4-regular 4-connected Hamiltonian graphs with a bounded number of Hamiltonian cycles

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Abstract. We prove that there exists an infinite family of 4-regular 4-connected Hamiltonian graphs with a bounded number of Hamiltonian cycles. We do not know if there exists such a family of 5-regular 5-connected Hamiltonian graphs.

Key words. Hamiltonian cycle; regular graph

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1 Introduction

There is a variety of 3-regular 3-connected graphs with no Hamiltonian cycles. Much less is known about 4-regular 4-connected graphs. Thus the Petersen graph (on 10 vertices) is the smallest 3-regular 3-connected non-Hamiltonian graph whereas it was an open problem of Nash-Williams if there exists a 4-regular 4-connected non-Hamiltonian graph until Meredith [9] gave an infinite family, the smallest of which has 70 vertices, see [11, p. 239]. Tait’s conjecture that every 3-regular 3-polyhedral graph has a Hamiltonian cycle was open from 1880 till Tutte [15] in 1946 found a counterexample, see [1, p. 161]. By Steinitz’ theorem, the Tutte graph (and subsequently many others) also show that there are infinitely many 3-regular 3-polyhedral graphs which are non-Hamiltonian, whereas it is a longstanding conjecture of Barnette that every 4-regular 4-polyhedral graph has a Hamiltonian cycle [5, p. 1145] (see also [6, p. 389a] and [4, p. 375]). There are infinitely many 3-regular hypohamiltonian graphs whereas it is a longstanding open problem if there exists a hypohamiltonian graph of minimum degree at least 4, see [12]. Starting with the complete graph on 4 vertices and successively replacing vertices by triangles we obtain an infinite family of graphs with precisely three Hamiltonian cycles. Cantoni’s conjecture says that these are precisely the planar 3-regular graphs with exactly three Hamiltonian cycles, see [16].

Recently, Haythorpe [8] conjectured that 4-regular graphs behave differently from the 3-regular graphs, also in this respect, in that the number of Hamiltonian cycles increases as a function of the number of vertices. The purpose of this note is to answer this in the negative. So in this respect, the 4-regular 4-connected graphs behave in a similar way as the 3-regular 3-connected graphs. We do not know if this also holds for the 5-regular 5-connected graphs.

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Andrew Thomason \[11\] proved that every Hamiltonian graph whose vertices are all of odd degree has a second Hamiltonian cycle. Thomason’s theorem was inspired by (and extends) Smith’s result stating that every cubic graph has an even number of Hamiltonian cycles through each edge, see \[15\]. Sheehan \[10\] conjectured that the same holds for 4-regular graphs: if they are Hamiltonian, then they contain at least two Hamiltonian cycles. In \[3\] this was shown to hold up to order 21. If true in general, this would imply that, for every natural number \(k \geq 3\), every \(k\)-regular Hamiltonian graph has a second Hamiltonian cycle. The latter statement was verified in \[14\] for all \(k > 72\) and subsequently in \[7\] for all \(k > 20\).

For restricted classes of 4-regular 4-connected graphs, it is still possible that the number of Hamiltonian cycles in a Hamiltonian graph must increase (perhaps even exponentially) as a function of the number of vertices. This may be true for planar graphs where it is known that the number of Hamiltonian cycles increases at least as a linear function, \[2\]. And it may hold for bipartite graphs where it is known that the number of Hamiltonian cycles in a Hamiltonian \(k\)-regular graph increases more than exponentially as a function of \(k\), \[13\].

2 4-regular 4-connected graphs with a bounded number of Hamiltonian cycles

As mentioned earlier, the Meredith graph is a 4-regular 4-connected non-Hamiltonian graph.

Figure 1 indicates an infinite family of 4-regular Hamiltonian graphs with a bounded number of Hamiltonian cycles.

\[
\begin{array}{c}
\text{Fig. 1: 4-regular graphs, each having exactly 216 Hamiltonian cycles.} \\
(\text{The left-most and right-most part of the graph are to be connected in the obvious way.)}
\end{array}
\]

All graphs in this infinite family have connectivity 2. We shall now describe 4-regular 4-connected graphs with a bounded (positive) number of Hamiltonian cycles.

**Theorem.** There exists a constant \(c > 0\) such that there are infinitely many 4-regular 4-connected graphs, each containing exactly \(c\) Hamiltonian cycles.

**Proof.** Assume \(G\) is a 4-regular 4-edge-connected graph containing a path \(abcd\) such that

(i) \(G\) has no Hamiltonian cycle;
(ii) \(G\) has a 2-factor consisting of two cycles \(C\) and \(C'\) such that \(C\) contains \(ab\) and \(C'\) contains \(cd\); and
(iii) \(G - bc\) has no Hamiltonian path joining two of \(a, b, c, d\). If \(v\) is any vertex in \(\{a, b, c, d\}\), then \(G - v - bc\) has no Hamiltonian path joining two of \(a, b, c, d\).

We construct the graph \(H_G\) as follows. Let \(G'\) be a copy of \(G\) and \(\ell \geq 1\) a natural number. Take the disjoint union of \(G, G'\); denote for a vertex \(v\) in \(G\) its copy in \(G'\) by \(v'\); delete the
edges $ab, bc, cd, a'b', b'c', c'd'$; and add four pairwise disjoint paths $P_a, P_b, P_c, P_d$ joining $a, c'$ and $b, b'$ and $c, a'$ and $d, d'$, respectively, where $P_a$ and $P_d$ have length $\ell + 1$, while $P_b$ and $P_c$ have length $\ell$. Thereafter, add a zig-zag path between $P_a$ and $P_b$, as well as a zig-zag path between $P_c$ and $P_d$. The construction of $H_G$ is illustrated in Fig. 2. Every Hamiltonian cycle in $H_G$ will contain the paths $P_a, P_b, P_c, P_d$. Note that the number of Hamiltonian cycles of $H_G$ is independent of $\ell$.

![Fig. 2: On the left, the graph $H_G$ is shown, with relevant 2-factors in $G$ and $G'$. On the right, we illustrate why conditions (i), (ii), and the first statement in (iii) would not suffice: if $G$ and $G'$ are traversed as shown, then the number of Hamiltonian cycles in $H_G$ would increase with $\ell$.](image1)

The required graph $G$ shall be the modification of the Petersen graph shown in Figure 3 where the bold edges form the 2-factor satisfying (ii). It is straightforward to infer from the non-Hamiltonicity of Petersen’s graph that $G$ satisfies (i), and it is left to the reader to verify that it satisfies (iii).

![Fig. 3: A useful modification of Petersen’s graph.](image2)

Using two copies of $G$, we construct $H_G$ as described above. One final issue remains: there are double edges occurring in $H_G$. We shall make use of the idea behind Meredith’s classical construction in which a vertex is replaced by a complete bipartite graph $K_{3,4}$, see [1, p. 161]. In the Meredith graph the operation is performed on each vertex of the Petersen graph (in which a 1-factor is replaced by double edges). In the present note the operation is performed only on both ends of each double edge in $H_G$. □
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References


