $\{x_0, x_1, x_2\}$-fragment of some hypohamiltonian graph.

Throughout this section, (a) and (b) will refer to conditions (a) and (b) of Lemma 5.

If F_i is a graph satisfying (a) and (b) with respect to an independent subset $X_i = \{x_{i0}, x_{i1}, x_{i2}\}$ of $V(F_i)$, $i \in \{1, 2\}$, then applying Thomassen's gluing procedure to F_1 and F_2 results in a hypohamiltonian graph. Unfortunately, in general we cannot guarantee that a non-trivial X-fragment of a hypohamiltonian *di*graph necessarily satisfies (a) and (b). Thus Theorem 4 does not immediately extend to digraphs. However, the second part of Lemma 5 holds for digraphs in general, because if F is a digraph satisfying (a) and (b), then we can obtain a hypohamiltonian digraph by applying the gluing procedure to F and $\overleftrightarrow{K_{1,3}}$.

In order to devise workable gluing procedures for the construction of hypohamiltonian digraphs, we define three conditions that are stronger than (b) for a digraph F containing an independent set $X = \{x_0, x_1, x_2\}$.

(b1) F satisfies (b) and additionally, $F - x_\ell$ has a hamiltonian $x_j x_k$-path for all pairwise distinct $j, k, \ell \in \{0, 1, 2\}$.

(b2) For every $v \in V(F)$ the subdigraph $F - v$ has a hamiltonian $x_i x_{i+1}$-path for some $i \in \{0, 1, 2\}$. We assume here that x_0, x_1, x_2 is an ordered triple, and we take indices mod 3.

(b3) For each $v \in V(F)$, there exist $j, k \in \{0, 1, 2\}$ with $j \neq k$ such that $F - v$ as well as $F - x_\ell$ has a hamiltonian $x_j x_k$-path, where $\{\ell\} = \{0, 1, 2\} \setminus \{j, k\}$.

Let F be a digraph containing an independent set $X = \{x_0, x_1, x_2\}$. Then we say that

F is X-*good* if F satisfies (a) and (b1),

F is X-*nice* if F satisfies (a) and (b2), and

F is X-*fair* if F satisfies (a) and (b3).

We note that (b1) implies (b3), and (b2) implies (b3), so X-good digraphs as well as X-nice digraphs are X-fair. The converse implications do not hold. For example, the oriented graph Z_6, labelled as in Figure 3, is X-nice (and hence also X-fair) but not X-good, since it satisfies the following condition:

(z) For $j \in \{0, 1, 2\}$, each of the subdigraphs $F - x_j$ and $F - v_j$ has a hamiltonian $x_{j+1} x_{j+2}$-path, but no other hamiltonian path that starts and ends in $\{x_0, x_1, x_2\}$.

If an X-good digraph is not X-nice with respect to a given ordering of the vertices in X, it might still be X-nice with respect to a different ordering. We do not know whether every X-good digraph is X-nice with respect to some labelling of the vertices.

Our next lemma provides three gluing results for the construction of hypohamiltonian digraphs. Therein, an X-fragment is called *arc-minimal* if every vertex in X has in-degree 1 and out-degree 1, and all other vertices have in-degree 2 and out-degree 2.

Lemma 6. *For $i \in \{1,2\}$, let F_i be a digraph of order at least 5 containing an independent set $\{x_{i0}, x_{i1}, x_{i2}\}$.*

(1) *If F_i is $\{x_{i0}, x_{i1}, x_{i2}\}$-good, $i \in \{1,2\}$, then the digraph H_1, obtained by identifying x_{1j} with x_{2j} for $j \in \{0,1,2\}$, is hypohamiltonian.*

(2) *If F_i is $\{x_{i0}, x_{i1}, x_{i2}\}$-nice, $i \in \{1,2\}$ then the digraph H_2, obtained by identifying x_{10} with x_{20}, x_{11} with x_{22}, and x_{12} with x_{21}, is hypohamiltonian.*

(3) *Let F be an $\{x_0, x_1, x_2\}$-fair digraph and denote by x_0', x_1', x_2' the vertices in F^{-1} corresponding to x_0, x_1, x_2, respectively. Then the digraph H_3, obtained by identifying x_j with x_j' for $j \in \{0,1,2\}$, is hypohamiltonian.*

In either statement, if both fragments are oriented graphs, then so is the resulting graph; if both fragments are plane digraphs and their attachments cofacial, the identification can be performed such that the resulting digraph is planar; and if both fragments are arc-minimal, then so is the resulting digraph.

Proof.

(1) We denote by x_j the vertex that results from the identification of x_{1j} with x_{2j}, $j \in \{0,1,2\}$, and we see F_1 and F_2 as subdigraphs of H_1. If $\mathfrak{h}$ is a hamiltonian cycle of H_1, then we may assume without loss of generality that the intersection of $\mathfrak{h}$ with F_1 is a hamiltonian x_1x_2-path of $F_1 - x_0$, and the intersection with F_2 is a hamiltonian x_2x_1-path of F_2. But since F_2 satisfies (a), this is not possible, so H_1 is non-hamiltonian.

Now let $v \in V(H_1)$. Without loss of generality we may assume that $v \in V(F_1)$. Then, since F_1 satisfies (b1), there exist $j, k \in \{0,1,2\}$ with $j \neq k$ such that $F - v$ has a hamiltonian x_jx_k-path $\mathfrak{p}_1$. Now let $\{\ell\} = \{0,1,2\} \setminus \{j,k\}$. Then, since F_2 satisfies (b1), the subdigraph $F_2 - x_\ell$ has a hamiltonian x_kx_j-path $\mathfrak{p}_2$. The paths $\mathfrak{p}_1$ and $\mathfrak{p}_2$ together form a hamiltonian cycle of $H_1 - v$.

(2) As in (1), the fact that F_1 and F_2 both satisfy (a) implies that H_2 is non-hamiltonian. Now suppose $v \in V(F_1)$. Then, since F_1 satisfies (b2), we may assume without loss of generality that $F_1 - v$ has a hamiltonian $x_{11}x_{12}$-path $\mathfrak{p}_1$. In the construction of H_2, the vertex x_{12} was identified with x_{21}, and x_{11} was identified with x_{22}. Since F_2 satisfies (b2), there is a hamiltonian $x_{21}x_{22}$-path $\mathfrak{p}_2$ in $F_2 - x_{20}$. The paths $\mathfrak{p}_1$ and $\mathfrak{p}_2$ together form a hamiltonian cycle of $H_2 - v$. Hence H_2 is hypohamiltonian.

(3) Since F, and hence also F^{-1}, satisfy (a), it follows that H_3 is non-hamiltonian. Now suppose $v \in V(F)$. Then, as F satisfies (b3), we may assume without loss of generality that $F - v$ has a hamiltonian x_1x_2-path $\mathfrak{p}_1$ and $F - x_0$ has a hamiltonian x_1x_2-path. Since F^{-1} is the converse of F, it follows that $F^{-1} - x_0'$ has a hamiltonian $x_2'x_1'$-path $\mathfrak{p}_2$. The paths $\mathfrak{p}_1$ and $\mathfrak{p}_2$ together form a hamiltonian cycle of $H_3 - v$.

□

The vertex-deleted Petersen graph P' is the smallest undirected graph that satisfies (a) and (b). It forms the basis for several constructions of hypohamiltonian and hypotraceable graphs found in the literature. The oriented graph Z_6, shown in Figure 3 may be regarded as the directed analogue of P', since it is the smallest digraph that satisfies (a) and (b). The smallest hypotraceable oriented graph can be obtained from Z_6 by adding a source and a sink (as shown in [1]) and, as will be shown in Section 3.2, the smallest planar hypohamiltonian oriented graph (depicted in Figure 3 on the right) is obtained by gluing together two copies of Z_6. Our next construction also uses Z_6 to construct hypohamiltonian oriented graphs from given ones.

2.3 Inserting Z_6 into a suitable triangle

In this section we describe another transformation, which we call S, for planar hypohamiltonian oriented graphs. Unfortunately, applying S to a planar hypohamiltonian oriented graph G does not guarantee that we can apply S to $S(G)$, as well. This was also the case for the operations replacing a θ-subdigraph, but these operations could add $6k$ vertices for any $k \geq 1$, while S only adds three vertices. Still, it will prove to be useful in the next section, and our computational experiments seem to indicate that a sufficient proportion of planar hypohamiltonian oriented graphs meet the requirements for applying S in order to warrant its inclusion (e.g. 14% for order 13 and 52% for order 14).

If $\Delta = x_0x_1x_2x_0$ is a 3-cycle in a hypohamiltonian oriented graph G, then we say that Δ is a *suitable triangle* of G if for every $v \in V(G)$, the subdigraph $G - v$ has a hamiltonian cycle that contains at least one arc of Δ.

Lemma 7. *Suppose G is a hypohamiltonian oriented graph of order n that contains a suitable triangle $\Delta = x'_0x'_1x'_2x'_0$. Let $S(G,\Delta)$ be the digraph obtained from the disjoint union of G and Z_6 (labelled as in Figure 3), by identifying the vertices x_i and x'_i for $i \in \{0,1,2\}$. Then $S(G,\Delta)$ is a hypohamiltonian oriented graph of order $n+3$. If G is planar and Δ a facial triangle in the planar embedding of G, then $S(G,\Delta)$ is planar, as well.*

Proof. We denote by F the copy of Z_6 in $S(G,\Delta)$. We see G as a subdigraph of $S(G,\Delta)$, but we change the label of x'_i in G to x_i, $i \in \{0,1,2\}$ so that F retains the labelling of Z_6. We use the fact that Z_6 satisfies the conditions (a) and (z) defined earlier.

Suppose $S(G,\Delta)$ has a hamiltonian cycle $\mathfrak{h}$. Then, since F satisfies (a) and (z), the intersection of $\mathfrak{h}$ with F is an $x_{i+1}x_{i+2}$-path $\mathfrak{p}_i$ that spans $F - x_i$ for some $i \in \{0,1,2\}$. But then replacing $\mathfrak{p}_i$ with the arc $x_{i+1}x_{i+2}$ yields a hamiltonian cycle of G, a contradiction. Thus $S(G,\Delta)$ is not hamiltonian.

If $u \in V(G)$, then since Δ is a suitable triangle in G, we have that $G-u$ has a hamiltonian cycle that contains the arc x_ix_{i+1} for some $j \in \{0,1,2\}$. By (z), this arc can be replaced in F with an x_ix_{i+1}-path that spans $F-x_{i-1}$. This yields a hamiltonian cycle of $S(G,\Delta)-u$.

Now consider $G - v_i$, $i \in \{0,1,2\}$. As Δ is a suitable triangle in G, the graph $G - x_i$ has a hamiltonian cycle that contains the arc $x_{i+1}x_{i+2}$. By (z), this arc can be replaced with an $x_{i+1}x_{i+2}$-path that spans $F - v_i$. This yields a hamiltonian cycle of $S(G,\Delta) - v_i$. □

3 Planar Hypohamiltonian Oriented Graphs

3.1 Covering all orders

We determined by computer the exact counts of planar hypohamiltonian oriented graphs on at most 15 vertices, see Table 1. In particular, we confirmed the result from [1] that no oriented graph on fewer than 9 vertices is hypohamiltonian, and we established that

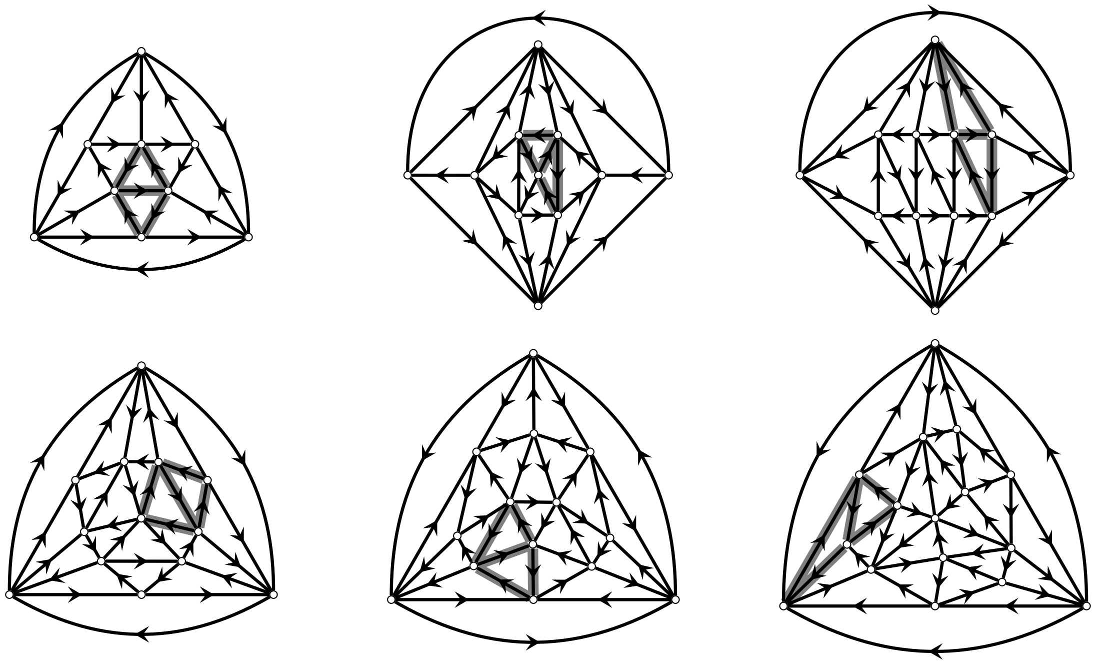

Figure 2: The planar hypohamiltonian oriented graphs of order 9, 11, 12, 13, 14, and 16 used in the proof of Theorem 8. The required θ-subdigraphs are shown in bold.

no planar hypohamiltonian oriented graph of order 10 exists (although a nonplanar hypohamiltonian oriented graph of order 10 is presented in [3]). The (undirected) underlying graphs were generated using `plantri` [6]. These graphs were then oriented using `directg` from Nauty [13] with some additional bounding. The oriented graphs were finally checked for being hypohamiltonian using a straightforward branch-and-bound algorithm that verifies the (non-)existence of certain cycles. For 15 vertices around 1.4×10^{13} oriented graphs needed to be checked for being hypohamiltonian. The programs and details on how to reconstruct these computations are available at [16].

We now characterise all orders for which planar hypohamiltonian oriented graphs exist.

Theorem 8. *There exists a planar hypohamiltonian oriented graph of order n if and only if $n = 9$ or $n \geq 11$.*

Proof. In Figure 2 we present for each $n \in \{9, 11, 12, 13, 14, 16\}$ a planar hypohamiltonian oriented graph G of order n that satisfies the conditions of Lemma 3. The required θ-subdigraphs are shown in bold. For the 9-vertex oriented graph we explicitly show in the Appendix that each of the vertex-deleted subdigraphs contains a hamiltonian cycle satisfying the conditions of Lemma 1. The remaining such verifications are left to the reader. By applying the operation from Lemma 3 (with A any subset of $\mathcal{A}$) to each of these six oriented graphs, we obtain a planar hypohamiltonian oriented graph of order n for every $n \geq 9$, except for $n = 10$. From our computational results mentioned above, we already know that there does not exist a hypohamiltonian oriented graph of order 10 or of order less than 9. ☐

Our computational experiments seem to indicate that the number of planar hypohamiltonian oriented graphs grows at least exponentially with respect to their order. However,

n	Oriented graphs	Underlying graphs	Time
9	25	9	0.03 seconds
10	0	0	0.76 seconds
11	4	3	34.58 seconds
12	10	4	28.4 minutes
13	367	71	1.1 days
14	6464	638	59.4 days
15	1422362	22767	9.3 years

Table 1: Overview of the number of planar hypohamiltonian oriented graphs for small orders and the total CPU time needed to generate them on a cluster of Intel Sandy Bridge (E5-2670) running at 2.6 GHz.

the best we have proved so far is the following.

Theorem 9. *For every* $n \geq 9$ *except* 10 *there exist at least* $\max\{1, 6 \cdot \lfloor \frac{n-11}{6} \rfloor - 1\}$ *pairwise non-isomorphic planar hypohamiltonian oriented graphs.*

Proof. We apply Lemma 3 exactly as in the proof of Theorem 8, but add the arcs of $\mathcal{A}$ one-by-one: If G is one of the graphs from Figure 2, then in $T_k(G)$ we can add any number of arcs from $\mathcal{A}$ and thus obtain graphs of order $|V(T_k(G))| = |V(G)| + 6k$ and size m for any $m \in \{|A(T_k(G))|, \ldots, |A(T_k(G))| + 6k - 2\}$, a set of cardinality $6k - 1$. From this the advertised counts follow. □

We have not found any result on the growth rate of undirected planar hypohamiltonian graphs in the literature, although Collier and Schmeichel [8] have shown that the growth rate of nonplanar hypohamiltonian graphs is at least exponential. Skupień [14] has shown that this even remains true for cubic hypohamiltonian graphs (in fact, even for hypohamiltonian snarks).

3.2 Maximising and minimising size

We call a planar hypohamiltonian oriented graph of order n *arc-minimal* if it has exactly $2n$ arcs (i.e., if it is 2-diregular) and *arc-maximal* if it has $3n - 6$ arcs (i.e., if it is an orientation of a maximal planar graph).

From Lemma 3, we deduce the following.

Theorem 10. *There exists an arc-maximal planar hypohamiltonian oriented graph of order* n *if and only if* $n = 9$ *or* $n \geq 11$.

Proof. The underlying graphs of all the oriented graphs in Figure 2 are triangulations of the plane, so the result follows by applying Lemma 3 to each of these six oriented graphs, with $A = \mathcal{A}$. □

Our search for *arc-minimal* planar hypohamiltonian oriented graphs has turned out to be less successful. We know only the following.

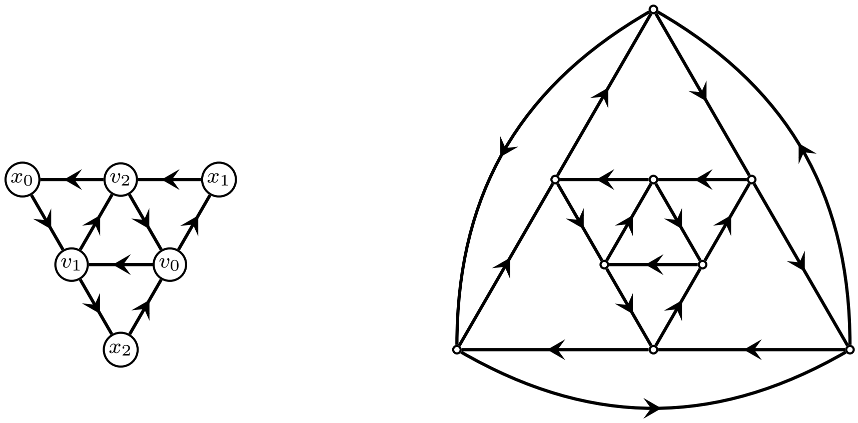

Figure 3: Left-hand side: The digraph Z_6. Right-hand side: The smallest planar hypohamiltonian oriented graph

Theorem 11. *There exists an arc-minimal planar hypohamiltonian oriented graph of order* 9*, and no other such graph of order less than* 25*. If there exists a planar arc-minimal hypohamiltonian oriented graph satisfying the properties of Lemma 2, then there exist infinitely many planar arc-minimal hypohamiltonian oriented graphs.*

Proof. It is easily seen that the oriented graph Z_6 (labelled as in Figure 3) is $\{x_0, x_1, x_2\}$-nice as well as $\{x_0, x_1, x_2\}$-fair and it is isomorphic to its converse. Thus, by applying either (2) or (3) of Lemma 6, we obtain that the oriented graph on the right in Figure 3 is hypohamiltonian. This proves the first part of the theorem.

The second part was obtained using a computer. The underlying graphs were generated using `plantri` [6]. These graphs were then oriented using `watercluster2` [7]. The oriented graphs were finally checked for being hypohamiltonian using a straightforward branch-and-bound algorithm that checks for the (non-)existence of certain cycles. To verify the case of 24 vertices in excess of 1.21×10^{12} oriented graphs needed to be checked for being hypohamiltonian and on a cluster of Intel Sandy Bridge (E5-2670) running at 2.6 GHz the computation took 3.9 CPU-years. The programs and details on how to reconstruct these computations are available at [16]. The final statement follows immediately from Lemma 2 since the planarity of $T_k(G)$ also implies the planarity of $T_k(G) - v_4v_2 - v_3v_1$. □

Unfortunately, at this point we know of only one arc-minimal planar hypohamiltonian oriented graph, and it does not satisfy the property (i)$^+$ of Lemma 2 with respect to any of its induced θ-subdigraphs. However, we now show that for infinitely many n there exist planar hypohamiltonian oriented graphs of order n and size $2n + 1$.

Theorem 12. *There exists an almost arc-minimal planar hypohamiltonian oriented graph of order* $3 + 6k$ *for every* $k \geq 1$*, but none of order* $10, 11, 12, 13, 14$ *or* 16*.*

Proof. We apply Lemma 2 to the graph shown in Figure 4 (see its caption for further details, in particular the location of the θ-subdigraph). This graph is constructed by adding an arc to the arc-minimal planar hypohamiltonian oriented graph shown in Figure 3. The second part was obtained using a computer. □

An overview of the number of planar hypohamiltonian oriented graphs with a specific number of arcs for small orders is given in Table 2. This table shows how, contrary to the

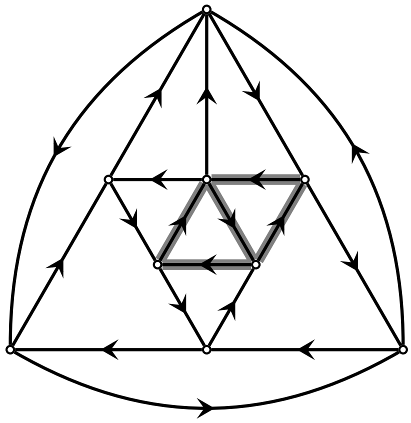

Figure 4: One of the two smallest almost arc-minimal planar hypohamiltonian oriented graphs. Together with the highlighted θ-subdigraph this forms the basis for an infinite family of almost arc-minimal planar hypohamiltonian oriented graphs.

non-oriented case, for planar hypohamiltonian oriented graphs the case of few arcs is the challenging part, not many arcs.

3.3 Planar hypohamiltonian oriented graphs without quartic vertices

As mentioned in Section 1, Thomassen [15] proved that every undirected planar hypohamiltonian graph has a cubic vertex, so it seems natural to ask whether every planar hypohamiltonian oriented graph has a quartic vertex. Our next result provides a negative answer to this question—its veracity is proved by the oriented graph shown in Figure 5.

Theorem 13. *There exists a planar hypohamiltonian oriented graph whose underlying graph has minimum degree* 5*.*

In fact, a computer search yielded eight such graphs on 15 vertices, but none on fewer vertices. It remains an open question whether an infinite family of such oriented graphs exists. Note that applying the operation from Lemma 3—which is the variant of the lemmas in Section 2.1 that gives the highest degrees—always adds a vertex of degree 4.

4 Planar Hypotraceable Oriented Graphs

In this section we fully characterise the orders for which planar hypotraceable oriented graphs exist.

As mentioned in Section 1, van Aardt, Burger and Frick [2] showed that there exist planar hypotraceable oriented graphs of order 10 and 12 and every even order greater than 14. They also proved that there exist strong planar hypotraceable oriented graphs of order $6k$ and $6k + 2$ for every $k \geq 3$. Their proofs rely on the following two lemmas. The first of these is an adaptation of a result of Grötschel, Thomassen, and Wakabayashi [10].

Lemma 14. [10] *For $i \in \{1, 2\}$, let T_i be a plane hypotraceable oriented graph of order n_i, with a source x_i and a sink z_i such that x_i and z_i are cofacial, and x_1 and z_1 are independent. In the disjoint union of T_1 and T_2, identify x_1 and z_2 to a single vertex and identify z_1 and x_2 to a single vertex. The result is a strong planar hypotraceable oriented graph of order $n_1 + n_2 - 2$.*

n	size 18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
9	1	2	10	12																					
10																									
11								1	2	1															
12												2	8												
13												1	14	75	162	115									
14															13	213	1265	2868	2105						
15														2	100	1913	18267	95993	286499	476908	406469	136211			
16																	?	?	?	?	?	?	?	?	≥ 179

Table 2: The number of planar hypohamiltonian oriented graphs of a specified order n (vertical) and size (horizontal). For each order the interval between the lower and upper bound on the size, i.e., $2n$ and $3n-6$, respectively, is emphasised. A blank entry signifies that there are no planar hypohamiltonian oriented graphs for the specified combination of order and size.

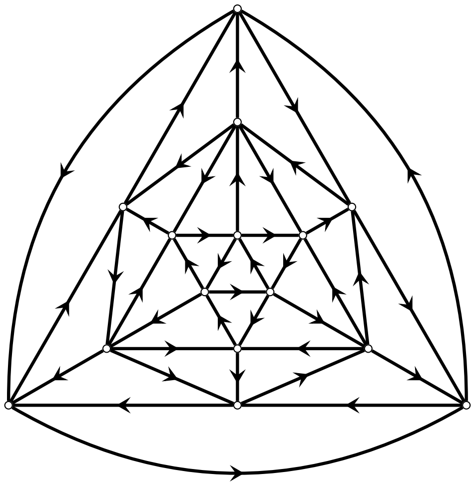

Figure 5: One of the eight smallest planar hypohamiltonian oriented graphs for which the underlying graph has minimum degree 5. It has 15 vertices.

Lemma 15. [2] *For every $k \geq 1$ there exists a plane hypotraceable oriented graph order $6k+4$ containing a source and a sink that are cofacial and independent. There also exists such an oriented graph of order* 12.

We now prove the following.

Theorem 16. *There exists a planar hypotraceable oriented graph of order n if and only if $n \geq 10$. Furthermore, up to* 13 *vertices all of these graphs have both a sink and a source. There also exists a strong planar hypotraceable oriented graph of order n for every $n \geq 18$.*

Proof. The non-existence in the first statement and the second statement were shown by computer (see Table 3). The underlying graphs were generated using `plantri` [6] with a custom plug-in to guarantee certain degree conditions. These graphs were then oriented using `directg` from Nauty [13] with some additional bounding. The oriented graphs were finally checked for being hypotraceable using a straightforward branch-and-bound algorithm that checks for the (non-)existence of certain paths. For 13 vertices around 1.12×10^{12} oriented graphs needed to be checked for being hypotraceable. The programs and details on how to reconstruct these computations are available at [16].

n	Oriented graphs	Underlying graphs	Time
10	12	9	2.2 minutes
11	103	51	1.7 hours
12	221	111	4.3 days
13	10412	2800	223.3 days

Table 3: Overview of the number of planar hypotraceable oriented graphs for small orders and the total CPU time needed to generate them on a cluster of Intel Sandy Bridge (E5-2670) running at 2.6 GHz.

Figure 6 presents plane hypotraceable oriented graph of orders 11, 13, 14, 15, and 17, each having a source and a sink that are cofacial. The graphs on 11 and 13 vertices were taken

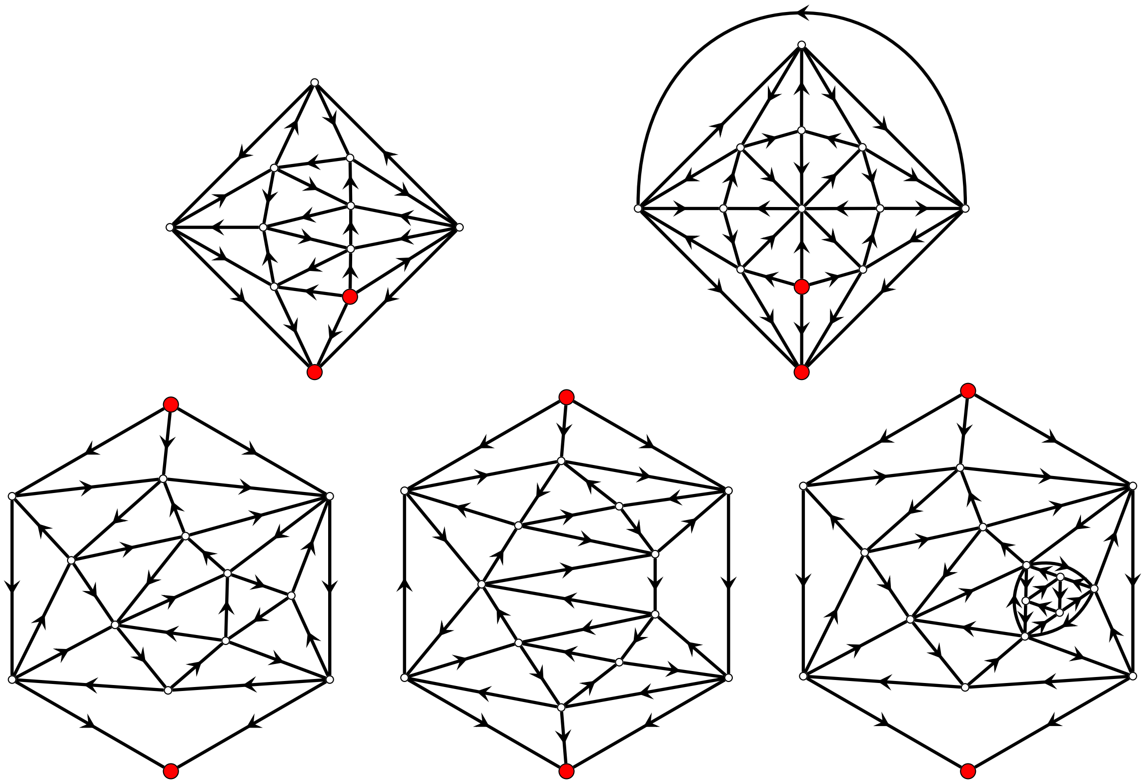

Figure 6: Planar hypotraceable oriented graphs of order 11, 13, 14, 15, and 17 as required in the proof of Theorem 16. In each the sink and source are emphasised.

from the exhaustive lists generated for the first part of this proof. The other three graphs were obtained from planar hypohamiltonian oriented graphs by splitting a suitable vertex (i.e., a vertex that can be split without destroying the planarity). We note that there is, up to its converse, only one planar hypohamiltonian oriented graph of order 13 containing a suitable vertex. By applying Lemma 7 to that oriented graph (i.e., by inserting a Z_6 into a suitable triangle), we obtain a planar hypohamiltonian oriented graph of order 16 that contains a suitable vertex, and splitting that vertex results in the hypotraceable oriented graph of order 17 shown in Figure 6.

Using Lemma 14 to combine the plane hypotraceable oriented graphs given in Figure 6 with those of order $6k + 4$ provided by Lemma 15, we obtain, for each $k \geq 3$, a strong planar hypotraceable oriented graphs of order $6k + 1, 6k + 3, 6k + 4$ and $6k + 5$. These, together with the strong planar hypotraceable oriented graphs of order $6k$ and $6k + 2$ for $k \geq 3$ found in [2], cover all orders from 18 upwards, so we have proved the result for the strong case. Now we add the planar hypotraceable oriented graphs shown in Figure 6 and those of order 10 and 12 presented in [2], and we have a planar hypohamiltonian oriented graph of order n for every $n \geq 10$. □

5 Discussion

1. Planar arc-minimal hypohamiltonian digraphs. We have discussed in this article planar hypohamiltonian oriented graphs of small size, showing that there exist infinitely many such graphs of order n and size $2n+1$. We recall that hypohamiltonian oriented graphs cannot have fewer than $2n$ arcs. Since we were only able to find one example of size $2n$, see Figure 3, we relax the problem and ask here whether an infinite family of planar arc-minimal hypohamiltonian *di*graphs with a sublinear number of 2-cycles exists. We note that Thomassen [15] showed that $\overrightarrow{C}_2 \times \overrightarrow{C}_k$, $k \geq 3$ odd, is a planar arc-minimal hypohamiltonian digraph containing exactly k 2-cycles.

2. Thomassen's problem. We recall Problem 10 from Thomassen's 1978 paper [15]: Does there exist a hypohamiltonian oriented graph whose underlying graph is also hypohamiltonian? Certainly, such a graph would be nonplanar, since every planar hypohamiltonian graph contains a cubic vertex [15], while the underlying graph of a hypohamiltonian oriented graph has minimum degree at least 4. We note that no hypohamiltonian graphs of minimum degree at least 4 are known ([15, Problem 4]).

3. Girth. For the undirected case, girth has been a widely studied property of hypohamiltonian graphs. Currently the literature contains infinite families of hypohamiltonian graphs of girth g for $3 \leq g \leq 7$, but no example of girth greater than 7 is known (see [9] and further references therein). Infinite families of planar hypohamiltonian graphs of girth 3, 4, and 5 exist, and no other girths are possible. Thus, planar hypohamiltonian digraphs of girth 3, 4 or 5 are immediately obtained. Our question is: Do planar hypohamiltonian oriented graphs of girth greater than 3 exist? Since the minimum degree of a hypohamiltonian oriented graph is at least 4, the Euler formula for plane graphs implies that the underlying graph of a planar hypohamiltonian oriented graph contains at least eight triangles. However those triangles need not be oriented cycles. We note that every oriented graph constructed in this paper has girth 3. Thomassen's constructions from [15] yield a toroidal hypohamiltonian oriented graph of girth g for any $g \geq 3$, e.g., $\overrightarrow{C_g} \times \overrightarrow{C}_{2g-1}$ is hypohamiltonian.

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6 Appendix

In Figure 7 we show all $(n-1)$-cycles for the 9-vertex graph from Figure 2 that are needed in order to apply Lemma 3.

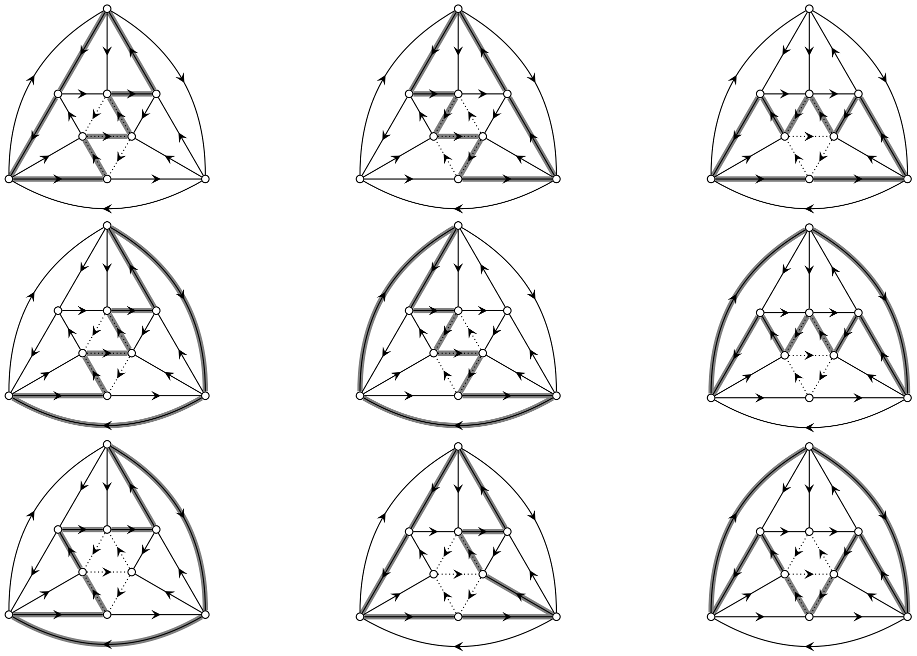

Figure 7: All $(n-1)$-cycles for the 9-vertex graph from Figure 2 that are needed in order to apply Lemma 3. Note that when missing the top or bottom vertex of the θ-subdigraph the cycle indeed uses one of the remaining edges, but not the central edge.