Vertex degrees and 2-cuts in graphs with many hamiltonian vertex-deleted subgraphs

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Abstract

A 2-connected non-hamiltonian graph G is a *k-graph* if for exactly $k < |V(G)|$ vertices in G, removing such a vertex yields a non-hamiltonian graph. We characterise k-graphs of connectivity 2 and describe structurally interesting examples of such graphs containing no cubic vertex or of minimum degree at least 4, with a special emphasis on the planar case. We prove that there exists a k_0 such that for every $k \geq k_0$ infinitely many planar k-graphs of connectivity 2 and minimum degree 4 exist. As a variation of a result of Thomassen, we show that certain planar 3-graphs must contain a cubic vertex.

Keywords: Hamiltonian, planar, vertex-deleted subgraph, hypohamiltonian.

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1. Introduction

We shall here investigate the hamiltonian properties of graphs and their vertex-deleted subgraphs, in particular in relation to planarity, connectivity, and vertex degrees. Our main focus will lie on a natural extension of hypohamiltonicity, a concept which has been applied in combinatorial optimisation—determining facets of the travelling salesman polytope [5, 6]—as well as coding theory [9]. Applications related to hypohamiltonicity also appear in the context of designing fault-tolerant networks, see for instance Chapter 12 of [7], [8], or, in this journal, [15]. Moreover, various graph generation algorithms have been designed for hypohamiltonian graphs and related families [1, 4].

Extending Tutte's famous theorem that every 4-connected planar graph is hamiltonian [13], Thomassen showed that a planar graph with minimum degree at least 4 in which every vertex-deleted subgraph is hamiltonian, must itself be hamiltonian [11]. We will refer to this result of Thomassen as (T1). The author recently extended (T1) and proved that a planar graph with minimum degree at least 4 in which at most five vertex-deleted subgraphs are non-hamiltonian, must itself be hamiltonian [23]. Now consider the following statement (T2): A planar graph without cubic vertices in which all vertex-deleted subgraphs are hamiltonian, must itself be hamiltonian. Since a graph in which every vertex-deleted subgraph is hamiltonian is 3-connected, (T1) and (T2) are equivalent. However, in (T2) "all" can be replaced by "all but one" [21], but *not* by "all but two", as illustrated by the join of K_2 and $3K_1$, see Fig. 1 (although we must admit that this is the only exception we are aware of). So if we allow vertices of degree 2, there is a qualitative difference between the extendability of (T1) and (T2), and it is this problem and more generally the connectivity 2 case we focus on here. Going back to (T1), we note that the general 3-connected case—i.e. the question whether a graph with minimum degree at least 4 in which every vertex-deleted subgraph is hamiltonian, must itself be hamiltonian—constitutes an old open problem of Thomassen [11].

Consider the following partition of the family of all 2-connected n-vertex graphs of circumference $n - 1$ introduced in [21]. Let G be such a graph and let $\text{exc}(G) \subset V(G)$ be the set of all vertices w in G such that the graph $G - w$ is non-hamiltonian. (We do not consider K_1 to be hamiltonian.) A vertex from $\text{exc}(G)$ is *exceptional*. Put $\text{nexc}(G) = V(G) \setminus \text{exc}(G)$. Throughout this paper, figures will show exceptional vertices in white and non-exceptional vertices in black unless explicitly stated otherwise. We call a 2-connected non-hamiltonian graph G with $|\text{exc}(G)| = k < |V(G)|$ a *k-graph*. A 0-graph is *hypohamiltonian*, and a 1-graph is *almost hypohamiltonian*. Planar 3-connected graphs will be called *polyhedral*. A graph is *plane* if it is planar and embedded in the Euclidean plane.

For $k \in \{0, 1\}$, k-graphs are 3-connected, but for $k \geq 2$, k-graphs of connectivity 2 exist. We investigate in this paper k-graphs of connectivity 2 with a special emphasis on the planar case and the cases for which $k \geq 2$ is small. Motivated by (T1), we study planar k-graphs of connectivity 2 and of minimum degree 4. Furthermore, in the light of (T2), we investigate planar k-graphs without cubic vertices. We will also discuss the general, i.e. not necessarily planar, versions of these problems. In preparation, we now introduce further notation and then describe the structure of k-graphs of connectivity 2. Thereafter, we show that for fixed k, there exist infinitely many (planar) k-graphs of connectivity 2 with various conditions imposed on their vertex degrees. The article ends with a structural result on the presence of a cubic vertex in certain planar 3-graphs, which complements (T2).

A vertex shall be called *k-valent* if it has degree k. When we speak of a cut, we always refer to a vertex-cut, i.e. a set of vertices whose removal increases the number of connected components. A *k-cut* shall be a cut containing exactly k vertices. Let G be a k-connected graph containing a k-cut X. The set of all components of $G - X$ will be denoted by $\mathcal{C}(G - X)$, and we put $c(G - X) := |\mathcal{C}(G - X)|$. For $C \in \mathcal{C}(G - X)$, we call $G[V(C) \cup X]$ a *k-fragment* of G with *attachments* X, which we sometimes shorten to *X-fragment*. An X-fragment is *trivial* if it contains exactly $k + 1$ vertices. A cut X of G is *trivial* if $K_1 \in \mathcal{C}(G - X)$. We call a component of $G - X$ or a fragment of G with attachments X *non-exceptional* if it contains a non-exceptional vertex of G, and *exceptional* otherwise. For $k = 2$, denote with $\mathcal{X}_2(G)$ the set of all 2-cuts of G whose removal splits G into exactly two components. A path with endpoints v and w is a *vw-path*. Let $\mathcal{F}_X$ be the set of all 2-fragments with attachments X. $F \in \mathcal{F}_{\{x,y\}}$ is *good* if it contains a hamiltonian xy-path, and F is *locally xy-hypohamiltonian* if F is not good, but for every vertex v in $F - x - y$, the graph $F - v$ is good. The *join* of two disjoint graphs G and G' is the graph denoted by $G + G'$, with vertex set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G') \cup \{vv' : v \in V(G), v' \in V(G')\}$.

Finally, we will also need the following definitions. Let G be a graph. We call a pair of vertices (a, b) in G *good* if there exists a hamiltonian ab-path in G. A pair of pairs of vertices $((a, b), (c, d))$ in G is *good* if there exists an ab-path P and a cd-path Q such that P and Q are disjoint, and $P \cup Q$ spans G. Following Hsu and Lin [7], for a graph H and vertices a, b, c, d in H, the quintuple (H, a, b, c, d) is a *J-cell* if (i) the pairs (a, d) and (b, c) are good in H; (ii) none of the pairs (a, b), (a, c), (b, d), (c, d), $((a, b), (c, d))$, and $((a, c), (b, d))$ are good in H; and (iii) for every vertex v in H there exists a good pair in $H - v$ among (a, b), (a, c), (b, d), (c, d), $((a, b), (c, d))$, and $((a, c), (b, d))$.

2. Structure and examples

Lemma 1. *Let G be a k-graph containing a 2-cut $X = \{x, y\}$. Then the following hold.*

(i) *The vertices x and y are exceptional. In particular, every neighbour of a vertex of degree 2 is exceptional, and $k \geq 2$.*

(ii) *If $X \in \mathcal{X}_2(G)$ is non-trivial, then $F \in \mathcal{F}_X$ is good if and only if F is exceptional. Thus, exactly one component of $G - X$ is non-exceptional.*

(iii) *Let $X \in \mathcal{X}_2(G)$ and $\mathcal{F}_X = \{F, F'\}$. If F is non-trivial and non-exceptional, then $G' := G \cup \mathcal{E}$ is a k-graph for every $\mathcal{E} \subset \{vw : v, w \in V(F')\}$.*

(iv) *A 2-connected graph H containing a 2-cut $\{x, y\}$ is a k-graph if and only if $H' := (V(H), E(H) \cup \{xy\})$ is a k-graph.*

Proof. (i) Obvious.

(ii) As $X \in \mathcal{X}_2(G)$, we have $\mathcal{F}_X = \{F, F'\}$. Let F contain a hamiltonian xy-path and assume there exists a non-exceptional vertex v in F. Since v is non-exceptional and X is non-trivial, there exists a hamiltonian xy-path in F'. But if F and F' have hamiltonian xy-paths, then G is hamiltonian, a contradiction. Now let every vertex in F be exceptional. As G has circumference $|V(G)| - 1$, there exists a vertex v in F' such that there is a hamiltonian cycle $\mathfrak{h}$ in $G - v$. Therefore, as X is non-trivial, $F \cap \mathfrak{h}$ is a hamiltonian xy-path in F. Thus, the advertised equivalence is proven. For the final statement, observe that at least one component must be non-exceptional, but that not both fragments can be good.

(iii) By (ii) there exists no hamiltonian xy-path in F, and F' is exceptional, so F' is good. This means that F' contains a hamiltonian xy-path, so no exceptional vertex of F was rendered non-exceptional by the addition of $\mathcal{E}$ to G. F remains unchanged (excluding the possible but inconsequential addition of the edge xy), so every vertex in F' remains exceptional after adding $\mathcal{E}$ to G. It is obvious that by adding edges no non-exceptional vertex can become exceptional. We can see F as a 2-fragment with attachments X in G', and since F is not good (because F' is good), G' is non-hamiltonian. Together with the preceding argument, G' is indeed a k-graph.

(iv) The arguments are similar to (iii), so we will be concise. Since $\{x, y\} = X$ is a cut in H it is clear that H is non-hamiltonian if and only if H' is non-hamiltonian, that no non-exceptional vertex has become exceptional through the addition of the edge xy, and that no exceptional vertex becomes non-exceptional by removing the edge xy. So assume that H' contains a vertex v such that $H' - v$ is hamiltonian, but $H - v$ is not. Clearly, $v \notin X$. Since every hamiltonian cycle of $H' - v$ uses the edge xy, the cut X must be trivial and a component of $H' - X$ is $\{v\}$. But then any hamiltonian xy-path of $H' - v$ together with the path xvy yields a hamiltonian cycle of H, a contradiction. The reasoning that an exceptional vertex of H remains exceptional in H' is very similar. □

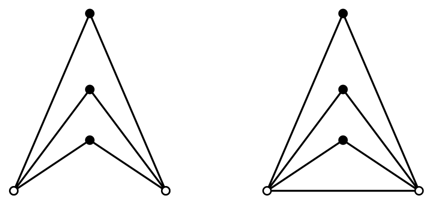

Fig. 1: $K_{2,3}$ and $K_2 + 3K_1$, respectively.
The latter is the only known 2-graph without a cubic vertex.

Theorem 1. *A graph G of connectivity 2 and order n is a k-graph if and only if either*

(1) *G contains exactly one 2-cut $X = \{x, y\}$ whose removal splits G into exactly three components, all other 2-cuts split G into exactly two components, and either*

(i) *$G \in \{K_{2,3}, K_2 + 3K_1\}$, in which case $k = 2$,*

(ii) *$\mathcal{F}_X$ consists of two trivial 2-fragments and one non-trivial good 2-fragment with attachments X, in which case $k = n - 2$ and the two non-exceptional vertices are the non-attachments of the two trivial 2-fragments and are non-adjacent, or*

(iii) *$\mathcal{F}_X$ consists of a trivial and two non-trivial good 2-fragments with attachments X, in which case $k = n - 1$ and the non-exceptional vertex is the non-attachment of the trivial 2-fragment; or*

(2) *every 2-cut splits G into exactly two components, and either*

(i) *there exists a 2-cut X in G such that $\mathcal{F}_X$ consists of a trivial 2-fragment containing a non-exceptional vertex and a 2-fragment that is not good, in which case $2 \leq k \leq n-1$, or*

(ii) *for every 2-cut in G whose removal yields a trivial component, this component consists of an exceptional vertex and*

$$\text{nexc}(G) \subset \bigcap_{X \in \mathcal{X}_2(G)} \{C \in \mathcal{C}(G - X) : C \cap \text{nexc}(G) \neq \emptyset\} =: I,$$

in which case $k \geq n - |I| \geq 3$.

Proof. We first show that for a graph G of connectivity 2 to be a k-graph, it must either (1) contain a 2-cut X such that $c(G-X) = 3$, in which case $K_1 \in \mathcal{C}(G - X)$ and all other 2-cuts split G into exactly two components, or (2) we have $c(G - Y) = 2$ for all 2-cuts Y in G.

Let now G be a k-graph of order n and connectivity 2 and $X = \{x, y\}$ a 2-cut in G. Whether or not x and y are adjacent makes no difference in the forthcoming arguments (see Lemma 1 (iv)), so this will be ignored in the remainder of this proof. Assume either $c(G - X) \geq 4$ or $c(G - X) = 3$ and every component of $G - X$ has at least two vertices. As G is a k-graph, by definition G contains at least one vertex v such that $G - v =: G'$ is hamiltonian; note that $v \notin X$. We then have that $c(G' - X) \geq 3$, but this is impossible, as in every hamiltonian graph, for any cut Y therein, the number of components after the removal of Y is at most $|Y|$. So if $c(G - X) = 3$ then certainly there exists at least one trivial component of $G-X$ consisting of a non-exceptional vertex. We distinguish between three cases.

Case 1. If all three components in $\mathcal{C}(G - X)$ are trivial, then $G \in \{K_{2,3}, K_2 + 3K_1\}$ (see Fig. 1) and $k \geq 2$ by Lemma 1 (i). By symmetry $k = 2$.

For Cases 2 and 3 we observe that, arguing as above, every non-trivial 2-fragment with attachments X contains only exceptional vertices.

Case 2. If there are exactly two trivial components in $\mathcal{C}(G-X)$, then each of these consists of a non-exceptional vertex. These vertices are non-adjacent, since they belong to different components of $G-X$. As mentioned above, the non-trivial 2-fragment F with attachments X must be exceptional, so $k = n - 2$. Note that on F no traversability condition is imposed except that it must contain a hamiltonian xy-path.

Case 3. If exactly one component of $G - X$ is trivial, then this component consists of the only non-exceptional vertex of G and $k = n - 1$. As in Case 2, the only condition imposed on the two non-trivial 2-fragments with attachments X is that each of them must contain a hamiltonian xy-path.

It remains to show that, if $c(G - X) = 3$, then X is the only 2-cut with this property. Suppose there exists a 2-cut $X' = \{x', y'\} \neq X$ with $c(G - X') = 3$. Let u be a non-exceptional vertex of G. In $G - u$, which is hamiltonian, no 2-cut whose removal splits the graph into more than two components may be present, so u must constitute a non-exceptional trivial component of both $G - X$ and $G - X'$. Necessarily, $ux, uy, ux', uy' \in E(G)$, of which at least three are pairwise distinct as $X \neq X'$; but this contradicts the fact that the degree of u is 2.

Finally, we deal with the case when for all 2-cuts Y in G we have $c(G - Y) = 2$. For G to be a k-graph, its order must be at least 5 (recall that k-graphs are 2-connected), so the removal of a 2-cut yields at most one trivial component. If there exists a 2-cut X in G whose removal yields a (unique) trivial component consisting of a non-exceptional vertex and $F \in \mathcal{F}_X$ is the non-trivial 2-fragment with attachments X, then the only traversability requirements imposed on F are that there is no hamiltonian xy-path in F, and that F itself is hamiltonian. Consequently, k may take any value between 2 and $n-1$. If for every trivial 2-cut X in G the trivial component of $G-X$ consists of an exceptional vertex, then by applying Lemma 1 (ii), we are done. □

Theorem 2. *Let $k \in \{2, 3\}$ and consider a k-graph G of connectivity 2. Then G contains a trivial 2-cut $X = \{x, y\}$. For $k = 2$, G contains no 2-cut other than X, and $\mathcal{F}_X$ consists of a trivial X-fragment and a hamiltonian locally xy-hypohamiltonian fragment.*

Furthermore, among graphs of connectivity 2, there exist(s)

(i) *for every $k \geq 2$ infinitely many k-graphs with neither cubic nor quartic vertices;*

(ii) *for $k = 2$ and every $k \geq 4$ a planar k-graph with no cubic vertices;*

(iii) *for every $k \geq 4$ infinitely many planar k-graphs of minimum degree 3 but no such graph for $k \leq 3$;*

(iv) *for every k and δ with $k > \delta \geq 4$ infinitely many k-graphs of minimum degree at least $\min\{\delta, k - \delta + 4\}$ but no such graph for $k \leq \delta$; and*

(v) *for every $k \geq 16$ a planar k-graph of minimum degree 4 but no such graph for $k \leq 5$.*

Proof. Let $X = \{x, y\}$ be a 2-cut of G. If X is non-trivial, by Theorem 1 we have $X \in \mathcal{X}_2(G)$, but then Lemmas 1 (i) and (ii) would imply $k \geq 4$, a contradiction. So $\mathcal{F}_X$ consists of one trivial X-fragment and another X-fragment F. If $k = 2$, then by Lemma 1 (i) the only exceptional vertices of G are x and y, and all other vertices are non-exceptional, in particular the non-attachment of the trivial fragment with attachments X, so F must be hamiltonian and locally xy-hypohamiltonian. This also implies that 2-cuts other than X cannot occur.

(i) For $k = 2$, consider a hypotraceable graph T' and let $T := T' + (\{x, y\}, \emptyset)$, where a graph G is *hypotraceable* if G itself is non-traceable (i.e. contains no spanning path) but every vertex-deleted subgraph of G is traceable. T is locally xy-hypohamiltonian: suppose there exists a hamiltonian xy-path $\mathfrak{p}$ in T. Then $\mathfrak{p} - x - y$ is a hamiltonian path in T', a contradiction. For any vertex v in T' there exists a hamiltonian path $\mathfrak{p}'$ in $T'-v$, say with endpoints u and w. Then $\mathfrak{p}'$ together with the edges ux and wy is a hamiltonian xy-path in $T - v$. Adding to this path the edges vx and vy proves that T is hamiltonian. Consider a path P on three vertices disjoint from T and identify x and y with the endpoints of P. The resulting graph is a 2-graph of connectivity 2 containing neither cubic nor quartic vertices as hypotraceable graphs have minimum degree at least 3 and order at least 5. There are infinitely many hypotraceable graphs [10], so the statement is shown. For $k = 3$ the same approach is used, but instead of hypotraceable graphs we use *almost hypotraceable* graphs—these are non-traceable graphs containing exactly one vertex w such that the removal of w yields a non-traceable graph, but removing any vertex other than w gives a traceable graph—, more specifically the infinitely many cubic ones described in [20]. We have shown the statement for $k \in \{2, 3\}$ and now present a more general approach based on the same idea, and similar to arguments used in [3].

We shall make use of so-called J-cells, as defined in the introduction, which occur naturally as hypohamiltonian graphs (of which there are infinitely many) minus a K_2. Let $J_i = (H_i, a_i, b_i, c_i, d_i)$ be pairwise disjoint J-cells for $i \in \{1, \ldots, k\}$ and put $G_k :=$

$$\left(\bigcup_{i=1}^{k} V(J_i), \ \bigcup_{i=1}^{k} E(J_i) \cup \bigcup_{i=1}^{k-1} b_i a_{i+1} \cup \bigcup_{i=1}^{k-1} c_i d_{i+1} \cup b_k a_1 \cup c_k d_1 \right).$$

As pointed out in [17], the graph G_k is 3-connected for all $k \geq 4$. We denote by $\mu(G)$ the minimum number of pairwise disjoint paths (which may consist of one vertex) that cover $V(G)$. A (possibly disconnected) graph G is *k-path-critical* if for any $v \in V(G)$ we have $\mu(G - v) + 1 = \mu(G) = k$. Note that 2-path-critical means hypotraceable. Wiener [16] showed that for every $k \geq 0$, the graph G_{4k+5} is $(k+2)$-path-critical. For $k' \geq 2$ denote the join of $k'K_1$ (i.e. k' disjoint copies of K_1) and a k'-path-critical graph T' by G, where x and y shall be vertices among the K_1's. Add to G a new vertex z and the edges xz and yz. Arguing as above, the resulting graph G' then contains neither cubic nor quartic vertices (in fact, excluding z, the graph G' has minimum degree at least $k' + 3$), every vertex added as a K_1 is exceptional, and every other vertex (including z) is non-exceptional. Assume G' contains a hamiltonian cycle $\mathfrak{h}$. As $\mathfrak{h}$ visits all k' copies of K_1 and must use the path xzy, $\mathfrak{h} \cap T'$ consists of $k' - 1$ pairwise disjoint paths which together span T'. But $\mu(T') = k'$, a contradiction. Thus, G' is a k'-graph.

(ii) For $k = 2$, consider $K_2 + 3K_1$. For $k \geq 4$, consider the wheel graph with k spokes, i.e. $C_k + K_1$. For every pair of adjacent vertices v, w, neither of which is the wheel's central vertex, add a path of length 2 with endpoints v and w. We obtain a planar k-graph of connectivity 2 with no cubic vertices. For $k \geq 5$, these graphs also do not contain quartic vertices. For $k = 2$, the graph is shown in Fig. 1, while for $k \in \{4, 5\}$ the graphs are depicted in Fig. 2.

(iii) Consider a plane hypohamiltonian graph (of which we know that there are infinitely many [10]) and a cubic vertex therein—such a vertex always exists by (T1). Removing this cubic vertex we obtain a plane non-trivial 3-fragment with attachments $\{u, v, w\}$, all of which lie on the boundary of the same face. We add the vertices x, y and edges ux, xv, vy, yw in order to obtain a plane locally xy-hypohamiltonian graph F with the property that x, y lie on the boundary of the same face. Let H be a wheel graph on $k \geq 4$ vertices, and x', y' adjacent vertices in H different from its central vertex. Identify x with x' and y with y'. We obtain a planar graph G of minimum degree 3. Due to the properties of locally xy-hypohamiltonian graphs, every vertex of $F - x - y$ is non-exceptional in G, the graph G is non-hamiltonian, and each of the k vertices of H is exceptional in G. We have shown that for every $k \geq 4$ there exists a planar k-graph of minimum degree 3. This result complements the statement that for $k \in \{2, 3\}$ a k-graph of connectivity 2 must contain a 2-valent vertex.

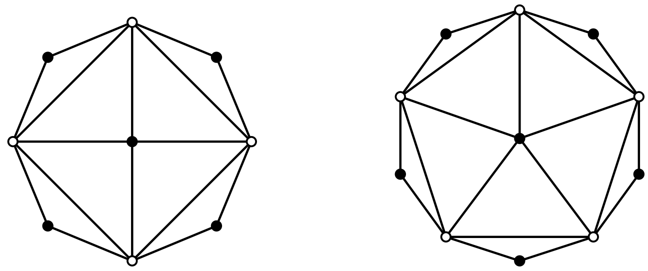

Fig. 2: A planar 4-graph without cubic vertices (left-hand side) and a planar 5-graph with neither cubic nor quartic vertices (right-hand side), each of connectivity 2.

For the second part of the statement, let G be a planar k-graph of connectivity 2 and minimum degree 3. Thus, no 2-cut of G is trivial. By Theorem 1, for every 2-cut X of G the removal of X from G yields exactly two (non-trivial) components. Hence, by Lemma 1 (ii), for every 2-cut X of G there is a non-trivial component C of $G - X$ containing exclusively exceptional vertices. As G has minimum degree 3, we have $|V(C)| \geq 2$, which combined with Lemma 1 (i) gives the bound.

(iv) Consider the graph G as constructed in the proof of (i), i.e. the join of $k'K_1$ and a k'-path-critical graph. Take a copy K of $K_{\delta+1}$ and distinct vertices v, w in K. Identify v with x and w with y, where x and y are vertices of G as defined in the proof of (i). We obtain a graph G'. This graph is a $(k'+\delta-1)$-graph, as all k' vertices constituting the k' copies of K_1 (occurring when constructing G in the proof of (i)) are exceptional, every vertex of K is exceptional, all other vertices in G' are non-exceptional, and G' is non-hamiltonian (this can be seen using the same arguments as provided at the end of the proof of (i)). By construction, the minimum degree of G' is at least $\min\{\delta, k' + 3\}$. Arguing as at the end of the proof of (iii) yields the non-existence statement given in (iv).

(v) See Fig. 3. An infinite family is obtained by replacing the 6-vertex 2-fragment with attachments x, y by appropriate (and easily determined) 2-fragments of larger order containing a hamiltonian xy-path. That indeed $k \geq 6$ can be shown as above, noting that here we have $|V(C)| \geq 4$—as before, C is a component of $G - X$ and X is an arbitrary 2-cut of G—as the minimum degree of G is 4 and G is planar. □

The constructions from Theorem 2 (ii) and (v) have a dramatic defect: for given k they only provide one example of a planar k-graph with certain desired properties. From these per-

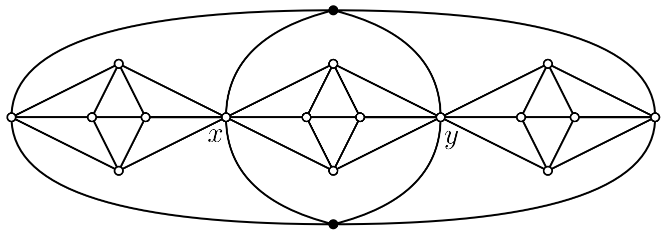

Fig. 3: A planar 16-graph of minimum degree 4 and connectivity 2.

haps several other such examples can be deduced with similar arguments, but infinite families of planar k-graphs for fixed k and of connectivity 2 seem out of reach with this strategy. We now change our approach and address this issue, but pay a price as our bounds are worse than the ones presented in Theorem 2.

3. Infinite families of planar k-graphs for fixed k

Thomassen [12] used the following operation to prove that there exist infinitely many planar cubic hypohamiltonian graphs. Let G be a graph containing a 4-cycle $C = v_1v_2v_3v_4$, and consider vertices $v'_1, v'_2, v'_3, v'_4 \notin V(G)$. We denote by $\mathrm{Th}(G_C)$ the graph obtained from G by deleting the edges v_1v_2, v_3v_4 and adding a new 4-cycle $C' = v'_1v'_2v'_3v'_4$ and the edges $v_iv'_i$, $1 \leq i \leq 4$. If we write "the graph $\mathrm{Th}(G_C)$" without specifying how C is labeled, then we refer to (an arbitrary but fixed) one of the two (possibly isomorphic) graphs obtained when applying Th. The following result is essentially due to Thomassen, who gives it (without proof) in [12]. A detailed proof for the planar case can be found in [19].

Proposition 1 (Thomassen [12])**.** *Let G be a hypohamiltonian graph containing a 4-cycle C whose vertices are cubic. Then $\mathrm{Th}(G_C)$ is hypohamiltonian, as well.*

Variations, generalisations, and applications of this proposition abound, see for instance [3], [22], and [23]. We here give yet another version, tailored to our needs. In the statement of Lemma 2, C' is as introduced in this section's first paragraph.

Lemma 2. *Let G be a polyhedral k-graph containing a facial 4-cycle $C = v_1v_2v_3v_4$ such that (i) C contains two cubic vertices and (ii) C contains two adjacent non-exceptional vertices. Then $T := \mathrm{Th}(G_C)+v_1v_2+v_3v_4$ is a polyhedral k-graph in which every vertex of C has degree at least 4, every vertex that was non-exceptional (exceptional) in G is non-exceptional (exceptional) in T, and every vertex of C' is cubic and non-exceptional.*

Proof. It was shown in the proof of [22, Lemma 5] that if G is non-hamiltonian, then so is T. (In that proof the vertices of C were assumed to be cubic, but exactly the same proof can be used without this restriction.) It is obvious that every vertex of C has degree at least 4 and every vertex of C' is cubic. We shall use the latter fact tacitly.

We consider G to be a subgraph of T. Let v be a non-exceptional vertex in G, and let $\mathfrak{h}_v$ be a hamiltonian cycle in $G - v$. By (i), $\mathfrak{h}_v$ contains an edge $e \in E(C)$. Replacing in $\mathfrak{h}_v$ the edge e by the appropriate path (e.g. replacing $e = v_1v_2$ by $v_1v'_1v'_4v'_3v'_2v_2$) visiting every vertex of C' proves that v is non-exceptional in T, as well. Assume there exists a non-exceptional vertex w in $T - C'$ with the property that w was exceptional in G. Let $\mathfrak{h}_w$ be a hamiltonian cycle in $T - w$. Then $\mathfrak{h}_w \cap C'$ has one or two components. Either way, it is straightforward to replace these with edges residing in $E(C)$, thus obtaining a hamiltonian cycle in $G - w$, a contradiction, since w was assumed to be exceptional in G. In conclusion, every vertex that was non-exceptional (exceptional) in G is non-exceptional (exceptional) in T.

By (ii) we know that C contains adjacent non-exceptional vertices, and without loss of generality we may assume that v_1 and v_2 are these two vertices. Suppose there exists a hamiltonian cycle in $G - v_1$ using the edge v_3v_4 (if no such cycle exists, it must use the edge v_2v_3 by (i), and the ensuing arguments are very similar). Replace v_3v_4 by $v_3v'_3v'_2v'_1v_1v_4$ and a hamiltonian cycle in $T - v'_4$ is obtained. Replacing in this cycle $v'_3v'_2v'_1$ by $v'_3v'_4v'_1$ yields a hamiltonian cycle in $T - v'_2$. The same reasoning allows us to infer from the hamiltonicity of $G - v_2$ that v'_1 and v'_3 are non-exceptional in T. □

We remark that although many polyhedral k-graphs satisfying condition (i) of Lemma 2 have been described (take for instance Herschel's graph, or graphs presented by Dillencourt in [2]), it is condition (ii)—in particular, the adjacency requirement—which is more difficult to satisfy.

We need the following lemma, the proof of which is straightforward and therefore omitted.

Lemma 3. *Let G be a k-graph containing a cubic vertex v which shall not be the only non-exceptional vertex in G. Replacing v by an octahedron O as shown in Fig. 4, we obtain an ℓ-graph G' with $\ell = k + 5$ if v is exceptional and $\ell = k + 6$ if it is not. Moreover, every vertex of $G' - O$ that was exceptional/non-exceptional in G remains so in G', and all six vertices in O are exceptional and of degree at least 4.*

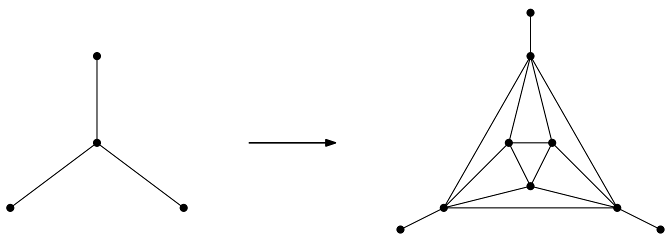

Fig. 4: Replacing a cubic vertex by an octahedron. (Vertices in this figure may be exceptional or non-exceptional.)

Henceforth, when we *replace a vertex by an octahedron* this means that we perform the operation depicted in Fig. 4. For a subgraph $H \subset G$, we denote with $V_3(H)$ the set of all cubic vertices in H. Using Lemmas 2 and 3, we can show the following result.

Theorem 3. *Let G be a polyhedral k-graph containing a facial 4-cycle C such that there exist two cubic vertices as well as two adjacent non-exceptional vertices in C. Then there exist for*

$$\ell := 6(|V_3(G)| - |V_3(C)| + 4) + |\mathrm{exc}(G) \cap (V(G) \setminus V_3(G))|$$

infinitely many polyhedral ℓ-graphs of minimum degree 4; infinitely many planar $(\ell + 1)$-graphs of connectivity 2 containing no cubic vertex; and infinitely many planar $(\ell + 4)$-graphs of connectivity 2 and minimum degree 4.

Proof. For the first statement, apply Lemma 2 to G and C in order to obtain a graph G'. By Lemma 2, G' is a polyhedral k-graph in which every vertex of C has degree at least 4, every vertex that was non-exceptional (exceptional) in G is non-exceptional (exceptional) in G', and every vertex of $C' = G'[V(G') \setminus V(G)]$ (we see G as a subgraph of G') is cubic and non-exceptional. Apply Lemma 3 to every cubic vertex in G' in order to obtain a graph G''. Clearly, G'' is a polyhedral graph and every vertex of G'' has degree at least 4. By Lemmas 2 and 3, every cubic vertex originally belonging to G and C' (exceptional or not) has been replaced by an octahedron of six exceptional non-cubic vertices in G'', while every exceptional (non-exceptional) non-cubic vertex originally belonging to G remains unchanged and is an exceptional (non-exceptional) non-cubic vertex in G''. That G'' indeed has minimum degree exactly 4, and not at least 4, follows from the fact that in G, the cycle C contained a cubic vertex, which after applying Lemma 2 to G becomes a quartic vertex, which is then not altered when applying Lemma 3 (as it applies only to cubic vertices).

In order to obtain an infinite family $\mathcal{G}$, note that Lemma 2 can be applied any number of times to G' (always producing a graph with the same number of exceptional vertices), giving a graph in which then Lemma 3 is applied to every cubic vertex.

For the statements concerning the connectivity 2 case, observe that there exists an edge e in G'' (and analogously in every member of $\mathcal{G}$) such that for every non-exceptional vertex v in G'' there is a hamiltonian cycle in $G''-v$ containing e: let e be the edge in a transformed cubic vertex (via Lemma 3) as shown in Fig. 5. (There always is such a vertex because every vertex of C' is cubic.) Fig. 5 shows all essentially different traversals. Now add a new, 2-valent vertex on e for the second statement of Theorem 3 (this new vertex will itself be exceptional, as its removal yields $G'' - e$, a non-hamiltonian graph), and for the final statement of Theorem 3 remove the edge $e = vw$, and consider an octahedron (disjoint from $G'' - e$), two distinct vertices of which are identified with v and w such that the resulting graph is planar. Noting that every vertex of the newly added octahedron is exceptional, the proof is complete. □

Consider the plane 3-connected 1-graph G constructed by Wiener [18] and reproduced in Fig. 6, where the required quadrilateral C (in order to apply Theorem 3) may be any of the quadrilaterals present in G. The exceptional vertex of G is marked white in Fig. 6.

We have $|V_3(G)| = 24$, $|V_3(C)| = 2$, and $|\mathrm{exc}(G) \cap (V(G) \setminus V_3(G))| = 1$. Therefore, by Theorem 3, there exist infinitely many polyhedral 157-graphs of minimum degree 4, infinitely many planar 158-graphs of connectivity 2 containing no cubic vertex, and infinitely many planar 161-graphs of connectivity 2 and minimum degree 4. We now improve these results.

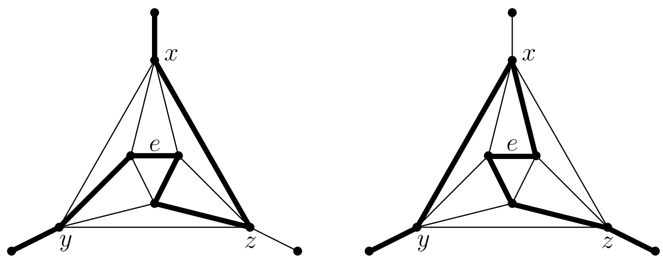

Fig. 5: Longest cycles using the edge e.
(Vertices in this figure may be exceptional or non-exceptional.)

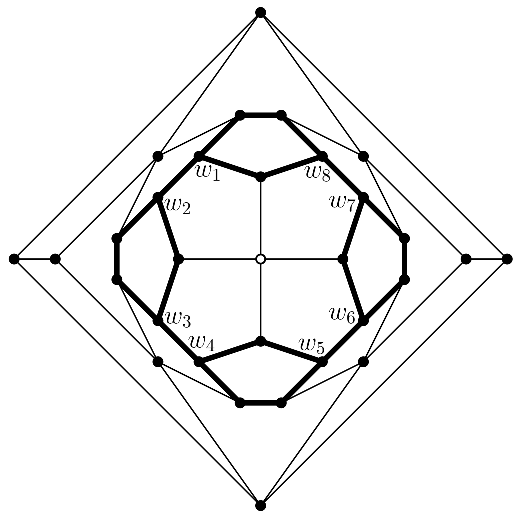

Fig. 6: Wiener's polyhedral 1-graph.

Lemma 4.

(i) *Let G be a k-graph in which adjacent cubic vertices were replaced by octahedra O_1, O_2, respectively. Let $v_i \in V(O_i)$ be adjacent. Then G/v_1v_2 is a $(k-1)$-graph.*
(ii) *Let H be an ℓ-graph in which adjacent cubic vertices v_1, v_2 and their cubic and pairwise distinct neighbours v_3, v_4, v_5, v_6 were replaced by octahedra O_i, respectively, and (i) was applied. Removing all six inner vertices of O_1 and O_2 yields an $(\ell - 6)$-graph.*

Proof. (i) Assume $G' := G/v_1v_2$ is hamiltonian. This yields a hamiltonian cycle in G, $G - v_1$ or $G - v_2$, all of which lead to a contradiction as G is a k-graph in which the vertices v_1 and v_2 are exceptional (as every vertex in $V(O_1) \cup V(O_2)$ is exceptional by Lemma 3). So G' is non-hamiltonian.

Let v be a non-exceptional vertex in G and $\mathfrak{h}_v$ a hamiltonian cycle in $G - v$. Note that $v \notin V(O_1) \cup V(O_2)$. If v_1v_2 lies on $\mathfrak{h}_v$, then after contracting v_1v_2 we have obtained a hamiltonian cycle in $G' - v$. If it does not, we reroute $\mathfrak{h}_v$ in O_1 so that it avoids

v_1 and only v_1 (among vertices in O_1), and becomes a hamiltonian cycle in $G - v - v_1$. Contracting v_1v_2 yields a hamiltonian cycle in $G' - v$. Thus, every vertex that is non-exceptional in G is also non-exceptional in G'.

Let u be the vertex in G' obtained when identifying v_1 and v_2. Let w be an exceptional vertex in G which is neither v_1 nor v_2, and assume that $G' - w$ contains a hamiltonian cycle $\mathfrak{h}_w$. If one neighbour of u on $\mathfrak{h}_w$ lies in $O_1 - u$ and the other lies in $O_2 - u$, we expand u back into v_1v_2 and obtain a hamiltonian cycle in $G - w$, a contradiction. So both neighbours of u on $\mathfrak{h}_w$ lie (without loss of generality) in $O_1 - u$. Expanding u back into v_1v_2, the cycle $\mathfrak{h}_w$ becomes a hamiltonian cycle $\mathfrak{h}$ in $G - w - v_2$. It is now easy to reroute $\mathfrak{h}$ in O_2 so that it visits v_2 as well, and we have obtained a hamiltonian cycle in $G - w$, once more a contradiction. Hence, excluding v_1 and v_2, every vertex that is exceptional in G is also exceptional in G'.

Assume u is non-exceptional in G', so there exists a hamiltonian cycle in $G' - u$. Expanding u back into v_1v_2, we obtain a hamiltonian cycle $\mathfrak{h}'$ in $G - v_1 - v_2$. As before, we can reroute $\mathfrak{h}'$ inside O_1 and O_2 such that a hamiltonian cycle in G is obtained, a contradiction.

(ii) The arguments are similar and therefore omitted. □

Consider once more Wiener's graph G from Fig. 6. We will be interested in the 20-vertex subgraph S emphasised by bold edges in Fig. 6. Replace every cubic vertex of G except for the two cubic vertices located on the boundary of the outer quadrilateral (see Fig. 6), which we call C, by an octahedron. Since there are $|V(S)| + 2 = 22$ such vertices, we obtain a polyhedral k-graph containing exactly two cubic vertices, and with $k = 22 \cdot 6 + 1 = 133$. Thereafter, contract all 24 edges of S, each of which now lies between two octahedra and satisfies the requirement stated in Lemma 4 (i). When performing this contraction, we also apply Lemma 4 (ii) to the vertex pairs (w_1, w_2), (w_3, w_4), (w_5, w_6), (w_7, w_8) as defined in Fig. 6. Hence, we are contracting 24 edges (and thus removing 24 exceptional vertices) as well as removing from eight octahedra their three interior vertices (via Lemma 4 (ii)), which removes another 24 exceptional vertices. So we obtain a graph G' with $133 - 24 - 24 = 85$ exceptional vertices. Thus, we have $|V_3(G')| = |V_3(C)| = 2$ and $|\mathrm{exc}(G') \cap (V(G') \setminus V_3(G'))| = 85$. Applying to G' and C Theorem 3, we obtain infinite families of polyhedral ℓ-graphs of minimum degree 4; $(\ell + 1)$-graphs of connectivity 2 containing no cubic vertex; and planar $(\ell + 4)$-graphs of connectivity 2 and minimum degree 4, where

$$\begin{aligned}\ell &= 6(|V_3(G')| - |V_3(C)| + 4) + |\mathrm{exc}(G') \cap (V(G') \setminus V_3(G'))| \\ &= 6(2 - 2 + 4) + 85 = 109.\end{aligned}$$

We now summarise our findings.

Theorem 4. *There exist infinitely many polyhedral* 109*-graphs of minimum degree* 4*, infinitely many planar* 110*-graphs of connectivity* 2 *containing no cubic vertex, and infinitely many planar* 113*-graphs of connectivity* 2 *and minimum degree* 4*.*

It was proven in [23] that in a polyhedral n-vertex k-graph of minimum degree at least 4 we have $k \leq n - 6$, and that for every $c < 1$ there exists a polyhedral non-hamiltonian n-vertex graph of minimum degree 4 with at least cn hamiltonian vertex-deleted subgraphs.

For a non-negative integer t, put $k(t) = 6(24 + 4t) + 1$. Applying the operation Th, defined in the beginning of this section, t times to Wiener's graph shown in Fig. 6, where C shall be the outer quadrilateral, we obtain by Theorem 3 an infinite family $\mathcal{G}$ of $k(t)$-graphs, each containing at least 24 edges (already contained in Wiener's graph, which is a subgraph of every graph in $\mathcal{G}$, and emphasised in Fig. 6) which can be contracted as described in Lemma 4 (i). Since $k(t + 1) - k(t) = 24$, using Theorem 3, contracting edges one-by-one, and slightly adapting the final part of the proof of Theorem 3, yields:

Theorem 5. *There exists a k_0 such that for every $k \geq k_0$ there are infinitely many polyhedral k-graphs of minimum degree* 4*, for every $k' \geq k_0 + 1$ infinitely many planar k'-graphs of connectivity* 2 *containing no cubic vertex, and for every $k'' \geq k_0 + 4$ infinitely many planar k''-graphs of connectivity* 2 *and minimum degree* 4*.*

We have already discussed the general (i.e. not necessarily planar) case in Theorem 2 (i) and (iv). We could now complement these results and give an alternative approach by adapting Theorem 3, applying Lemma 2 as stated above, and modifying Lemma 3 by using K_5 instead of the octahedron. However, even after applying a suitable variation of Lemma 4, this would produce worse bounds than what we have seen in Theorem 2 and is therefore omitted.

4. On a theorem of Thomassen

In the light of Thomassen's result (T2) that a planar n-vertex graph with no cubic vertices in which n vertex-deleted subgraphs are hamiltonian, must itself be hamiltonian, and our strengthening stating that the latter "n" can be replaced with "$n - 1$" [22], we ask the natural question whether this can be further lowered to $n - 2$ (if we exclude $K_2 + 3K_1$) or $n - 3$; in other words, whether a planar k-graph without cubic vertices, other than $K_2 + 3K_1$, exists for $k \in \{2, 3\}$. We shall call such a graph *naughty*.

In [23] we proved that naughty graphs of minimum degree at least 4 do not exist, so a naughty graph must contain a 2-valent vertex. We remark that there are no small naughty graphs other than $K_2 + 3K_1$: Van Cleemput [14] verified, using a computer, that up to order 10 the only planar k-graphs with $k \leq 2$, irrespective of connectivity or degree conditions, are $K_{2,3}$ and $K_2 + 3K_1$. Even dropping the planarity requirement only adds the obvious candidate, Petersen's graph, which is hypohamiltonian, i.e. a 0-graph. His computations also show that planar 3-graphs of order at most 10 exist only for order 7, and that all of them contain a cubic vertex (these graphs are structurally similar to the ones constructed in the proof of Theorem 2 (ii)). We conclude this article with the following result on the non-existence of certain planar 3-graphs, complementing (T2).

Theorem 6. *Let G be a planar graph of order at least 4 and without cubic vertices, in which there exist exactly three vertices, each of whose deletion yields a non-hamiltonian graph, one of which is 2-valent. Then G is hamiltonian.*

Proof. The conditions imposed on G imply that G must be 2-connected. Assume that G is non-hamiltonian, i.e. a 3-graph. Denote the exceptional vertices of G with $X = \{x, y, z\}$ and let z have degree 2. The neighbours of z must be exceptional, so they are x and y, and these together form the only 2-cut of G (otherwise G would not be a 3-graph as every vertex of a 2-cut is exceptional). Thus, the only 2-valent vertex of G is z. Put $X = \{x, y\}$. By Theorem 1, $G - X$ has exactly two components, one of which is $\{z\}$.

We first treat the case that xy is not an edge of G. As z is exceptional and G is non-hamiltonian, $G' := (V(G) \setminus \{z\}, E(G) \cup \{xy\})$ is a planar k-graph of minimum degree at least 4 and $k \in \{0, 1, 2\}$, as x and y may have become non-exceptional in G', but for every vertex v that was non-exceptional in G there exists a hamiltonian xy-path in $G' - v$ and thus a hamiltonian cycle in $G' - v$. By the result from [23] that if in a planar n-vertex graph with minimum degree at least 4 at least $n - 5$ vertex-deleted subgraphs are hamiltonian, then the graph is hamiltonian, we have that G' must be hamiltonian. As G is non-hamiltonian, $G - z$ contains no hamiltonian xy-path, so no hamiltonian cycle of G' uses the edge xy. Thus, z is non-exceptional in G, a contradiction.

Let us now assume that x and y are adjacent in G. The graph $G'' := G - z$ is non-hamiltonian because z is exceptional in G. For every vertex v that was non-exceptional in G there exists a hamiltonian xy-path in $G'' - v$. Therefore, the vertices x and y cannot have a common neighbour u in G'', as we could extend the hamiltonian xy-path in $G'' - u$ to a hamiltonian cycle (by adding the path xuy) in G'', a contradiction. Let $G''' := G''/xy$, i.e. the graph obtained after contracting the edge xy in G''. The non-hamiltonicity of G, exceptionality of x in G, and exceptionality of y in G together yield that G''' is non-hamiltonian.

Denoting the vertex obtained through the contraction of the edge xy by w, we have that w may be exceptional or non-exceptional in G''', but certainly its degree is at least 4 by above arguments. Therefore G''' is planar, has minimum degree at least 4, and is either hypohamiltonian or almost hypohamiltonian, contradicting the aforementioned result from [23]. □

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