K_2-hamiltonian graphs: I

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Abstract. Motivated by a conjecture of Grünbaum and a problem of Katona, Kostochka, Pach, and Stechkin, both dealing with non-hamiltonian n-vertex graphs and their $(n-2)$-cycles, we investigate K_2-*hamiltonian* graphs, i.e. graphs in which the removal of any pair of adjacent vertices yields a hamiltonian graph. In this first part, we prove structural properties, and show that there exist infinitely many cubic non-hamiltonian K_2-hamiltonian graphs, both of the 3-edge-colourable and the non-3-edge-colourable variety. In fact, cubic K_2-hamiltonian graphs with chromatic index 4 (such as Petersen's graph) are a subset of the critical snarks. On the other hand, it is proven that non-hamiltonian K_2-hamiltonian graphs of any maximum degree exist. Several operations conserving K_2-hamiltonicity are described, one of which leads to the result that there exists an infinite family of non-hamiltonian K_2-hamiltonian graphs in which, asymptotically, a quarter of vertices has the property that removing such a vertex yields a non-hamiltonian graph. We extend a celebrated result of Tutte by showing that every planar K_2-hamiltonian graph with minimum degree at least 4 is hamiltonian. Finally, we investigate K_2-traceable graphs, and discuss open problems.

Keywords. Hamiltonian cycle, snark, vertex-deleted subgraph, hypohamiltonian, planar

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1 Introduction

Grünbaum [18] defined $\Gamma(j,k)$ with $k \geq j$ as the family of all graphs whose order and circumference differ by k and in which any j vertices are missed by some longest cycle. $\Gamma(1,1)$ are exactly the *hypohamiltonian* graphs, i.e. non-hamiltonian graphs in which every vertex-deleted subgraph is hamiltonian. These have been studied extensively, see [20] for a survey and [24, 40, 42] for recent contributions.

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In 1974, Grünbaum [18] conjectured that $\Gamma(j, j)$ is empty for all $j \geq 2$. Very little is known about the veracity of this conjecture, which in its general form seems disconcertingly difficult. In this article, we focus on a class of graphs related to $\Gamma(2, 2)$. Thomassen agrees with Grünbaum that $\Gamma(2, 2)$ is empty, and points out that every member of $\Gamma(2, 2)$ has the property that each of its vertex-deleted subgraphs must be hypohamiltonian [35]. A relaxation of the question whether $\Gamma(2, 2)$ is empty was raised in 1989 by Katona, Kostochka, Pach, and Stechkin [25]. They asked whether an n-vertex graph in which any $n - 2$ vertices induce a hamiltonian graph, must itself be hamiltonian. An equivalent problem was raised by van Aardt, Burger, Frick, Llano, and Zuazua, see [1, Question 1]. The only difference between these graphs and Grünbaum's $\Gamma(2, 2)$ is the fact that, if n is a graph's order, the former allow $(n - 1)$-cycles, while the latter do not.

Reformulating the Katona et al. question, we can ask whether a graph in which the removal of any pair of vertices yields a hamiltonian graph, must itself be hamiltonian. The problems of Grünbaum and Katona et al. restricted to pairs of *non*-adjacent vertices can be solved by considering the join of K_t and $\overline{K}_{t+2}$. In this paper, we concentrate on the Katona et al. question restricted to adjacent vertices, which we abbreviate to ($\mathfrak{K}$), and Grünbaum's conjecture restricted to pairs of adjacent vertices, which we abbreviate to ($\mathfrak{G}$). We will call K_2-*hamiltonian* graphs in which the removal of any pair of adjacent vertices is hamiltonian. Applications of the hamiltonian properties of K_2-deleted subgraphs include Faulkner and Younger's use in [12], which inspired the application in [26]; Horton's use [21]—as well as Thomassen's generalisation [34]—to construct hypotraceable graphs (we come back to this in Section 5); results on the widely used dot product, for instance in the context of snarks; and Chvátal's so-called "flip-flops" [7], generalised by Hsu and Lin [22], a generalisation which in turn was used by Wiener to study a criticality notion for spanning trees [39].

The problems ($\mathfrak{K}$) and ($\mathfrak{G}$) vary greatly in difficulty: on the one hand, we shall provide many solutions to ($\mathfrak{K}$), but on the other hand, we will only "scratch the surface" concerning ($\mathfrak{G}$), reaching in a certain sense a quarter of a solution. In Section 2 we present properties and provide examples of non-hamiltonian K_2-hamiltonian graphs—these solve ($\mathfrak{K}$) and yield potential counterexamples to ($\mathfrak{G}$); in Section 3 we discuss the hamiltonian properties of vertex-deleted subgraphs of K_2-hamiltonian graphs and three operations conserving K_2-hamiltonicity; and in Section 4 we study *planar* K_2-hamiltonian graphs, which we motivate in the next paragraph. In Section 5 we treat K_2-*traceable* graphs, i.e. graphs in which the removal of any pair of adjacent vertices yields a graph containing a hamiltonian path, and in Section 6 the article concludes with open problems and directions for future research.

In [18], Grünbaum also conjectured that planar hypohamiltonian graphs do not exist. Restated in a perhaps more attractive fashion: a planar graph in which every vertex-deleted subgraph is hamiltonian, must itself be hamiltonian. Although Thomassen refuted this [34], he also showed an elegant way to *save* Grünbaum's conjecture by proving that a planar graph of minimum degree at least 4 in which every vertex-deleted subgraph is hamiltonian, must itself be hamiltonian [35]. Searching for a different manner in which Grünbaum's conjecture can be saved, one could ask whether it is true that a planar K_2-hamiltonian graph must be hamiltonian. This question will be addressed in the continuation of this article [15]. Equivalent to Thomassen's aforementioned result is the statement that every planar hypohamiltonian graph contains a cubic vertex. We will here prove a K_2-hamiltonian analogue of this result. It can be inferred from results of Tutte [37], Thomas and Yu [32], and Sanders [30], that every planar 4-connected graph of order n contains a cycle of length n, $n - 1$, $n - 2$, and $n - 3$. In Section 4, we will extend these results and show that every planar K_2-hamiltonian of minimum degree at least 4 contains a cycle of length $n, n - 1$, $n - 2$, and $n - 3$.

In this paper, all graphs are finite, simple, and connected, unless otherwise stated. Our notation follows Diestel's book [10], with a few exceptions and additions: here, a complete n-vertex graphs will be denoted by K_n, and a set of vertices (edges) whose removal disconnects a given graph is a *cut* (*edge-cut*). A k-*cut* (k-*edge-cut*) is a cut (an edge-cut) of cardinality k. The number of components of a possibly disconnected graph G is denoted by $c(G)$. Let G be a non-complete graph of connectivity k and order greater than k, X a k-cut in G, and C a component of $G - X$. Then $G[V(C) \cup X]$ is a k-*fragment* of G with *attachments* X—we here use Wiener's definition [40] which slightly differs from Thomassen's [34]—but sometimes we will suppress specifying the attachments, or shorten this and simply write X-*fragment*. A k-fragment is *trivial* if it contains exactly $k + 1$ vertices. Let F, F' be disjoint 3-fragments of graphs of connectivity 3, and let F have attachments x_1, x_2, x_3 and F' have attachments x'_1, x'_2, x'_3. Identifying x_i with x'_i for all i, we obtain the graph $(F, \{x_1, x_2, x_3\}) \vdots (F', \{x'_1, x'_2, x'_3\})$. When the vertices that are being identified (always using a bijection) are clear from context, we simply write $F \vdots F'$. A cut X of G is *trivial* if $G - X$ has exactly two components and X is the set of attachments of a trivial k-fragment. A 3-connected graph in which every 3-cut is trivial is called *essentially* 4-*connected*. A path with end-vertex v is a v-*path*, and a v-path with end-vertex $w \neq v$ is a vw-*path*. For $t \in \{1, 3\}$ we say that a graph G is K_t-*hamiltonian* if the deletion of any copy of K_t present in G yields a hamiltonian graph.

2 Fundamental properties and examples

Every K_2-hamiltonian graph is 3-connected. It is also 1-tough: let G be a K_2-hamiltonian graph and $S \subset V(G)$ have cardinality k. Consider $v \in S$. For $w \in N(v)$ we denote with $\mathfrak{h}$ a hamiltonian cycle in $G - v - w$. Let $u \in N(w) \setminus \{v\}$. We remove from $\mathfrak{h}$ an edge incident with u and add the path uw in order to obtain a path $\mathfrak{p}$. The vertices in $S \setminus \{v\}$ are on $\mathfrak{p}$ and determine at most k subpaths of $\mathfrak{p}$. Each such subpath visits at most one component of $G - S$, so $G - S$ has at most k components.

Moreover, if G is K_2-hamiltonian and triangle-free, then its girth is at most

$$3 + \min_{vw \in E(G)} \frac{|V(G)| - 2}{\deg(v) + \deg(w) - 2},$$

which we now show. Consider adjacent $v, w \in V(G)$, put $N := N(v) \cup N(w) \setminus \{v, w\}$, and let $\mathfrak{h}$ be a hamiltonian cycle in $G - v - w$. The vertices of N split $\mathfrak{h}$ into paths $S_1, \ldots, S_k$, where $k = \deg(v) + \deg(w) - 2$ (since G is triangle-free). For arbitrary but fixed $i \in \{1, \ldots, k\}$, let S_i have end-vertices $x, y \in N$. Then either $S_i \cup xvy$, $S_i \cup xwy$, $S_i \cup xvwy$, or $S_i \cup xwvy$ is a cycle Q_i in G. We have

$$\sum_{i=1}^{k} |E(Q_i)| \leq |E(\mathfrak{h})| + 3k = n - 2 + 3k.$$

Thus, the shortest of these cycles has length at most $\frac{n-2+3k}{k}$.

We shift our focus to *non-hamiltonian* K_2-hamiltonian graphs, for which we present a series of basic structural results which will be useful in later arguments. Several of these are variations of results on hypohamiltonian graphs, e.g. Bondy's observation that triangles in hypohamiltonian graphs do not contain cubic vertices (see [7]).

Proposition 1. *Let G be a non-hamiltonian K_2-hamiltonian graph of order n. Then the following hold.*

(i) *The vertices of a 3-cycle or a 4-cycle in G have degree at least 4. If G contains a 5-cycle with two non-adjacent cubic vertices, then it has circumference $n-1$. Hence, cubic K_2-hamiltonian graphs of order n and circumference $n-2$, and thus any cubic counterexample to ($\mathfrak{G}$), must have girth at least 6.*

(ii) *The vertices of a 3-cut X of G are independent and $c(G-X)=2$. If X is non-trivial, then each vertex in X has at least two neighbours in each component of $G-X$; in particular, the vertices in X have degree at least 4 in G. If Y is a 4-cut of G and $E(G[Y]) \neq \emptyset$, then $c(G-Y)=2$, and the edges in $G[Y]$ are independent.*

(iii) *For a non-trivial 3-cut $X=\{x_1,x_2,x_3\}$ of G, both X-fragments F_1 and F_2 contain an x_ix_j-path that is hamiltonian in the fragment minus x_k, where i,j,k are pairwise different. Moreover, for $i \in \{1,2\}$ and adjacent vertices $a,b \in V(F_i)\setminus X$ there exists a hamiltonian path between two vertices of X in $F_i - a - b$.*

(iv) *G is cyclically 4-edge-connected.*

(v) *If G has circumference $n-2$, then it must be essentially 4-connected. In particular, any counterexample to ($\mathfrak{G}$) is essentially 4-connected.*

Proof. (i) Let uvw be a 3-cycle in G with v cubic. Removing u and w from G gives a graph of connectivity 1, which cannot be hamiltonian. Let $uvwx$ be a 4-cycle in G with v cubic. Removing w and x from G, we obtain a graph containing a hamiltonian cycle using the edge uv. Replacing this edge with the path $uxwv$ yields a hamiltonian cycle in G, a contradiction. Let $uvwxy$ be a 5-cycle in G with v and x cubic. Consider the hamiltonian cycle $\mathfrak{h}$ in $G-u-y$. Certainly, $\mathfrak{h}$ contains the path vwx. Replacing, in G, vwx with the path $vuyx$, we obtain a cycle of length $n-1$ in G.

(ii) Assume two vertices of a 3-cut X are adjacent. Removing these two vertices we would obtain a graph of connectivity 1, which cannot be hamiltonian, a contradiction. Suppose $c(G-X) \geq 3$ and let F_1,F_2,F_3 be pairwise different X-fragments of G. For $i \in \{1,2,3\}$ there exists an edge $xy \in E(G)$ such that $x \in X$ and $y \in V(F_i)\setminus X$. Since $G-x-y$ is hamiltonian, $c(G-X)=3$ and $F_i - X = K_1$. Thus, as G is 3-connected, we have $G=K_{3,3}$, which is hamiltonian. But this yields a contradiction, since G is assumed to be non-hamiltonian.

Suppose G has a non-trivial 3-cut $X=\{u,v,w\}$ such that v is cubic. We have seen that there are exactly two X-fragments F_1 and F_2. Without loss of generality v has only one neighbour x in F_2. Since X is non-trivial, F_1 and F_2 contain at least five vertices each. Removing x and a neighbour of x residing in $V(F_2)\setminus X$, we obtain a hamiltonian uw-path $\mathfrak{p}$ in F_1 unless F_2 has exactly five vertices—if this is the case, it follows from the fact that G has minimum degree 3 that a 3-cycle with a cubic vertex must occur, contradicting (i). Removing v and a neighbour of v lying in F_1, we obtain a hamiltonian uw-path $\mathfrak{p}'$ in $F_2 - v$. However, this implies that $\mathfrak{p} \cup \mathfrak{p}'$ is a hamiltonian cycle in G, a contradiction.

Finally, let $Y=\{u,v,w,x\}$ be a 4-cut of G and assume that $uv \in E(G[Y])$. Since $G-u-v$ is hamiltonian and of connectivity 2, we have $c(G-Y)=2$. Suppose $G[Y]$ contains incident edges uv and vw. Let F_1 and F_2 be Y-fragments of G. Since $G-u-v$ and $G-v-w$ are hamiltonian, F_1-u-v contains a hamiltonian wx-path and F_2-v-w contains a hamiltonian ux-path $\mathfrak{p}'$. Then $\mathfrak{p} \cup \mathfrak{p}' \cup uvw$ is a hamiltonian cycle in G, a contradiction.

(iii) By (ii) we have $c(G-X)=2$. Denote these two fragments by F_1 and F_2. Their order is greater than 4, since X is non-trivial. In order to obtain the desired hamiltonian path in F_1, we use the fact that G is K_2-hamiltonian and remove x_k (defined in the statement above) and a vertex from $F_2 - X$ adjacent to x_k. (Observe that we cannot guarantee the advertised property if we allow X to be trivial.)

Let $a, b \in V(F_1) \setminus X$. Since $G - a - b$ is hamiltonian, there exists a hamiltonian path $\mathfrak{p}_1$ between two vertices of X in $F_1 - a - b$ or there exists a hamiltonian path $\mathfrak{p}_2$ between two vertices of X in F_2. It is easy to see that the latter is impossible: by the paragraph above there is a hamiltonian $x_i x_j$-path $\mathfrak{p}_3$ in $F_1 - x_k$ if $\{i, j, k\} = \{1, 2, 3\}$, and (choosing $\mathfrak{p}_3$ appropriately) $\mathfrak{p}_2 \cup \mathfrak{p}_3$ would be a hamiltonian cycle of G.

(iv) Assume G contains a 3-edge-cut $M = \{uu', vv', ww'\}$, such that the components of $G - M$, which we call F, F', satisfy $u, v, w \in V(F)$ and $u', v', w' \in V(F')$, and each contain a cycle. By (i), F and F' contain at least four vertices each. Thus, we can remove from F the vertex v and a neighbour x of v which also lies in F. Since $G - v - x$ is hamiltonian, we obtain a hamiltonian $u'w'$-path $\mathfrak{p}'$ in F'. We can similarly obtain a hamiltonian uw-path $\mathfrak{p}$ in F. But then $\mathfrak{p} \cup \mathfrak{p}' \cup uu' \cup ww'$ is a hamiltonian cycle in G, a contradiction.

(v) Suppose G contains a non-trivial 3-cut $X = \{u, v, w\}$. Let F_1 and F_2 be the two X-fragments of G. Consider $x \in (N(u) \cap V(F_1)) \setminus X$ and $x' \in (N(u) \cap V(F_2)) \setminus X$. As $G - u - x$ is hamiltonian and $F_1 \neq K_{1,3}$, there exists a hamiltonian vw-path $\mathfrak{p}'$ in $F_2 - u$ and as $G - u - x'$ is hamiltonian and $F_2 \neq K_{1,3}$, there is a hamiltonian vw-path $\mathfrak{p}$ in $F_1 - u$. But then $\mathfrak{p} \cup \mathfrak{p}'$ is a hamiltonian cycle in $G - u$, so G has circumference $n - 1$, a contradiction. □

Clearly, every bipartite K_2-hamiltonian graph must be balanced, but we do not know whether bipartite non-hamiltonian K_2-hamiltonian graphs actually exist. Allowing hamiltonian graphs, examples are easily found, e.g. any prism over a cycle of even length. Although hypohamiltonian graphs cannot be bipartite, this is similar in spirit with a question of Grötschel asking whether there is a bipartite *hypotraceable* graph (a graph admitting no hamiltonian path, but in which every vertex-deleted subgraph does contain a hamiltonian path), see [17, Problem 4.56].

A *snark* is a cubic bridgeless graph which is not 3-edge-colourable (and thus non-hamiltonian). Moreover, in order to avoid degenerate situations, one assumes it to have girth at least 5 and to be cyclically 4-edge-connected. A snark is *critical* if the removal of any two adjacent vertices results in a 3-edge-colourable graph [27]. Let G be a cubic K_2-hamiltonian graph with chromatic index 4. Then, by Proposition 1, G is a snark, and since for every pair of adjacent vertices v, w in G the graph $G - v - w$ contains a bichromatic cycle of length $|V(G)| - 2$, we can colour the remaining edges with a third colour, so G is a critical snark. Recall that by Vizing's Theorem, the chromatic index of a cubic graph is either 3 or 4. We summarise our findings:

Corollary 1. *Every cubic K_2-hamiltonian graph which is not* 3*-edge-colourable is a critical snark.*

In the light of Corollary 1 it is natural to investigate cubic K_2-hamiltonian graphs in terms of their edge-colourability. In the following, we show that two famous infinite families of cubic non-hamiltonian graphs are K_2-hamiltonian, the first having chromatic index 3, the second chromatic index 4.

2.1 Generalised Petersen graphs

Coxeter [9] introduced the family of *generalised Petersen graphs*

$$\mathrm{GP}(n,k) = (\{u_i, v_i\}_{i=0}^{n-1}, \{u_i u_{i+1}, u_i v_i, v_i v_{i+k}\}_{i=0}^{n-1}),$$

indices mod. n, and $k < n/2$ (graphs in this paper are simple). Due to a certain depiction of generalised Petersen graphs, it is common to speak of edges $u_i u_{i+1}$ ($v_i v_{i+k}$) as belonging to the *inner rim* (*outer rim*), and edges $u_i v_i$ to be called *spokes*. Let $\mathcal{P}$ be the family of all generalised Petersen graphs $\mathrm{GP}(n,k)$ with $n = 5$ mod. 6 and $k = 2$.

Proposition 2. *Every member of $\mathcal{P}$, in particular Petersen's graph, is non-hamiltonian and K_2-hamiltonian.*

Proof. Robertson [29] (and, independently, Bondy [2]) showed that every member of $\mathcal{P}$ is non-hamiltonian. It remains to prove that the deletion of any copy of K_2 produces a hamiltonian graph. The symmetry group of the graph $\mathrm{GP}(n,2)$ is transitive on the edges if $n = 5$ (this case, i.e. Petersen's graph, is left to the reader), while for $n > 5$ it has three edge orbits: one containing an edge of the inner rim, one containing a spoke, and one containing an edge of the outer rim—for each of these cases we give in Fig. 1 a solution, where the three dots indicate any number of copies of six spokes, traversable in the obvious and unique way inferred from the already given partial cycle. □

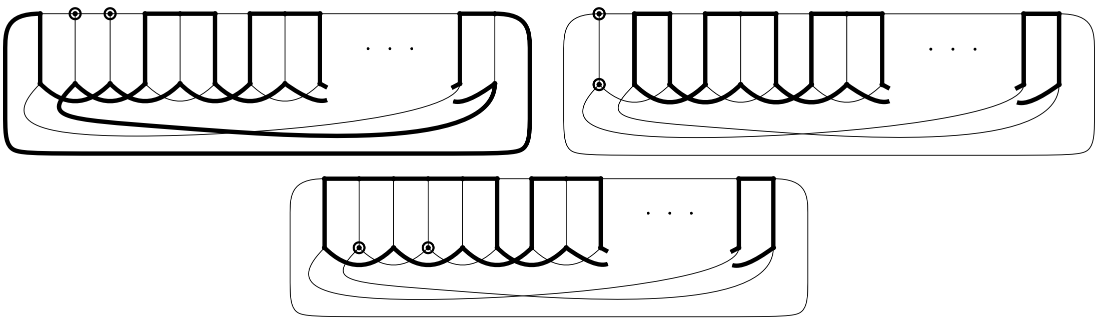

Fig. 1: Every member of $\mathcal{P}$ is K_2-hamiltonian.

Since every generalised Petersen graph except for Petersen's graph itself is 3-edge-colourable [5], we can deduce the following from Proposition 2:

Corollary 2. *There exist infinitely many cubic non-hamiltonian K_2-hamiltonian graphs which have chromatic index* 3 *and girth* 5.

Lemma 1. *If in a cubic K_2-hamiltonian graph G every vertex lies on some* 5*-cycle, then G is K_1-hamiltonian.*

Proof. By hypothesis, for every $v_1 \in V(G)$ there exists a 5-cycle $v_1v_2v_3v_4v_5$. Since G is K_2-hamiltonian, $G - v_3 - v_4$ contains a hamiltonian cycle $\mathfrak{h}$. As G is cubic, $\mathfrak{h}$ contains the path $P := v_2v_1v_5$. In $\mathfrak{h}$, replace P by the path $v_2v_3v_4v_5$. We obtain a hamiltonian cycle in $G - v_1$. □

From Proposition 2 and Lemma 1 we obtain:

Corollary 3 (Bondy [2]). *Every member of $\mathcal{P}$ is hypohamiltonian.*

Motivated by a conjecture of Grünbaum (different from the already mentioned $\Gamma(j,j) = \emptyset$ for all $j \geq 2$), we shall investigate in Section 4 planar K_2-hamiltonian graphs—however, planar non-hamiltonian K_2-hamiltonian graphs elude us. The situation changes dramatically on the torus:

Corollary 4. *There exist infinitely many toroidal non-hamiltonian K_2-hamiltonian graphs.*

Proof. This immediately follows from Proposition 2, taking into account that every member of $\mathcal{P}$ is non-planar [11] yet toroidal, as shown in Fig. 2. □

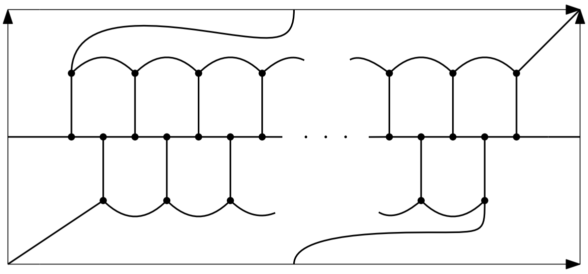

Fig. 2: Every member of $\mathcal{P}$ is toroidal.

It is natural to wonder whether small perturbations—such as connecting a pair of non-adjacent vertices by an edge—of a graph from a given class leaves the graph in the class. If a hypohamiltonian graph is not *maximally non-hamiltonian*, i.e. non-hamiltonian, but adding any edge from the graph's complement renders the graph hamiltonian, then we can add at least one edge and the resulting graph will be hypohamiltonian, and as long as the graph has not become maximally non-hamiltonian, we can continue this procedure. This is not necessarily true for non-hamiltonian K_2-hamiltonian graphs: we do know that the resulting graph is non-hamiltonian, but by adding an edge we must also verify that removing this new edge and its end-vertices yields a hamiltonian graph. This discussion motivates the following observation.

Proposition 3. *For every non-negative integer k there is a non-hamiltonian K_2-hamiltonian graph to which k edges can be added such that the resulting graph is non-hamiltonian and K_2-hamiltonian. Furthermore, for every integer $d \geq 3$ there exists a non-hamiltonian K_2-hamiltonian graph of maximum degree d. In particular, there exist infinitely many non-cubic non-hamiltonian K_2-hamiltonian graphs.*

Proof. In this entire proof, we consider $n = 5$ mod. 12. Put

$$G_n := \mathrm{GP}(n,2) + \sum_{\substack{i=8 \bmod . 12, \\ 8 \leq i < n/2}} v_0 v_i.$$

We call the edges we add to $\mathrm{GP}(n,2)$ *new*. Suppose G_n does contain a hamiltonian cycle $\mathfrak{h}$ (reductio ad absurdum). As discussed above, $\mathrm{GP}(n,2)$ is non-hamiltonian, so $\mathfrak{h}$ contains at least one new edge, and since all new edges are incident with v_0, it contains at most two new edges. We treat the former case first.

Let $v_0 v_k$ be the new edge in $\mathfrak{h}$. Ignoring the spokes $u_0 v_0$ and $u_k v_k$, a straightforward case analysis reveals that there are five ways in which $\mathfrak{h}$ traverses G_n, see Figs. 3 and 4. The cases from Fig. 3 will be called *periodic*, and the cases from Fig. 4 *non-periodic.*

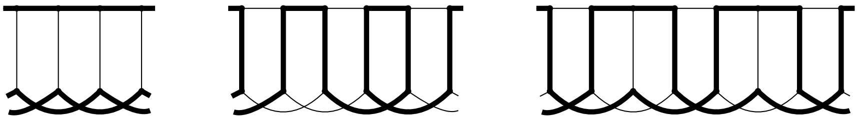

Fig. 3: The periodic traversals π_1, π_2, and π_3 of G_n by $\mathfrak{h}$.

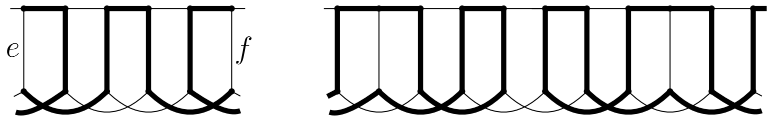

Fig. 4: The non-periodic traversals ν_1 and ν_2 of G_n by $\mathfrak{h}$.
In ν_1, the edges e and f may, but need not be in $\mathfrak{h}$.

Periodic traversals as shown in Fig. 3 repeat every 2, 4, and 6 spokes, respectively. These can be contracted in the obvious way; for example, if the path

$$P := v_{j-4}v_{j-2}u_{j-2}u_{j-1}v_{j-1}v_{j+1}v_{j+3}u_{j+3}u_{j+2}u_{j+1}u_jv_jv_{j+2}v_{j+4}u_{j+4}u_{j+5}$$

is contained in $\mathfrak{h}$ for some j, we contract the two spokes $u_{j+1}v_{j+1}$ and $u_{j+2}v_{j+2}$ (traversal π_1) and P becomes $v_{j-4}v_{j-2}u_{j-2}u_{j-1}v_{j-1}v_{j+3}u_{j+3}u_jv_jv_{j+4}u_{j+4}u_{j+5}$ in which we contract the four spokes $u_{j-1}v_{j-1}$, u_jv_j, $u_{j+3}v_{j+3}$, and $u_{j+4}v_{j+4}$ (traversal π_2) so that P becomes $v_{j-4}v_{j-2}u_{j-2}u_{j+5}$ and we may relabel u_{j+5} as u_{j-1}, etc. Thus, it suffices to show that $G_{17} = \mathrm{GP}(17, 2) + v_0v_8$ is non-hamiltonian due to the following arguments.

The non-periodic traversal ν_2 can occur at most once, and the non-periodic traversal ν_1 transitions into the periodic traversal π_1 (π_2) if the edge f, as defined in Fig. 4, does not lie in $\mathfrak{h}$ (does lie in $\mathfrak{h}$). Furthermore, let us call a subgraph of $\mathfrak{h}$ containing π_1, π_2 or ν_1 of *type 1* and a subgraph of $\mathfrak{h}$ containing π_3 or ν_2 of *type 2*. The cycle $\mathfrak{h}$ may contain subgraphs of both types, but it is easy to see that switching from one type to the other may only occur at indices 0 or k (due to the edge $v_0v_k \in E(\mathfrak{h})$). If between indices 0 and k the subgraph of $\mathfrak{h}$ is of type 2, then after a suitable amount of contractions and relabelling, we must have $k = 8$. If between indices 0 and k the subgraph of $\mathfrak{h}$ is of type 1 and $k < 8$, then $k = 6$ and we add, between indices 0 and k, a copy of the periodic traversal π_1 to the graph; it is straightforward to verify that for a subgraph of type 1 this is possible. The same reasoning is applied between indices k and $n - 1$.

That G_{17} is indeed non-hamiltonian can be verified with a trivial computer program (for instance in SageMath). A similar argumentation can be used for the case when $\mathfrak{h}$ contains two new edges. In this situation, it suffices to show that $G_{41} = \mathrm{GP}(41, 2) + v_0v_8 + v_0v_{20}$ is non-hamiltonian. Again, this can be verified easily and within a reasonable amount of time with a computer program. We have proven that G_n is non-hamiltonian.

The graph $\mathrm{GP}(n, 2) - v_1 - v_3$ has a hamiltonian cycle containing the paths

$$v_{10}v_8u_8u_7u_6v_6v_4v_2v_0u_0u_1u_2u_3u_4u_5v_5v_7v_9u_9u_{10}$$

and $v_{n-1}u_{n-1}$, a situation illustrated in Fig. 1 (bottom). Replacing these paths by $v_{10}u_{10}$ and

$$v_{n-1}v_1v_3u_3u_4u_5v_5v_7v_9u_9u_8u_7u_6v_6v_4v_2u_2u_1u_0u_{n-1},$$

respectively, we obtain a hamiltonian cycle $\mathfrak{h}$ in $G_n - v_0 - v_8$. For $n \geq 41$, $\mathfrak{h}$ contains the paths $v_9u_9u_8$ and

$$v_{22}v_{20}u_{20}u_{19}u_{18}v_{18}v_{16}v_{14}u_{14}u_{13}u_{12}v_{12}v_{10}u_{10}u_{11}v_{11}v_{13}v_{15}u_{15}u_{16}u_{17}v_{17}v_{19}v_{21}u_{21}u_{22}$$

which we replace by

$$v_9u_9u_{10}u_{11}v_{11}v_{13}v_{15}u_{15}u_{16}u_{17}v_{17}v_{19}v_{21}u_{21}u_{20}u_{19}u_{18}v_{18}v_{16}v_{14}u_{14}u_{13}u_{12}v_{12}v_{10}v_8u_8$$

and $v_{22}u_{22}$ in order to obtain a hamiltonian cycle in $G_n - v_0 - v_{20}$. This replacement can be iterated as often as needed in order to obtain, together with the fact that every member of $\mathrm{GP}(n, 2) \subset \mathcal{P}$ is K_2-hamiltonian (see Proposition 2), the K_2-hamiltonicity of G_n. □

2.2 Flower snarks

For $k \geq 3$ odd, the graphs

$$F_{4k} := (\{u_i, v_i, v_{i,1}, v_{i,2}\}_{i=0}^{k-1}, \{u_i u_{i+1}, u_i v_i, v_i v_{i,1}, v_i v_{i,2}, v_{i,1} v_{i+1,2}, v_{i,2} v_{i+1,1}\}_{i=0}^{k-1}),$$

indices mod. k, are the so-called *flower snarks* introduced by Isaacs [23]. We restrict snarks to have girth at least 5, so we define $\mathcal{F} := \{F_{4k}\}_{k \geq 5 \text{ odd}}$. Gutt proved that a class of graphs including the flower snarks are hypohamiltonian [19]. Another proof of their hypohamiltonicity can be found in [8].

Proposition 4. *Every member of $\mathcal{F}$ is K_2-hamiltonian.*

Proof. As in the proof of Proposition 2, we can restrict ourselves to four cases; these are shown in Fig. 5. □

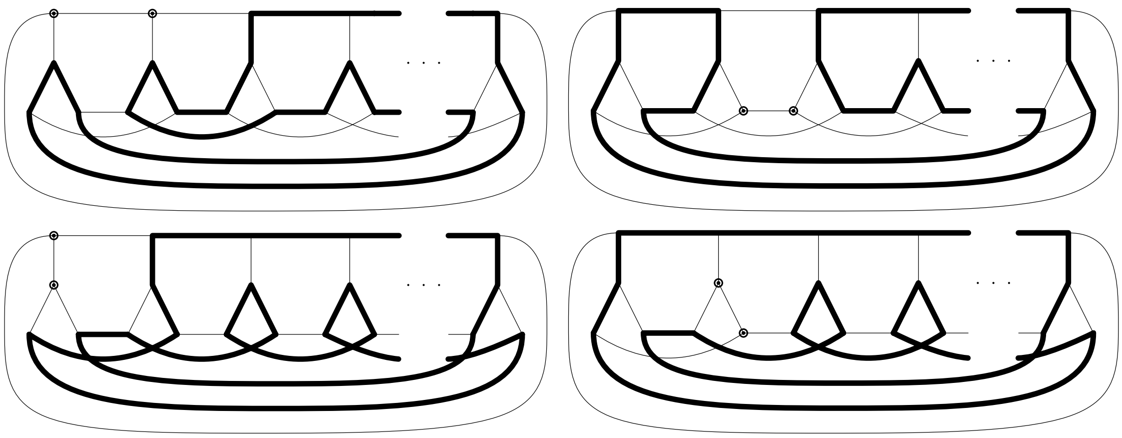

Fig. 5: Every member of $\mathcal{F}$ is K_2-hamiltonian.

In Corollary 2 we have already seen that there are infinitely many cubic non-hamiltonian K_2-hamiltonian graphs of girth 5. By Proposition 1 no cubic non-hamiltonian K_2-hamiltonian graphs of girth 3 or 4 exist, while no such graphs of girth greater than 6 are known—Coxeter's graph, which has girth 7, is not K_2-hamiltonian. From Proposition 4 we obtain the following result, as Clark and Entringer [8] proved that every member of $\mathcal{F}$ is maximally non-hamiltonian.

Corollary 5. *There exist infinitely many maximally non-hamiltonian K_2-hamiltonian critical snarks of girth* 6.

3 Operations preserving K_2-hamiltonicity

We present three operations on graphs designed to obtain new non-hamiltonian K_2-hamiltonian from given graphs with certain hamiltonian properties. The first such operation is a variation of a gluing result on hypohamiltonian graphs due to Thomassen [35] and motivated by our desire to describe non-hamiltonian K_2-hamiltonian graphs with as few hamiltonian vertex-deleted subgraphs as possible: as discussed above, every member of $\mathcal{P} \cup \mathcal{F}$ is non-hamiltonian and K_2-hamiltonian, but is K_1-hamiltonian as well, which was perhaps to be expected. Thus, these graphs constitute solutions to ($\mathfrak{K}$), but cannot be counterexamples to ($\mathfrak{G}$) as this would require the circumference to be 2 less than the graph's order. With this operation, we show that non-hamiltonian K_2-hamiltonian graphs are not necessarily K_1-hamiltonian. Thereafter, we discuss under which conditions the well-known "dot product", which is frequently used in the context of snarks but has also been applied to settle a question of McKay on planar hypohamiltonian graphs [16], can be used for K_2-hamiltonicity. Lastly, we present an operation based on the hamiltonian properties of the dodecahedron which not only preserves 3-regularity, but also yields non-hamiltonian K_2-hamiltonian graphs with fewer hamiltonian vertex-deleted subgraphs than in the graphs inferred from the first operation.

3.1 A variation of a gluing lemma of Thomassen

In a 2-connected non-hamiltonian graph G, we call $\mathrm{exc}(G) \subset V(G)$, which contains every vertex v of G such that $G - v$ is non-hamiltonian, the set of *exceptional vertices* of G. A 2-connected graph G is *k-hypohamiltonian* if G is non-hamiltonian and $|\mathrm{exc}(G)| = k < |V(G)|$. Every non-hamiltonian K_2-hamiltonian graph G with $\mathrm{exc}(G) = V(G)$ is a counterexample to ($\mathfrak{G}$).

Lemma 2. *Let G_1 and G_2 be disjoint non-hamiltonian K_2-hamiltonian graphs. For $i \in \{1,2\}$, let G_i contain a 3-cut X_i and X_i-fragments F_i and F'_i such that for each $x \in X_i$ there is a hamiltonian path in $F_i - x$ and in $F'_i - x$ between the two vertices of $X_i - x$. This is fulfilled e.g. when X_i is non-trivial, or $\mathrm{exc}(G_i) \cap X_i = \emptyset$. Then, if both F_1 and F_2 are non-trivial, or both F_i and F'_{3-i} are trivial, then $(F_1, X_1) : (F_2, X_2)$ is K_2-hamiltonian, but not hamiltonian.*

Proof. Put $X_i = \{x_{i1}, x_{i2}, x_{i3}\}$ such that the bijection between X_1 and X_2 identifies x_{1j} and x_{2j}. Let G denote the graph we obtain through this identification. We see F_1 and F_2 as subgraphs of G, write x_j for the vertex obtained when identifying x_{1j} and x_{2j}, and put $X = \{x_1, x_2, x_3\}$. Suppose (reductio ad absurdum) that G contains a hamiltonian cycle $\mathfrak{h}$. In the following we shall make frequent and tacit use of Proposition 1 (iii), and of the fact that if F_i is trivial, F'_i is not. Either $\mathfrak{h}[V(F_1)]$ is a hamiltonian path of F_1 between two vertices of X or $\mathfrak{h}[V(F_2)]$ is a hamiltonian path of F_2 between two vertices of X. Without loss of generality we may suppose that F_1 contains a hamiltonian x_2x_3-path $\mathfrak{p}_1$. As G_1 is non-hamiltonian and K_2-hamiltonian, we have $c(G_1 - X_1) = 2$ (Proposition 1 (ii)), so the X_1-fragments of G_1 are F_1 and F'_1. In G_1, by hypothesis, $F'_1 - x_{11}$ contains a hamiltonian $x_{12}x_{13}$-path $\mathfrak{p}'_1$. But then, seeing F_1 now as lying in G_1, we obtain the hamiltonian cycle $\mathfrak{p}_1 \cup \mathfrak{p}'_1$ in G_1, a contradiction.

Next we show that G is K_2-hamiltonian. Let ab be an arbitrary edge of G. We need to prove that $G - a - b$ is hamiltonian. We distinguish between two cases depending on the position of the vertices a, b. By the definition of G, it is not possible that one of a and b is in $F_1 - X$ and the other is in $F_2 - X$. By Proposition 1 (ii), $a, b \in X$ is also impossible.

Case 1. $a, b \notin X$. We may suppose without loss of generality that $a, b \in V(F_2) \setminus X$. Since G_2 is K_2-hamiltonian, there is a hamiltonian cycle $\mathfrak{h}$ in $G_2 - a - b$. Hence $\mathfrak{h}[V(F_2) \setminus \{a, b\}]$ is either a hamiltonian path in $F_2 - a - b$ between two vertices of X_2, say x_{21} and x_{23}, or the disjoint union of a path between two vertices of X_2, say x_{21} and x_{23}, and the isolated vertex x_{22}. In fact, the latter case is impossible, since this would imply that $\mathfrak{p} = \mathfrak{h}[V(F'_2)]$ is a hamiltonian $x_{21}x_{23}$-path in F'_2, and therefore the union of $\mathfrak{p}$ and a hamiltonian $x_{21}x_{23}$-path in $F_2 - x_{22}$ (guaranteed to exist by the conditions of the lemma) would be a hamiltonian cycle of G_2. (Note that while we essentially used Proposition 1 (iii), we needed the above argument, since X could be a trivial 3-cut.) Thus, we know that $\mathfrak{q}_2 = \mathfrak{h}[V(F_2) \setminus \{a,b\}]$ is a hamiltonian $x_{21}x_{23}$-path of $F_2 - a - b$. By the conditions of the lemma, $F_1 - x_{12}$ contains a hamiltonian $x_{11}x_{13}$-path $\mathfrak{q}_1$. Now $\mathfrak{q}_1 \cup \mathfrak{q}_2$ is a hamiltonian cycle of $G - a - b$.

Case 2. $a \in X$, $b \notin X$. Without loss of generality we may assume that $b \in V(F_2) \setminus X$. There are two subcases we need to distinguish: first, assume that F_2 is non-trivial. Note that in this situation F_1 may be trivial or non-trivial. Since G_2 is K_2-hamiltonian, there is a hamiltonian cycle $\mathfrak{h}$ in $G_2 - a - b$. Then $\mathfrak{p}_2 = \mathfrak{h}[V(F_2) \setminus \{a,b\}]$ is a hamiltonian $x_{2i}x_{2j}$-path of $F_2 - a - b$ for appropriate i, j. By the conditions of the lemma, $F_1 - a$ contains a hamiltonian $x_{1i}x_{1j}$-path which together with $\mathfrak{p}_2$ yields a hamiltonian cycle of $G - a - b$. Second, suppose that F_2 is trivial. Then, by hypothesis, F_1' is trivial as well. Thus, as G_1 is K_2-hamiltonian, $F_1 - a$ contains a hamiltonian cycle $\mathfrak{h}'$. Considering $\mathfrak{h}'$ now as lying in G we have proven that $G - a - b$ is hamiltonian. □

Denote the family of all k-hypohamiltonian graphs by $\mathcal{H}_k$. The next lemma is a generalisation of Thomassen's [35, Corollary 1] and a slightly stronger version of the author's [42, Theorem 6]—the proof of this strengthening is identical to the one given in [42], so we omit it.

Lemma 3. *Let i, j be non-negative integers and consider disjoint graphs $G \in \mathcal{H}_i$ and $H \in \mathcal{H}_j$. We require G and H to contain cubic vertices x and y, respectively, such that $N(x) \cap \text{exc}(G) = \emptyset$ and $N(y) \cap \text{exc}(H) = \emptyset$. Let F_G (F_H) be the non-trivial $N(x)$-fragment ($N(y)$-fragment) of G (H). Then $(F_G, N(x)) \colon (F_H, N(y)) \in \mathcal{H}_{i+j}$.*

Proposition 5 (Goedgebeur [14]). *The smallest non-hypohamiltonian K_2-hamiltonian snark has 26 vertices (see Fig. 6). It has two exceptional vertices. Furthermore, there exists a K_2-hamiltonian snark with 28 vertices which has exactly one exceptional vertex.*

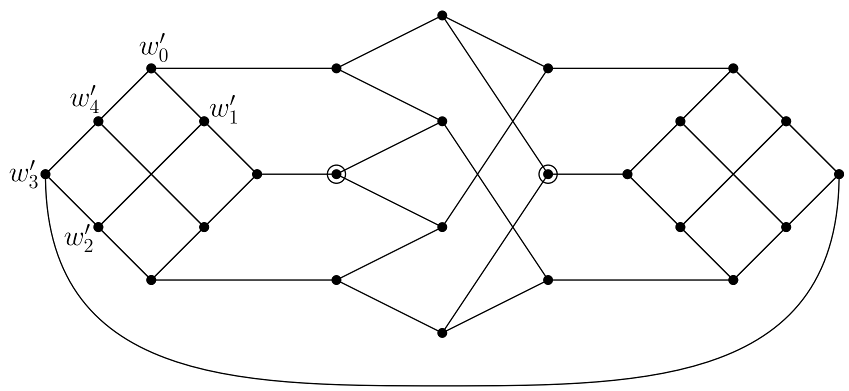

Fig. 6: A K_2-hamiltonian snark on 26 vertices.
Its two exceptional vertices are circled.

Theorem 1. *For every non-negative integer k there exists an infinite family of non-hamiltonian K_2-hamiltonian graphs G with $|\text{exc}(G)| = k$.*

Proof. For $k = 0$ the statement is given by Proposition 2 and Corollary 3, so henceforth we consider the case $k \geq 1$. For this purpose we first describe for every $k \geq 1$ a non-hamiltonian K_2-hamiltonian graph G with $|\text{exc}(G)| = k$.

By Proposition 5, there exists a K_2-hamiltonian snark Γ with 28 vertices which has exactly one exceptional vertex. Consider a copy Γ' of Γ, and denote by w' and w their respective exceptional vertices. Consider $x \in (V(\Gamma) \setminus N[w])$ and $y \in (V(\Gamma') \setminus N[w'])$. Let F (F') be the non-trivial $N(x)$-fragment ($N(y)$-fragment) of Γ (Γ'). Then, by Lemma 3, $(F, N(x)) \vdots (F', N(y)) \in \mathcal{H}_2$ and by Lemma 2, this graph is also K_2-hamiltonian. Since in each step many non-exceptional vertices are added, we can iterate this procedure and obtain for every $k \geq 1$ a K_2-hamiltonian graph $\Gamma_k \in \mathcal{H}_k$. (In order to obtain Γ_k, we apply the procedure $k-1$ times.)

Let $\{G_\ell\}_{\ell \in \mathbb{N}}$ be an infinite family of hypohamiltonian K_2-hamiltonian graphs, each containing a cubic vertex x_ℓ (e.g. $\mathcal{P}$). Let F_ℓ be the non-trivial $N(x_\ell)$-fragment of G_ℓ. By construction, Γ_k contains a cubic vertex y_k such that no vertex of $N(y_k)$ is exceptional. Let F'_k be the non-trivial $N(y_k)$-fragment of Γ_k. Then, once more invoking Lemmas 2 and 3, $\{(F_\ell, N(x_\ell)) \vdots (F'_k, N(y_k))\}_{\ell,k \in \mathbb{N}}$ yields the statement. □

The graphs described in the proof of Theorem 1 provide an additional (to the ones from Proposition 3) infinite family of non-hamiltonian K_2-hamiltonian graphs which are not cubic.

3.2 The dot product

The second operation we want to discuss is the well-known dot product of two graphs. Let G and H be disjoint graphs on at least six vertices. For independent edges ab, cd in G and adjacent cubic vertices x and y in H, consider $G' = G - ab - cd$ and $H' = H - x - y$, and let a', b' be the neighbours of x in $H - y$ and c', d' be the neighbours of y in $H - x$. Then the *dot product* $G \cdot H$ is defined as the graph

$$(V(G) \cup V(H'), E(G') \cup E(H') \cup \{aa', bb', cc', dd'\}).$$

Proposition 6. *Let G and H be disjoint non-hamiltonian graphs with $a, b, c, d \in V(G)$ and $a', b', c', d', x, y \in V(H)$ as introduced above—in particular, x and y are cubic—, $a, b \notin N_G(c) \cup N_G(d)$, $a' \notin N_H(b')$, and $c' \notin N_H(d')$. If*

(i) *for any $vw \in E(G)$ there exists in $G - v - w$ a hamiltonian ab-path not containing cd or a hamiltonian cd-path not containing ab;*

(ii) *for any $v \in \{a, b\}$ and $w \in \{c, d\}$, the graph G admits a hamiltonian vw-path containing neither ab nor cd;*

(iii) *$G - a$ and $G - b$ contain a hamiltonian cycle through cd, and $G - c$ and $G - d$ contain a hamiltonian cycle through ab;*

(iv) *$H - x$ and $H - y$ are hamiltonian, and*

(v) *for any $vw \in E(H)$ with $v, w \notin \{x, y\}$ there exists in $H - v - w$ a hamiltonian st-path with $s \in \{a', b'\}$ and $t \in \{c', d'\}$,*

then $G \cdot H$ is non-hamiltonian and K_2-hamiltonian.

Proof. We see $G - ab - cd$ and $H - x - y$ as subgraphs of $G \cdot H$. We first show that $G \cdot H$ is non-hamiltonian. Assume $G \cdot H$ contains a hamiltonian cycle $\mathfrak{h}$. $\mathfrak{h} \cap (H - x - y)$ must be either an $a'b'$-path or a $c'd'$-path, as otherwise we would obtain a contradiction: either a simple modification of $\mathfrak{h} \cap (H - x - y)$ would yield a hamiltonian cycle in H, or $\mathfrak{h} \cap (H - x - y)$ consists of an $a'b'$-path P and a $c'd'$-path Q such that $P \cap Q = \emptyset$ and $P \cup Q$ spans $H - x - y$, but then $(\mathfrak{h} \cap (G - ab - cd)) + ab + cd$ is a hamiltonian cycle in G. Without loss of generality let $\mathfrak{h} \cap (H - x - y)$ be an $a'b'$-path. This implies that there exists a hamiltonian ab-path $\mathfrak{p}$ in $G - ab - cd$. In this case $\mathfrak{p} + ab$ would be a hamiltonian cycle in G, yet again a contradiction.

In the remainder of the proof it is shown that $G \cdot H$ is K_2-hamiltonian. Consider $vw \in E(G)$. By (i) there exists in $G - v - w$ a hamiltonian ab-path not containing cd or a hamiltonian cd-path not containing ab; assume the former and call it $\mathfrak{p}$. By (iv), $H - x - y$ contains a hamiltonian $a'b'$-path $\mathfrak{q}$. Now $\mathfrak{p} \cup \mathfrak{q}$ is a hamiltonian cycle in $G \cdot H - v - w$. The case when there exists in $G - v - w$ a hamiltonian cd-path not containing ab is analogous.

By (iii) there is a hamiltonian cd-path $\mathfrak{p}$ in $G - a$, and (iv) implies that there is a hamiltonian $c'd'$-path $\mathfrak{q}$ in $H - x - y$. Hence $\mathfrak{p} \cup \mathfrak{q}$ is a hamiltonian cycle in $G \cdot H - a - a'$. The cases when removing b, b' or c, c' or d, d' can be dealt with in the same way.

Consider $vw \in E(H)$ with $v, w \notin \{x, y\}$. By (v) there exists in $H - v - w$ a hamiltonian $s't'$-path $\mathfrak{p}$ with $s' \in \{a', b'\}$ and $t' \in \{c', d'\}$. Let $N_{G \cdot H}(s') \cap V(G) = \{s\}$ and $N_{G \cdot H}(t') \cap V(G) = \{t\}$. By (ii) there exists a hamiltonian st-path containing neither ab nor cd, which together with $\mathfrak{p}$ forms a hamiltonian cycle in $G \cdot H - v - w$. □

The many requirements imposed on G and H in Proposition 6 make it impractical. The most natural choice would be to consider the dot product of two copies of Petersen's graph. There are two ways to do so, yielding two non-isomorphic graphs: the Blanuša snarks. Unfortunately, not all of the lemma's conditions are met, and these two snarks turn out not to be K_2-hamiltonian. Lemma 2 is much more widely applicable but does not conserve 3-regularity. We therefore now introduce an operation without too restrictive requirements, yet maintaining 3-regularity.

3.3 Inserting a dodecahedron

Let $C = v_0 \dots v_4$ be a 5-cycle in the 1-skeleton of the dodecahedron. In this and the next paragraph, indices are to be considered mod. 5. Insert on each edge $v_i v_{i+1}$ a new vertex w_i. Let G be a graph containing a 5-cycle $w'_0 \dots w'_4$. For each i, identify $w'_i w'_{i+1}$ with the path $w_i v_{i+1} w_{i+1}$ in order to obtain $P_C(G)$. The relevant part of $P_C(G)$ is illustrated in Fig. 7.

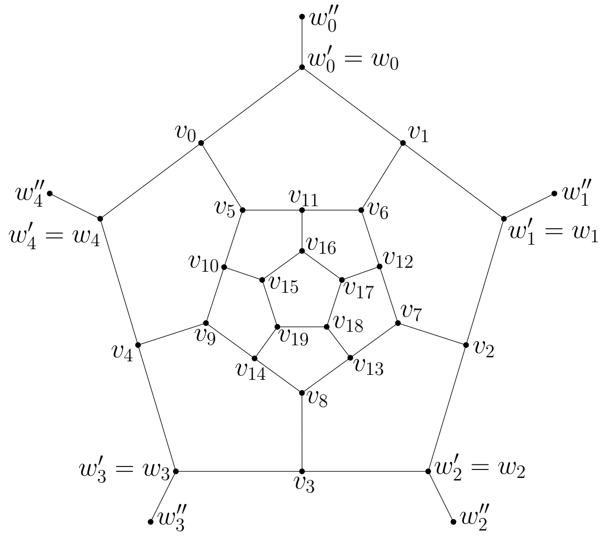

Fig. 7: The image of the 5-cycle $w'_0 \dots w'_4$ under the operation P_C.

In a graph G, a 5-cycle $C' = w'_0 \dots w'_4$ composed of cubic vertices, and such that w''_i is the neighbour of w'_i not on C' (see Fig. 7) is called *extendable* if for any i, (i) there exists a hamiltonian cycle $\mathfrak{h}$ in $G - w'_i$ with $w'_{i-2}w'_{i+2} \notin E(\mathfrak{h})$ and (ii) there exists a hamiltonian cycle $\mathfrak{h}'$ in $G - w''_i$ with $\mathfrak{h}' \cap C' = w'_{i-2}w'_{i-1}w'_iw'_{i+1}w'_{i+2}$.

Theorem 2. *Let G be a K_2-hamiltonian graph containing an extendable 5-cycle C. Then $P_C(G)$ is a K_2-hamiltonian graph containing an extendable 5-cycle. If G is non-hamiltonian or cubic, then so is $P_C(G)$, respectively. If G has girth at least 5, then $P_C(G)$ has girth 5. If G is plane and C a facial cycle in G, then $P_C(G)$ is planar.*

Proof. In the entirety of this proof, we refer to Fig. 7 and the notation introduced therein. Moreover, when considering a cycle D in G, we shall frequently switch to seeing it as a cycle in $P_C(G)$, where if an edge $w'_iw'_{i+1}$ of C belongs to D it shall be replaced by its corresponding path $w'_iv_{i+1}w'_{i+1}$ in $P_C(G)$, indices mod. 5. We first show that $P_C(G)$ is K_2-hamiltonian. Consider $vw \in E(P_C(G))$. With the following seven cases, we cover all situations (up to symmetry).

Case 1. $v, w \notin \{w_0, \ldots, w_4, v_0, \ldots, v_{19}\}$. Let $\mathfrak{h}$ be a hamiltonian cycle in $G-v-w$. There are two essentially different cases to consider: Case 1.1. $\mathfrak{h} \cap C$ is the path $w'_0w'_1w'_2w'_3w'_4$, which, if we see $\mathfrak{h}$ as lying in $P_C(G)$, becomes the path $w'_0v_1w'_1v_2w'_2v_3w'_3v_4w'_4$. In $P_C(G)$, replace in this path the edge w'_0v_1 by

$$w'_0v_0v_5v_{10}v_9v_{14}v_8v_{13}v_7v_{12}v_{17}v_{18}v_{19}v_{15}v_{16}v_{11}v_6v_1.$$

Case 1.2. $\mathfrak{h} \cap C$ is the union of the paths $w'_0v_0w'_4$ and $w'_1v_2w'_2v_3w'_3$. Replace in the former the edge w'_0v_0 and in the latter the edge $v_3w'_3$ by

$$w'_0v_1v_6v_{12}v_7v_{13}v_{18}v_{17}v_{16}v_{11}v_5v_0 \text{ and } v_3v_8v_{14}v_{19}v_{15}v_{10}v_9v_4w'_3,$$

respectively.

Case 2. $v = w'_0$, $w = w''_0$. By property (ii) of C, there exists a hamiltonian cycle $\mathfrak{h}$ in $G - w''_0$ such that $\mathfrak{h} \cap C = w'_3w'_4w'_0w'_1w'_2$ which corresponds to $w'_3v_4w'_4v_0w'_0v_1w'_1v_2w'_2$ in $P_C(G)$. Replace in $\mathfrak{h}$ the path $v_0w'_0v_1w'_1v_2w'_2$ by

$$v_0v_5v_{11}v_{16}v_{17}v_{12}v_6v_1w'_1v_2v_7v_{13}v_{18}v_{19}v_{15}v_{10}v_9v_{14}v_8v_3w'_2$$

and we obtain a hamiltonian cycle in $P_C(G) - w'_0 - w''_0$ as desired.

Case 3. $v = v_0$, $w = w'_0$. As G is K_2-hamiltonian, there exists a hamiltonian cycle in $G - w'_2 - w'_3$ whose intersection with C is the path $w'_1w'_0w'_4$ which corresponds to $w'_1v_1w'_0v_0w'_4$ in $P_C(G)$. Replace in $\mathfrak{h}$ the path $v_1w'_0v_0w'_4$ by

$$v_1v_6v_{12}v_{17}v_{18}v_{19}v_{15}v_{16}v_{11}v_5v_{10}v_9v_{14}v_8v_{13}v_7v_2w'_2v_3w'_3v_4w'_4.$$

Case 4. $v = v_0$, $w = v_5$. By property (i) of C, there exists a hamiltonian cycle $\mathfrak{h}$ in $G - w'_1$ such that $\mathfrak{h} \cap C$ is composed of the paths $w'_0w'_4$ and $w'_2w'_3$ which correspond to $w'_0v_0w'_4$ and $w'_2v_3w'_3$ in $P_C(G)$, respectively. Replace in $\mathfrak{h}$, now seen as lying in $P_C(G)$, the former path by

$$w'_0v_1w'_1v_2v_7v_{13}v_8v_{14}v_{19}v_{18}v_{17}v_{12}v_6v_{11}v_{16}v_{15}v_{10}v_9v_4w'_4,$$

while the latter path remains unchanged.

Case 5. $v = v_5$, $w = v_{11}$. As G is K_2-hamiltonian, there exists a hamiltonian cycle in $G - w'_1 - w'_2$ whose intersection with C is the path $w'_0w'_4w'_3$ which corresponds to $w'_0v_0w'_4v_4w'_3$ in $P_C(G)$. Replace in this path the edge $v_4w'_3$ by

$$v_4v_9v_{10}v_{15}v_{16}v_{17}v_{18}v_{19}v_{14}v_8v_{13}v_7v_{12}v_6v_1w'_1v_2w'_2v_3w'_3.$$

Case 6. $v = v_{11}$, $w = v_{16}$. By property (i) of C, there exists a hamiltonian cycle $\mathfrak{h}$ in $G - w'_2$ such that $\mathfrak{h} \cap C$ is composed of the paths $w'_0 w'_1$ and $w'_3 w'_4$ which correspond to $w'_0 v_1 w'_1$ and $w'_3 v_4 w'_4$ in $P_C(G)$. Replace in $\mathfrak{h}$ the edges $w'_0 v_1$ and $w'_3 v_4$ by

$$w'_0 v_0 v_5 v_{10} v_{15} v_{19} v_{18} v_{17} v_{12} v_6 v_1 \text{ and } w'_3 v_3 w'_2 v_2 v_7 v_{13} v_8 v_{14} v_9 v_4,$$

respectively.

Case 7. $v = v_{15}$, $w = v_{16}$. As G is K_2-hamiltonian, there exists a hamiltonian cycle in $G - w'_0 - w'_1$ whose intersection with C is the path $w'_2 w'_3 w'_4$ which corresponds to $w'_2 v_3 w'_3 v_4 w'_4$ in $P_C(G)$. Replace in this path the edge $v_4 w'_4$ by

$$v_4 v_9 v_{10} v_5 v_{11} v_6 v_{12} v_{17} v_{18} v_{19} v_{14} v_8 v_{13} v_7 v_2 w'_1 v_1 w'_0 v_0 w'_4.$$

We now show that $P_C(G)$ contains an extendable 5-cycle, namely $C' = v_{15} \ldots v_{19}$. We first verify property (i) for C', i.e. prove that there exists a hamiltonian cycle $\mathfrak{h}$ in $P_C(G) - v_{16}$ such that $v_{18} v_{19} \notin E(\mathfrak{h})$. (The treatment of the other vertices of C' is analogous.) By property (i) of C, there exists a hamiltonian cycle in $G - w'_1$ whose intersection with C are the paths $w'_0 w'_4$ and $w'_2 w'_3$ which correspond to $w'_0 v_0 w'_4$ and $w'_2 v_3 w'_3$ in $P_C(G)$. In $P_C(G)$, replace in these paths the edges $w'_0 v_0$ and $v_3 w'_3$ by

$$w'_0 v_1 w'_1 v_2 v_7 v_{13} v_{18} v_{17} v_{12} v_6 v_{11} v_5 v_0 \text{ and } v_3 v_8 v_{14} v_{19} v_{15} v_{10} v_9 v_4 w'_3,$$

respectively. Let us prove property (ii) for C' by showing that there exists a hamiltonian cycle in $P_C(G) - v_{11}$ whose intersection with C' is $v_{18} v_{17} v_{16} v_{15} v_{19}$. (The treatment of the vertices $v_{10}, v_{12}, v_{13}, v_{14}$ is analogous.) As above, since C satisfies property (i), there exists a hamiltonian cycle $\mathfrak{h}$ in $G - w'_1$ such that $w'_3 w'_4 \notin E(\mathfrak{h})$. In $P_C(G)$, replace in $w'_0 v_0 w'_4$ and $w'_2 v_3 w'_3$ the edges $v_0 w'_4$ and $w'_2 v_3$ by

$$v_0 v_5 v_{10} v_9 v_4 w'_4 \quad \text{and} \quad w'_2 v_2 w'_1 v_1 v_6 v_{12} v_7 v_{13} v_{18} v_{17} v_{16} v_{15} v_{19} v_{14} v_8 v_3,$$

respectively.

It remains to prove that if G is non-hamiltonian, then so is $P_C(G)$. Put $R := P_C(G)[\{v_5, \ldots, v_{19}\}]$ and $R' := P_C(G)[\{v_0, \ldots, v_{19}\}]$. Assume $P_C(G)$ does contain a hamiltonian cycle $\mathfrak{h}$ (reductio ad absurdum). We leave to the reader the straightforward verification of the fact that $\mathfrak{h}$ can only intersect R in two ways (up to symmetry): either in (a) a spanning $v_5 v_6$-path P or (b) two disjoint paths Q_1 and Q_2 which together span R, the former a $v_5 v_6$-path and the latter a $v_7 v_8$-path.

(a) As $\mathfrak{h}$ is a hamiltonian cycle, without loss of generality $v_6 v_1 w'_0$ and $v_5 v_0 w'_4 v_4 w'_3 v_3 w'_2 v_2 w'_1$ lie in $\mathfrak{h}$. Then $\mathfrak{h} - R'$ together with the path $w'_0 w'_4 w'_3 w'_2 w'_1$ yields a hamiltonian cycle in G, a contradiction.

(b) Either the path $v_5 v_0 w'_0$ or the path $v_6 v_1 w'_0$ must lie in $\mathfrak{h}$. Suppose the former is true (treating the latter case is analogous). This implies that the paths $v_8 v_3 w'_3 v_4 w'_4$, $v_6 v_1 w'_1$, and $v_7 v_2 w'_2$ must lie in $\mathfrak{h}$. But then, by considering $\mathfrak{h} - R'$ together with the edge $w'_0 w'_1$ and the path $w'_2 w'_3 w'_4$ we would obtain a hamiltonian cycle in G, a contradiction. □

Variations of condition (ii) in the definition of an extendable 5-cycle C' are possible, e.g. one might relax it to: (ii') there exists a hamiltonian cycle $\mathfrak{h}'$ in $G - w''_i$ such that $\mathfrak{h}' \cap C'$ is one of the paths

$$w'_{i-2} w'_{i+2} w'_{i+1} w'_i w'_{i-1}, \; w'_{i-2} w'_{i-1} w'_i w'_{i+1} w'_{i+2}, \; w'_{i+1} w'_i w'_{i-1} w'_{i-2} w'_{i+2}$$

or there exists a hamiltonian cycle $\mathfrak{h}''$ in $G - w'_i - w''_i$ such that $\mathfrak{h}'' \cap C'$ has one component. However, this did not seem to substantially increase its applicability, so we opted for the present version which is easier to handle.

An example of a graph to which Theorem 2 can be applied is Goedgebeur's snark and its 5-cycle $w'_0 \dots w'_4$, see Fig. 6. It is however noteworthy that when applying the operation P_C to a cubic graph with chromatic index 4, the resulting graph may have chromatic index 3. This occurs for instance if P_C is applied to (any 5-cycle of) Petersen's graph.

We now present an application of Theorem 2. Let $\mathcal{G}$ be the family of all non-hamiltonian K_2-hamiltonian graphs. In the light of ($\mathfrak{G}$), we set

$$\rho := \sup_{G \in \mathcal{G}} \frac{|\mathrm{exc}(G)|}{|V(G)|}.$$

Furthermore, let ρ_3 be defined in the same way but restricted to cubic graphs. By Proposition 5 there is a K_2-hamiltonian snark Γ on 26 vertices containing exactly two exceptional vertices, and proceeding as in the proof of Theorem 1, from Γ we obtain Γ_k, which has $2k$ exceptional vertices and order $21k + 5$. Therefore $\rho \geq \frac{2}{21}$. But with the operation P_C a better bound can be proven:

Corollary 6. *We have $\rho_3 \geq \frac{1}{4}$.*

Proof. Applying, iteratively, the operation P_C to any 5-cycle C of the Petersen graph G—the straightforward verification that C indeed satisfies conditions (i) and (ii) is left to the reader—by Theorem 2 an infinite family of cubic non-hamiltonian K_2-hamiltonian graphs is obtained.

Let $P_C^k(G)$ be the graph resulting from a k-fold application of P_C to G (with $P_C^0(G) = G$). We now investigate the occurrence of non-hamiltonian vertex-deleted subgraphs in $P_C^k(G)$ for $k \geq 1$. Using the notation from Fig. 7, we show that $P_C^k(G) - v_0$ is non-hamiltonian. Assume this graph does contain a hamiltonian cycle $\mathfrak{h}$ (reductio ad absurdum). We argue as in the proof of Theorem 2. Define $R := P_C^k(G)[\{v_5, \dots, v_{19}\}]$ and $R' := P_C^k(G)[\{v_0, \dots, v_{19}\}]$. Then $\mathfrak{h}$ can only intersect R in three ways (up to symmetry): either in (a) a spanning v_6v_7-path P or (b) a spanning v_7v_8-path P' or (c) two disjoint paths Q_1 and Q_2 which together span R, the former a v_6v_7-path and the latter a v_8v_9-path.

(a) As $\mathfrak{h}$ is a hamiltonian cycle in $P_C^k(G) - v_0$, we have $v_6v_1w'_0 \subset \mathfrak{h}$ and thus $v_7v_2w'_1 \subset \mathfrak{h}$. But then the path $w'_2v_3w'_3v_4w'_4$ is also contained in $\mathfrak{h}$, and by adding to $\mathfrak{h} - R'$ (seen as lying in $P_C^{k-1}(G)$) the edges $w'_0w'_1, w'_2w'_3, w'_3w'_4$, we obtain a hamiltonian cycle in $P_C^{k-1}(G)$, a contradiction.

(b) We either have (1) $v_7v_2w'_1v_1w'_0 \subset \mathfrak{h}$ and thus $v_8v_3w'_2 \subset \mathfrak{h}$ and $w'_3v_4w'_4 \subset \mathfrak{h}$, or (2) $v_7v_2w'_2 \subset \mathfrak{h}$ and thus $w'_0v_1w'_1 \subset \mathfrak{h}$ and $v_8v_3w'_3v_4w'_4 \subset \mathfrak{h}$. In the former case, by adding to $\mathfrak{h} - R'$ the edges $w'_0w'_1, w'_1w'_2, w'_3w'_4$, we obtain a hamiltonian cycle in $P_C^{k-1}(G)$, a contradiction. Case (2) can be dealt with in the same way.

(c) We have that $v_6v_1w'_0$ is a subpath of $\mathfrak{h}$, and thus also $v_7v_2w'_1$, $v_8v_3w'_2$, and either $v_9v_4w'_3$ or $v_9v_4w'_4$, whence $w'_4 \notin V(\mathfrak{h})$ or $w'_3 \notin V(\mathfrak{h})$, respectively, which is absurd.

We have proven that $P_C^k(G) - v_0$ is non-hamiltonian, and with analogous arguments it can be shown that $P_C^k(G) - v_i$ is non-hamiltonian for $i \in \{1, 2, 3, 4\}$. Therefore, as exceptional vertices remain exceptional after applying P_C, we have $|\mathrm{exc}(P_C^k(G))| \geq 5k$, and since $|V(P_C^k(G))| = 10 + 20k$, the proof is complete. $\square$

4 The planar case

In this section, we investigate the structural properties of planar K_2-hamiltonian graphs. By Euler's formula and Proposition 1, we have:

Corollary 7. *Every planar cubic K_2-hamiltonian graph of order n contains an $(n-1)$-cycle or an n-cycle. Equivalently, planar cubic K_2-hamiltonian graphs of order n and circumference $n-2$ do not exist. In particular, among planar cubic graphs there exist no counterexamples to* ($\mathfrak{G}$).

The motivation for the following theorem is threefold. (1) In 1978 Thomassen showed that every planar hypohamiltonian graph contains a cubic vertex [35]. With a different approach than his, we now prove a K_2-hamiltonian analogue of this result. (2) Thomas and Yu [32] showed that in a planar 4-connected graph the removal of any pair of vertices yields a hamiltonian graph, so the family of all planar 4-connected graphs is a subclass of the family $\mathcal{K}$ of all planar K_2-hamiltonian graphs of minimum degree at least 4, and it is not difficult to see that these families do not coincide. In the following theorem we settle affirmatively the natural question whether every member of $\mathcal{K}$ is hamiltonian, thus extending Tutte's classic theorem that planar 4-connected graphs are hamiltonian. (3) Nelson [28] observed that it follows from Tutte's paper [37] that planar 4-connected graphs are not only hamiltonian, but K_1-hamiltonian. This was extended by Thomas and Yu [32] who showed that the removal of any set of at most two vertices from a planar 4-connected graph yields a hamiltonian graph. Sanders [30] proved that every planar 4-connected graph is K_3-hamiltonian. By Euler's formula, every planar 4-connected graph contains a triangle, so these results imply that every planar 4-connected graph contains a cycle of length $n-1, n-2$, and $n-3$. We will show that the same holds for every graph in $\mathcal{K}$.

Our proof strategy follows the same lines as the one used by the author in [43]. The following lemma is a special case of a central lemma from [4]. Alternatively, its statement (i) can also be inferred from the Sanders-Thomas-Yu "Three Edge Lemma" (see (2.7) in [32] or [30]) while its statement (ii) follows from Sanders' result that in a planar 4-connected graph there exists a hamiltonian cycle through any two of its edges [31].

Lemma 4. *Let G be a plane 4-connected graph and let $\Delta = xyz$ be a triangular face in G. Then there is a hamiltonian yz-path*

(i) *in $G - x$ containing no edge of Δ, and*

(ii) *in G containing no edge of Δ.*

Theorem 3. *If in a planar graph G of order n, size m, and minimum degree at least 4 at least $m-8$ K_2-deleted subgraphs are hamiltonian, then G is hamiltonian. If at least $m-2$ K_2-deleted subgraphs of G are hamiltonian, then G must also contain an $(n-1)$-cycle and an $(n-3)$-cycle.*

Proof. We begin by proving the first statement. By Tutte's result that planar 4-connected graphs are hamiltonian [37], only the connectivity 2 and connectivity 3 cases remain to be settled. Consider the former. Then G contains a 2-cut $\{v, w\}$. The removal of v or w and one of its neighbours (which may be v or w) cannot yield a hamiltonian graph. v and w are incident with at least seven edges. It is easy to see that each component of $G - v - w$ contains at least two edges and therefore at least three vertices. Thus, each of these components contains adjacent vertices whose removal yields a hamiltonian graph in which $\{v, w\}$ is a 2-cut. So we have a hamiltonian vw-path in each of the $\{v, w\}$-fragments (of which, thus, there must be exactly two). Their union yields the desired hamiltonian cycle in G.

We now treat the 3-connected case. It is well-known (for a proof, see [4]) that a planar graph of connectivity 3 contains a 3-cut $X = \{x, y, z\}$ such that for at least one of the X-fragments F and F' (as $K_{3,3}$ is non-planar there are exactly two such fragments), say F, the graph $\overline{F} = F + xy + yz + zx$ is either K_4 or 4-connected. As G has minimum degree at least 4, the former case is impossible, so $\overline{F}$ is 4-connected and clearly planar. Such a planar 4-connected graph has size at least 12. However, we do not know whether the edges xy, yz, zx are present in G, so the size of F is at least 9. Therefore, F contains an edge vw such that $G - v - w$ is hamiltonian. Since X is a 3-cut, the hamiltonicity of $G - v - w$ implies, ignoring analogous cases, that there exists either a hamiltonian yz-path $\mathfrak{p}'$ in $F' - x$ (for $v = x$ or $w = x$) or F' (for $v, w \notin X$). Let us now treat these two situations.

Case 1. *There is a hamiltonian yz-path $\mathfrak{p}'$ in $F' - x$.* By Lemma 4 (ii) we have a hamiltonian yz-path $\mathfrak{p}$ in $\overline{F}$ using none of the edges xy, yz, zx. Then $\mathfrak{p} \cup \mathfrak{p}'$ is a hamiltonian cycle in G.

Case 2. *There is a hamiltonian yz-path $\mathfrak{p}'$ in F'.* By Lemma 4 (i) there is a hamiltonian yz-path $\mathfrak{p}$ in $\overline{F} - x$. Since none of the edges xy, yz, zx lie in $\mathfrak{p}$, the path $\mathfrak{p}$ lies in $F - x$. As above, $\mathfrak{p} \cup \mathfrak{p}'$ is a hamiltonian cycle in G.

Thus, the first statement is proven.

Let us show the second statement and begin our reasoning as above. However, we now know that without loss of generality $F' - x - v$ contains a hamiltonian yz-path $\mathfrak{q}'$ for some vertex v in $G - F$ adjacent to x. Using Lemma 4 (ii) as above, we obtain a hamiltonian cycle in $G - v$.

As $G - x - w$ contains a hamiltonian cycle for some vertex w in $G - F'$ adjacent to x, we obtain a hamiltonian yz-path $\mathfrak{q}$ in $F - x - w$. Now $\mathfrak{q} \cup \mathfrak{q}'$ is a hamiltonian cycle in $G - x - v - w$. □

It was shown by Bondy and Jackson [3] that a planar graph containing exactly one hamiltonian cycle has at least two vertices of degree 2 or 3. For the graph G from the proof of Theorem 3 we have shown that G is hamiltonian, and as G has minimum degree at least 4, by the theorem of Bondy and Jackson it must in fact contain at least two hamiltonian cycles.

It follows directly from Theorem 3 that every planar non-hamiltonian K_2-hamiltonian graph contains a cubic vertex. (This is not true if the non-hamiltonicity requirement is dropped, as planar 4-connected graphs are K_2-hamiltonian [32].) Using an approach of Thomassen [35], we now present another proof of this fact and also extend it, showing that at least four cubic vertices must be present.

Theorem 4. *Every planar non-hamiltonian K_2-hamiltonian graph contains four cubic vertices.*

Proof. In this proof we make use of the theorem stating that a planar 3-connected graph containing at most three 3-cuts must be hamiltonian [4], so a planar non-hamiltonian K_2-hamiltonian graph contains at least four 3-cuts. Let G be a smallest planar non-hamiltonian K_2-hamiltonian graph with minimum degree at least 4, so no 3-cut of G is trivial. Since G has more than one 3-cut, by Thomassen's [35, Lemma 3] there exists a non-trivial 3-fragment F of G with fewer than $\frac{n+3}{2}$ vertices, where n denotes the order of G.

Gluing F and a copy of F using Lemma 2 (by identifying, using a bijection, their respective attachments) we obtain a planar non-hamiltonian K_2-hamiltonian graph with minimum degree at least 4 of order smaller than n, which contradicts the minimality of G. (By Proposition 1 (ii), every attachment of F has degree at least 2 in F, and every vertex in F excluding its attachments has degree at least 4 as G has minimum degree at least 4.) We have proven that a planar non-hamiltonian K_2-hamiltonian graph contains at least one cubic vertex.

Now let G be a smallest planar non-hamiltonian K_2-hamiltonian graph containing at most two cubic vertices. As above, we know that G contains a non-trivial X-fragment F of order less than $\frac{n+3}{2}$ and that no cubic vertex of G resides in X. We distinguish between three cases: either F contains no cubic vertex in which case gluing F to a copy of F we obtain a planar non-hamiltonian K_2-hamiltonian graph of minimum degree at least 4, a contradiction to the above paragraph; or F contains exactly one cubic vertex, in which case gluing F to a copy of F yields a planar non-hamiltonian K_2-hamiltonian graph containing at most two cubic vertices, contradicting the minimality of G; or F contains exactly two cubic vertices (which cannot be attachments of F), in which case consider the other fragment of G with attachments X, say F', and glue F' to a copy of F', producing a planar non-hamiltonian K_2-hamiltonian graph of minimum degree at least 4, in contradiction with the above paragraph. We have proven that every planar non-hamiltonian K_2-hamiltonian graph contains at least three cubic vertices.

Finally, assume there exists a planar non-hamiltonian K_2-hamiltonian graph containing exactly three cubic vertices. Then there must exist a 3-fragment in G containing at most one cubic vertex. Gluing this fragment to its copy yields a planar non-hamiltonian K_2-hamiltonian graph containing at most two cubic vertices, a contradiction to the above paragraph. □

5 K_2-traceable graphs

A well-known Helly-type result is that if T is a tree and $\mathcal{T}$ a set of pairwise intersecting subtrees of T, then there is a vertex $v \in V(T)$ such that every tree in $\mathcal{T}$ contains v. If we widen our scope to graphs containing cycles, the original statement does not hold, but it is well-known that every two longest paths in a graph intersect. In 1966, Gallai [13] asked whether every connected graph has a vertex that appears in *all* longest paths. On the one hand, Walther [38] proved that this is not the case, and every graph which is hypotraceable (i.e. non-traceable, but every vertex-deleted subgraph is traceable) also provides a negative answer to Gallai's question. On the other hand, substantial efforts have been made to prove that in certain classes of graphs we can guarantee that the intersection of all longest paths is non-empty, see [6] for a recent contribution.

As an application of the results presented above, and as a natural counterpart to non-hamiltonian K_2-hamiltonian graphs, we study in this section non-traceable K_2-traceable graphs which, we recall, are non-traceable graphs in which the removal of any copy of K_2 yields a traceable graph. We shall see that among non-traceable K_2-traceable graphs, there exist examples satisfying Gallai's aforementioned condition as well as examples which do not satisfy it. But first, we present an infinite family of non-traceable K_2-traceable graphs which includes the smallest known such graph.

5.1 Searching for a smallest example

The following result is a variation of a technique of Thomassen [33], initially designed to obtain hypotraceable graphs from hypohamiltonian graphs.

Proposition 7. *Let* $G_1, ..., G_4$ *be pairwise disjoint non-hamiltonian* K_2*-hamiltonian graphs containing cubic vertices* $v_i \in V(G_i)$ *with* $N(v_i) = \{v_{i1}, v_{i2}, v_{i3}\}$ *such that for every* v_{ij} *the graph* $G_i - v_{ij}$ *is hamiltonian. In* $\bigcup_i G_i - v_i$ *identify* v_{11} *with* v_{21} *and* v_{31} *with* v_{41}*, and add the edges* $v_{12}v_{32}, v_{22}v_{42}, v_{13}v_{33}, v_{23}v_{43}$*. The resulting graph* Γ *is non-traceable and* K_2*-traceable.*

Proof. The non-traceability of Γ can be shown as in Thomassen's proof [33] and is therefore omitted. We see each $G_i - v_i$ as a subgraph of Γ. Let $vw \in E(G_1 - v_1)$. Note that, as v_1 is cubic, by Proposition 1 (i), it is impossible for both v and w to be neighbours of v_1. Since $G_1 - v - w$ is hamiltonian, there exist i, j such that $G_1 - v - w - v_1$ contains a hamiltonian $v_{1i}v_{1j}$-path $\mathfrak{p}_1$. There are two essentially different situations.

Case 1. $1 \in \{i, j\}$. Without loss of generality put $i = 1$ and $j = 2$. In $G_3 - v_3 - v_{32}$ there is a hamiltonian cycle $\mathfrak{h}_3$. Let x be a neighbour of v_{32} on $\mathfrak{h}_3$. Remove from $\mathfrak{h}_3 \cup \{v_3\}$ an edge incident with x and add the edge xv_{32} to obtain a hamiltonian path $\mathfrak{p}_3$ in $G_3 - v_3$ having v_{32} as an end-vertex. As $G_2 - v_{22}$ is hamiltonian, there exists a hamiltonian $v_{21}v_{23}$-path $\mathfrak{p}_2$ in $G_2 - v_2 - v_{22}$. Since $G_4 - v_{41}$ is hamiltonian, there is a hamiltonian $v_{42}v_{43}$-path $\mathfrak{p}_4$ in $G_4 - v_4 - v_{41}$. We now see $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$, and $\mathfrak{p}_4$ as paths in Γ and recall that $v_{11} = v_{21}$ and $v_{31} = v_{41}$. Adding to $\mathfrak{p}_3 \cup \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \mathfrak{p}_4$ the edges $v_{32}v_{12}$, $v_{23}v_{43}$, $v_{42}v_{22}$, we obtain a hamiltonian path in $\Gamma - v - w$.

Case 2. $1 \notin \{i, j\}$. Without loss of generality put $i = 2$ and $j = 3$. As $G_3 - v_{33}$ is hamiltonian, there is a hamiltonian $v_{32}v_{31}$-path $\mathfrak{p}'_3$ in $G - v_3 - v_{33}$. Similarly, in $G_4 - v_4 - v_{42}$ there exists a hamiltonian $v_{41}v_{43}$-path $\mathfrak{p}'_4$ and in $G_2 - v_2 - v_{21}$ there exists a hamiltonian $v_{22}v_{23}$-path $\mathfrak{p}'_2$. We now see $\mathfrak{p}_1, \mathfrak{p}'_2, \mathfrak{p}'_3$, and $\mathfrak{p}'_4$ as paths in Γ and recall that $v_{11} = v_{21}$ and $v_{31} = v_{41}$. Adding to $\mathfrak{p}_1 \cup \mathfrak{p}'_3 \cup \mathfrak{p}'_4 \cup \mathfrak{p}'_2$ the edges $v_{33}v_{13}$, $v_{12}v_{32}$, $v_{43}v_{23}$, $v_{22}v_{42}$, we obtain a hamiltonian path in $\Gamma - v - w$.

The treatment of pairs of adjacent vertices in G_2, G_3, and G_4 is analogous. Finally, we show that $\Gamma - v_{13} - v_{33}$ is traceable. For $k \in \{1, 3\}$, as $G_k - v_{k3}$ is hamiltonian, there exists a hamiltonian $v_{k1}v_{k2}$-path $\mathfrak{p}''_k$ in $G_k - v_k - v_{k3}$. Then $\mathfrak{p}''_3 \cup \mathfrak{p}''_1 \cup \mathfrak{p}_2 \cup \mathfrak{p}_4$ together with the edges $v_{32}v_{12}$, $v_{23}v_{43}$, and $v_{42}v_{22}$ form a hamiltonian path in $\Gamma - v_{13} - v_{33}$. Dealing with the removal of the pairs of adjacent vertices $\{v_{12}, v_{32}\}$, $\{v_{22}, v_{42}\}$, and $\{v_{23}, v_{43}\}$ is very similar. □

In the above proof we did not make use of the fact that $G_i - v_i - v_{ij}$ is hamiltonian for any i, j, so the requirements on the graphs G_i could be slightly relaxed, at the expense of a more cumbersome statement.

Originally, Thomassen [33] had applied this procedure to four copies of the Petersen graph in order to construct a 34-vertex hypotraceable graph, shown in Fig. 8. No smaller hypotraceable graph is known. It turns out that we can apply Proposition 7 to four copies of the Petersen graph as well, so the graph given by Thomassen is also K_2-traceable. No smaller non-traceable K_2-traceable graph is known.

Corollary 8. *There exists a hypotraceable* K_2*-traceable graph of order* 34*, see Fig. 8.*

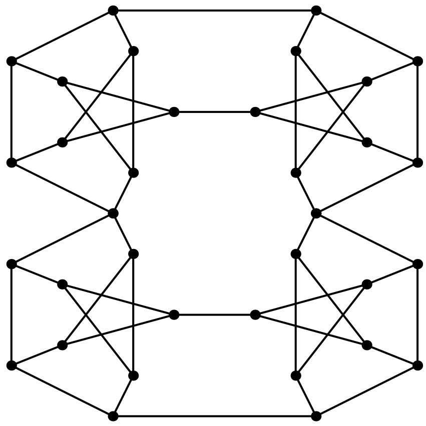

Fig. 8: A hypotraceable graph due to Thomassen. It is K_2-traceable.

5.2 Two families of 3-connected non-traceable K_2-traceable graphs

We omit the proofs of Propositions 8 and 9—they are similar to what has been presented above, as well as the proofs of the results of which they are variants. First, we note a variation of another result of Thomassen [34] (Thomassen's result generalises an observation of Horton [21]).

Proposition 8. *For $1 \leq i \leq 5$, let G_i be pairwise disjoint non-hamiltonian K_2-hamiltonian graphs containing adjacent cubic vertices x_i, y_i such that the neighbours of x_i (y_i) distinct from y_i (x_i) are a_i and b_i (c_i and d_i), and $G_i - a_i, G_i - b_i, G - c_i, G_i - d_i$ are hamiltonian. The graph resulting from adding to $\bigcup_i G_i - x_i - y_i$ the edges c_1a_2, c_2a_3, c_3a_4, c_4a_5, c_5a_1, d_1b_2, d_2b_3, d_3b_4, d_4b_5, and d_5b_1, is 3-connected, non-traceable and K_2-traceable.*

Consider disjoint graphs G and H, each containing a k-valent vertex v and w, respectively. We say that we *replace v with $G - w$* if in the disjoint union of $H - v$ and $G - w$ we connect the vertices of $N_H(v)$ to the vertices of $N_G(w)$. We now present a variation of a result of Wiener and the author [41]:

Proposition 9. *Let G_1, G_2, G_3 be pairwise disjoint non-hamiltonian K_2-hamiltonian graphs containing cubic vertices $w_i \in V(G_i)$ with neighbours w_{i1}, w_{i2}, w_{i3} such that for every w_{ij} the graph $G_i - w_{ij}$ is hamiltonian. Consider K_4 and put $V(K_4) = \{v_1, v_2, v_3, v_4\}$. For all $i \in \{1, 2, 3\}$, replace v_i with $G_i - w_i$. Thus, a 3-connected non-traceable K_2-traceable graph containing a non-traceable vertex-deleted subgraph is obtained.*

Corollary 9. *There exist infinitely many non-traceable K_2-traceable graphs in which the intersection of all longest paths is empty, as well as infinitely many non-traceable K_2-traceable graphs in which the intersection of all longest paths is non-empty.*

Proof. For the first statement, apply Proposition 8 to quintuples of graphs from an infinite family of cubic hypohamiltonian K_2-hamiltonian graphs, e.g. the family $\mathcal{P}$ from Section 2.1. We obtain an infinite family of non-traceable K_2-traceable graphs, which by a result of Thomassen [34] are in fact hypotraceable. Therefore, in each such graph the intersection of all longest paths is empty. For the second statement, apply Proposition 9 to triples of graphs from an infinite family of cubic hypohamiltonian K_2-hamiltonian graphs. The resulting graphs are non-traceable and K_2-traceable, but contain both traceable and non-traceable vertex-deleted subgraphs by a result from [41], so the intersection of their longest paths is non-empty. □

6 Notes

1. Our investigation naturally leads to the study of the properties of K_3-hamiltonian graphs. Already Grünbaum presented a planar cubic non-hamiltonian K_3-hamiltonian graph G_1 in [18]. What Grünbaum was in fact interested in was the observation that every vertex in the 124-vertex graph G_1 is missed by a cycle of length 121, which coincides with the circumference of G_1. He obtains G_1 by replacing in the planar and cubic graph G_0, see [18, Figure 1], suitable 40 of its 44 vertices by a triangle. Since none of the four non-replaced vertices of G_0 lie on a triangle, we can conclude that G_1 is K_3-hamiltonian. With the advent of planar cubic hypohamiltonian graphs [36], infinitely many planar cubic non-hamiltonian K_3-hamiltonian became readily available, as replacing every vertex of a member of the former family with a triangle yields a member of the latter family. Thus, structurally, the relationship between K_1- and K_3-hamiltonian graphs seems much stronger than with K_2-hamiltonian graphs.

2. Relaxing ($\mathfrak{G}$) in a different direction, one might ask for n-vertex graphs in which every pair of vertices is avoided by a longest cycle—when that cycle has length $n-2$, i.e. the graph's circumference is $n-2$, we arrive at a counterexample to ($\mathfrak{G}$). Grünbaum [18] provided a cubic graph of order 90 in which every pair of vertices is avoided by a longest cycle: insert a vertex-deleted Petersen graph into every vertex of a Petersen graph P. It has circumference 72. Zamfirescu [44] gave such a graph of order 75 and circumference 63, but it is not cubic: contract all "old" edges, i.e. edges which originally belonged to P, in the aforementioned construction. It would be very interesting to decrease the difference between order and circumference, in both the general and cubic case.

3. Another conjecture of Grünbaum [18] is that $\Gamma(1,2)$ contains no planar 3-connected graph. The author is not aware of a published solution to this problem. Grünbaum's original conjecture did not mention the word "3-connected", but in private communication between him and Thomassen, the latter showed that $\Gamma(1,k)$ contains infinitely many planar graphs for every $k \geq 2$ (see [18] for details) and, in a later paper, that $\Gamma(1,1)$ contains infinitely many planar 3-connected graphs [34]. This result leads to a solution: consider such a graph G and a cubic vertex therein (such a vertex always exists [35]). Let F be the non-trivial $N(v)$-fragment of G. Connect the attachments of F to the attachments of a copy of F, disjoint from F, by three edges such that the resulting graph is planar and 3-connected. Thus we obtain infinitely many counterexamples to the conjecture that $\Gamma(1,2)$ contains no planar 3-connected graph.

4. In the introduction we noted that for the problems of Grünbaum and Katona et al. restricted to pairs of non-adjacent vertices, it is easy to give infinitely many solutions by considering $K_t + \overline{K}_{t+2}$. Every such graph has circumference 2 less than its order, and satisfies that the removal of any pair of vertices at distance at least 2 yields a hamiltonian graph. However, every member of this family has diameter 2, so the natural follow-up question is whether we can show this result for larger diameters.

5. A continuation of this article is in preparation [15]. It will combine results from this paper with new theoretical insights and computational tools in order to further study, for n-vertex graphs, the interplay between K_2-hamiltonicity and $(n-1)$-cycles, in particular with regards to planar graphs.

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