On 3-polytopes with non-hamiltonian prisms

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Abstract. Špacapan recently showed that there exist 3-polytopes with non-hamiltonian prisms, disproving a conjecture of Rosenfeld and Barnette. By adapting Špacapan's approach we strengthen his result in several directions. We prove that there exists an infinite family of counterexamples to the Rosenfeld-Barnette conjecture, each member of which has maximum degree 37, is of girth 4, and contains no odd-length face with length less than k for a given odd integer k. We also show that for any given 3-polytope H there is a counterexample containing H as an induced subgraph. This yields an infinite family of non-hamiltonian 4-polytopes in which the proportion of quartic vertices tends to 1. However, Barnette's conjecture stating that every 4-polytope in which all vertices are quartic is hamiltonian still stands. Finally, we prove that the Grünbaum-Walther shortness coefficient of the family of all prisms of 3-polytopes is at most 59/60.

Keywords. Longest cycle, prism, polytope

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1 Introduction

This article concerns the study of vertex degrees and longest cycles in the Cartesian product of the 1-skeleton of a 3-polytope with K_2, i.e. its *prism*. We shall generally use graph-theoretical notation but frame our results in polytopal terminology, since many problems were originally formulated thusly. When we here speak of a d-polytope (i.e. a d-dimensional polytope), we are always referring to its 1-skeleton. A d-polytope is *simple* if each of its vertices is incident with d edges. We recall that 3-polytopes

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have been characterised by Steinitz [19] and coincide with planar 3-connected graphs, and that by a theorem of Balinski, every d-polytope is d-connected [1]. For classical references on graphs and polytopes we refer to Grünbaum's works [5, 6], and for a more recent contribution, including a good overview, see [14].

Rosenfeld and Barnette [17] showed that the Four Colour Theorem implies that simple 3-polytopes have hamiltonian prisms, and Fleischner [4] gave a proof avoiding the use of the Four Colour Theorem. Paulraja [13] strengthened this by showing that the same conclusion holds even if planarity is not assumed, i.e. the prism over any 3-connected cubic graph is hamiltonian. Čada, Kaiser, Rosenfeld, and Ryjáček gave a new, simplified proof of this result [3].

The conjecture that any 3-polytope is *prism-hamiltonian*, i.e. has a hamiltonian prism, is generally attributed to the 1973 article [17] of Rosenfeld and Barnette, and Špacapan [18] writes that Rosenfeld mentioned the conjecture during many of his talks. It appears explicitly in [8, Conjecture 1]. In support of the conjecture, various classes of 3-polytopes were shown to be prism-hamiltonian, see [18] for a recent summary. In particular, in 2008 Biebighauser and Ellingham [2] proved that triangulations and bipartite 3-polytopes are prism-hamiltonian. However, in the general case, Špacapan [18] recently showed that the conjecture does not hold.

On the other hand, the question whether all simple 4-polytopes are hamiltonian remains open. This question is the title of Rosenfeld's 1983 article [15]. Therein, he constructs extensive families of non-hamiltonian 4-regular 4-connected graphs none of which are the 1-skeleton of a 4-polytope. Already in 1970 Barnette formulated the conjecture that the answer to this question is positive [6, p. 1145]. In stark contrast, Mohar conjectures that for every integer $d \geq 3$, there exists a simple d-polytope that is not hamiltonian [11]. For $d = 3$ this was proved by Tutte [21], for $d > 3$ it is open. If one drops the simplicity requirement, non-hamiltonian d-polytopes are easy to describe for any d.

As in many questions related to hamiltonicity, vertex degrees play a crucial role. In the next section we shall therefore adapt Špacapan's solution in order to better understand the structural properties of the counterexamples to the Rosenfeld-Barnette conjecture. Concerning the combinatorial properties of polytopes, Kalai writes in [9]: "it seems that overall, we are short of examples. The methods for coming up with useful examples (or counterexamples for commonly believed conjectures) are even less clear than the methods of proving."—our contribution is very much driven by this sentiment. We will address certain natural questions concerning vertex degrees and short longest cycles in prisms of 3-polytopes. Concretely, we prove that there exists an infinite family of counterexamples to the conjecture, each member having maximum degree 37, girth 4, and no odd-length face with length less than k for a given odd integer k, whereas the maximum degree in Špacapan's infinite family grows with the graphs' order, and many triangles are present. We also show that for any given 3-polytope H there is a counterexample containing H as an induced subgraph, immediately yielding an infinite family of non-hamiltonian 4-polytopes in which the proportion of quartic vertices tends to 1. However, Barnette's conjecture stating that simple 4-polytopes are hamiltonian still stands. Finally, we prove that the Grünbaum-Walther shortness coefficient of the family of all prisms of 3-polytopes is at most 59/60. This establishes that there exist prisms of 3-polytopes in which every longest cycle misses a number of vertices that is linear in the order of the prism.

2 Results

All graphs in this paper are assumed to be connected. A graph is *plane* if it is planar and embedded in the Euclidean plane. Let G be a graph and K_2 the complete graph on two vertices. Put $G^\square := G\square K_2$, and for a set of vertices W, set $W^\square := W \times V(K_2)$. For $W \subset V(G)$, we write $G[W]$ for the subgraph of G induced by W. For a vertex $v \in V(G)$, in $G^\square$ the two copies of v will be denoted by v' and v''. For $X, Y \subset V(G)$ we call a path with one endpoint in X and one endpoint in Y an *XY-path*. When $X = Y$, we simply write *X-path*. Abusing notation, single-element sets will be denoted by the element they contain. In this paper, all cuts are vertex-cuts, and a *k-cut* is a cut on k vertices. We call a path (cycle) on k vertices a *k-vertex-path* (*k-cycle*). For a vertex v we denote the set of vertices adjacent to v by $N(v)$. Sets $A, B \subset S$ *partition* S if $A \cap B = \emptyset$ and $A \cup B = S$. We begin with a simple fact:

Lemma 1. *Let b, x be the end-vertices of a 3-vertex-path P and Q a bipartite graph disjoint from P. Let G be the graph obtained from P and Q by adding the two edges bc and xy, where c, y are contained in different partite sets of Q. In $G^\square$, put $B := \{b', b''\}$ and $X := \{x', x''\}$. Then there exists no pair of non-empty BX-paths $\mathfrak{p}, \mathfrak{q}$ such that $V(\mathfrak{p})$ and $V(\mathfrak{q})$ partition $V(G^\square)$.*

Proof. Suppose that there exist non-empty BX-paths $\mathfrak{p}, \mathfrak{q}$ such that $V(\mathfrak{p})$ and $V(\mathfrak{q})$ partition $V(G^\square)$. It is easy to see that one of $\mathfrak{p}$ and $\mathfrak{q}$, say $\mathfrak{p}$, contains no vertex of $Q^\square$. Moreover, we may assume that $b', x'' \in V(\mathfrak{p})$. We show that there is no $b''x'$-path in $\widehat{Q} := G^\square[V(Q^\square) \cup \{b'', x'\}]$ passing through all vertices in $Q^\square$. Since $G[Q \cup \{b, x\}]$ is a bipartite graph, $\widehat{Q}$ is a bipartite graph as well. As c and y are contained in different partite sets of Q, we can infer that b'' and x' are contained in the same partite set of $\widehat{Q}$. Since $Q^\square$ has partite sets of equal cardinality, the cardinality of the partite set of $\widehat{Q}$ containing b'' and x' is by 2 greater than the cardinality of its other partite set. Thus, there is no $b''x'$-path in $\widehat{Q}$ spanning $Q^\square$. □

Let Q be a bipartite 3-polytope and G be as introduced in Lemma 1. We define the graph $R_{H,Q}$ as shown in Fig. 1, where for z being the common neighbour of b and x, the graph induced by $\{b, x, z\} \cup V(Q)$ is isomorphic to G, and H either is an arbitrary 3-polytope, in which case the two edges between H and $R_{H,Q} - H$ are non-incident, or the graph $(\emptyset, \emptyset)$, in which case these two edges are suppressed. Whenever the graph Q does not change in an argument, we write R_H instead of $R_{H,Q}$. Lemma 1 and the fact that X (as defined in Lemma 1) is a 2-cut in $R_H^\square$ imply the next lemma.

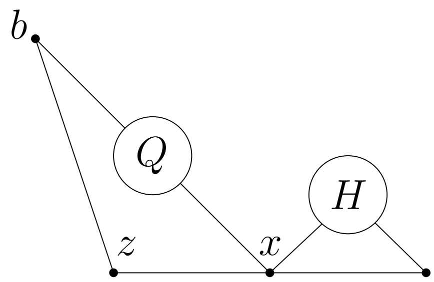

Fig. 1: The graph R_H.

Lemma 2. *Let R_H be the graph from Fig. 1 and b the vertex indicated in Fig. 1. Then there exists no hamiltonian $b'b''$-path in $R_H^\square$.*

Let H be a 3-polytope. By combining four copies of R_H we obtain the graph T depicted in Fig. 2, where c is adjacent to a vertex in H (resp. Q) which is different from the two vertices in H (resp. Q) adjacent to a vertex outside of H (resp. Q).

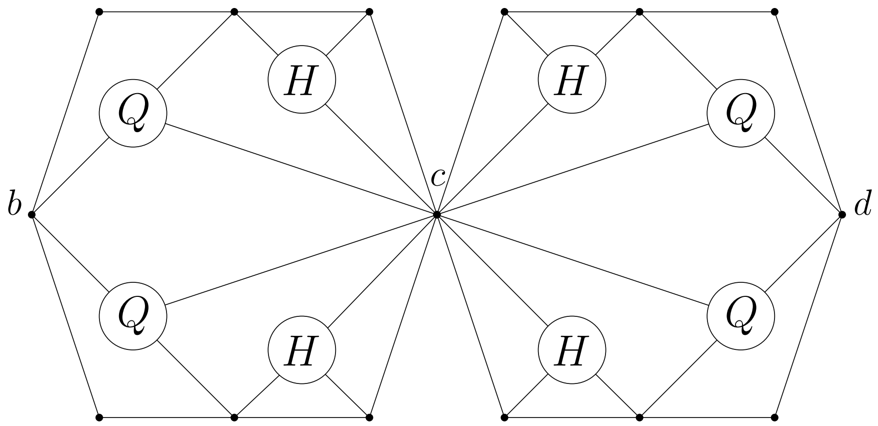

Fig. 2: The graph T.

Lemma 3. *Consider the graph T as well as vertices b and d in T as shown in Fig. 2. In $T^\square$, put $B := \{b', b''\}$ and $D := \{d', d''\}$. Then there is no BD-path $\mathfrak{p}$ in $T^\square$ such that $\mathfrak{p}$ contains all vertices of $T^\square - B - D$.*

Proof. Assume there exists such a path $\mathfrak{p}$. Referring to Fig. 2, since $\{c', c''\} =: C$ is a 2-cut of $T^\square$, $\mathfrak{p}$ must visit all vertices of the, with respect to C, left-hand side L of $T^\square$ before entering the right-hand side R (with the possible exception of c' or c'').

Let $\mathfrak{q}$ be a BC-path in L such that $\mathfrak{q}$ visits all vertices of $L - B - C$. We now show that then $V(C) \subset V(\mathfrak{q})$. Assume $\mathfrak{q}$ visits only one of c', c''. Then however either the bottom half of $L - C$ or the top half of $L - C$, both isomorphic to $R_H^\square$, contains a hamiltonian $b'b''$-path, contradicting Lemma 2.

Thus, we know that $\mathfrak{p}$, when entering R, must have already visited c' and c''. By symmetry we can now assume that $\mathfrak{p}$ enters the lower half of R, visits all of its vertices, and exits it through a vertex of D. But $\mathfrak{p}$ must end in D, so it contains as a sub-path a hamiltonian $d'd''$-path in the top half of $R - C$, which is isomorphic to $R_H^\square$. This contradicts Lemma 2. □

Consider a 3-vertex-path P whose end-vertices are v, w and a bipartite 3-polytope Q. Let $\widehat{R}_Q$ be a planar graph obtained from P and Q by adding two edges vx and wy where x and y are contained in different partite sets of Q. Note that we can choose x and y from $V(Q)$ such that $\widehat{R}_Q$ is a planar graph. Consider the disjoint union of T and two copies of $\widehat{R}_Q$ which we call C^1 and C^2. Let a, b be the end-vertices of the copy of P in C^1 and d, e be the end-vertices of the copy of P in C^2. Identify b in T (see Fig. 2) with b in C^1 and identify d in the resulting graph with d in C^2 in order to obtain the graph U. We have depicted a planar embedding of U in Fig. 3. We introduce white and square vertices as defined in Fig. 3 and its caption. Choose a vertex v_1 (v_2) from

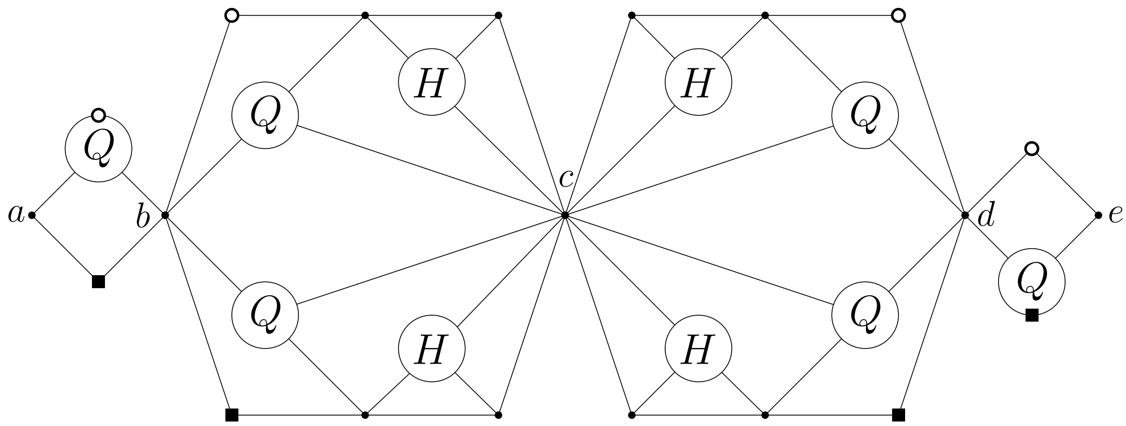

Fig. 3: The graph U. Vertices shown as white discs will be called *white* and vertices depicted as squares will be called *square*.

the boundary of the unbounded face of Q in C^1 (C^2) as a white vertex (square vertex) such that $v_1 \notin N(a) \cup N(b)$ and $v_2 \notin N(d) \cup N(e)$.

Lemma 4. *Consider U as defined above. In $U^\square$, put $A := \{a', a''\}$ and $E := \{e', e''\}$. Then, in $U^\square$,*
(i) *there is no AE-path spanning $U^\square - A - E$;*
(ii) *there are no two AE-paths whose vertices partition $V(U^\square)$;*
(iii) *there is no A-path $\mathfrak{p}$ and E-path $\mathfrak{q}$ such that $V(\mathfrak{p})$ and $V(\mathfrak{q})$ partition $V(U^\square)$; and*
(iv) *there is no A-path spanning $U^\square - A - E$ and no E-path spanning $U^\square - A - E$.*

Proof. (i) follows directly from Lemma 3, and (ii) from Lemma 1. Now assume that the converse of (iii) is true. By Lemma 1, $\mathfrak{p}$ cannot go beyond $\{b', b''\}$, and $\mathfrak{q}$ cannot go beyond $\{d', d''\}$, which means that $c' \notin V(\mathfrak{p}) \cup V(\mathfrak{q})$, a contradiction. This includes the case that either $\mathfrak{p}$ or $\mathfrak{q}$ is isomorphic to $(\emptyset, \emptyset)$. For (iv), we argue as for (iii). □

Consider nine pairwise disjoint copies of U, each embedded in the plane as shown in Fig. 3, where for a vertex v of U its i-th copy will be denoted by v_i, as well as two additional vertices x and y disjoint from these copies of U. We identify vertex e_i (in the i-th copy of U) with vertex a_{i+1} (in the $(i+1)$-th copy of U) for all $i \in \{1, \ldots, 8\}$ as well as e_9 with a_1. In each copy of U, assume $H \neq \emptyset$ and connect all white vertices of every copy of U with x and all square vertices of every copy of U with y. Clearly, this can be done so that the obtained graph G_9 is a plane graph.

Lemma 5. *G_9 is 3-connected.*

Proof. It is easy to see that G_9 and $G_9 - \{x, y\}$ are 2-connected. Suppose that G_9 contains a 2-cut S. Since $G_9 - \{x, y\}$ is 2-connected, $x, y \notin S$.

We call a block of U which is not isomorphic to $\widehat{R}_Q$ an *octagonal fragment*. For each $i \in \{1, \ldots, 9\}$, U_i contains two octagonal fragments. In the rest of the proof, we consider only the octagonal fragment O_i containing b_i and c_i. For any two vertices $u, v \in V(O_i) \setminus \{b_i, c_i\}$ with $\{u, v\} \neq \{b_i, c_i\}$, $G_9 - \{x, y, u, v\}$ is connected and there is no 2-cut in O_i except for $\{b_i, c_i\}$. Thus, we obtain the following claim.

Claim. *For an octagonal fragment O_i in U_i, the following statements hold.*

- *For $u, v \in V(O_i)$, if $G_9 - \{x, y, u, v\}$ is disconnected, then $\{u, v\} = \{b_i, c_i\}$ and*
- *$\{b_i, c_i\}$ is the unique 2-cut of O_i.*

Suppose that every block of $G_9 - \{x, y\} - S$ is connected. Since every block of $G_9 - \{x, y\}$ has both a vertex adjacent to x and a vertex adjacent to y, there exists a block B of $G_9 - \{x, y\}$ such that $B - S$ has no vertex adjacent to either x or y. This implies that B is an octagonal fragment and S consists of a square vertex and a white vertex contained in B. Then $S \cap \{b_i, c_i\} = \emptyset$, which contradicts the Claim. Hence, there exists a block B of $G_9 - \{x, y\} - S$ that is disconnected. Then $S \subseteq B$. Suppose that B is an octagonal fragment. Then $S = \{b_i, c_i\}$ by the Claim. Since each component of $B - S$ has a vertex adjacent to either x or y, the graph $G_9 - S$ is connected, a contradiction. Hence B is isomorphic to $\widehat{R}_Q$.

If each component of $B - S$ has a vertex adjacent to either x or y, we can deduce a contradiction as above. So there exists a component of $B - S$ which has no vertex adjacent to either x or y. This implies that $\{a_i, b_i, d_i, e_i\} \cap S = \emptyset$. Then for each vertex z in $B - S$, there exists either a path connecting x and z in $G_9 - S$ or a path connecting y and z in $G_9 - S$, but this implies that $G_9 - S$ is connected, a contradiction. $\square$

As mentioned in the introduction, Špacapan [18] has shown that there exist infinitely many 3-polytopes whose prisms are not hamiltonian. However, in his construction the maximum degree increases with the graph's order. Motivated by this as well as Barnette's aforementioned conjecture, we present the following results, where $V_k(G)$ shall denote the set of all k-valent vertices present in a given graph G. We shall make use of the so-called *shortness coefficient* ρ and *shortness exponent* σ, defined by Grünbaum and Walther [7] as follows. For an infinite family $\mathcal{G}$ of graphs, let

$$\rho(\mathcal{G}) := \liminf \frac{\operatorname{circ}(G_n)}{|V(G_n)|} \quad \text{and} \quad \sigma(\mathcal{G}) := \liminf \frac{\log \operatorname{circ}(G_n)}{\log |V(G_n)|},$$

with the limit inferior taken over all sequences of graphs G_n in $\mathcal{G}$ such that $|V(G_n)| \to \infty$ as $n \to \infty$. Here, $\operatorname{circ}(G)$ stands for the *circumference* of the graph G, i.e. the length of a longest cycle in G. Already in the sixties Moon and Moser proved [12] that there exist infinitely many n-vertex d-polytopes the longest paths of which have length $< 3dn^{\log_d 2}$. Thus, if $\mathcal{P}_d$ is the family of all d-polytopes, for $d \geq 3$ we have $\sigma(\mathcal{P}_d) \leq \log_d 2 < 1$, so $\rho(\mathcal{P}_d) = 0$. Their proof is based on the following observation: let G be a d-polytope containing strictly more facets (i.e. $(d-1)$-faces) than vertices. For each of its facets, add a *new* vertex to G and connect it to every vertex of the facet. It is easy to see that this can be done such that the resulting graph G' is also a d-polytope. We call a non-new vertex *old*. In G', there is no edge between two new vertices, and there can be many more new vertices than old vertices, hence the large discrepancy between circumference and order.

Let $\mathcal{P}_3^\square$ denote the family of all prisms over 3-polytopes. For a graph G, the maximum degree in G is denoted by $\Delta(G)$. We now present our results.

Theorem. (i) *For any* 3*-polytope* H *there exists a* 3*-polytope* G *containing* H *as an induced subgraph such that the prism over* G *is non-hamiltonian and has maximum degree* $\max\{37, \Delta(H)+1\}$*. Thus, every simple* 4*-polytope obtained as the prism over a simple* 3*-polytope, is contained in some non-hamiltonian* 4*-polytope.*

(ii) *For any odd integer* $k \geq 3$*, there is an infinite family of* 3*-polytopes whose prisms* Π *are non-hamiltonian, have maximum degree* 37*, are of girth* 4*, contain no odd-length face with length less than* k*, and satisfy*

$$\lim_{|V(\Pi)|\to\infty} |V_4(\Pi)|/|V(\Pi)| = 1.$$

(iii) *We have* $\rho(\mathcal{P}_3^\square) \leq \frac{59}{60}$*. In particular, there exist prisms of* 3*-polytopes in which every longest cycle misses a number of vertices that is linear in the order of the prism.*

Proof. Clearly, x and y attain the maximum degree among all vertices of G_9 not contained in a copy of H or Q, which is 36. Since we can choose a bipartite 3-polytope Q with $\Delta(Q) \leq 36$ (for instance the prism over an even cycle), $G_9^\square$ has maximum degree $\max\{37, \Delta(H)+1\}$. We now show that $G_9^\square$ is non-hamiltonian. Suppose that $G_9^\square$ has a hamiltonian cycle $\mathfrak{h}$. Then the number of edges of $\mathfrak{h}$ incident with a vertex in x', x'', y', y'' is at most eight. Since $G_9 - \{x, y\}$ contains nine copies of U, there exists an $i \in \{1, \ldots, 9\}$ such that $U_i^\square$ has no vertex adjacent to either x', x'', y', or y'' in $\mathfrak{h}$. Every way in which $\mathfrak{h}$ might traverse $U_i^\square$—depicted diagrammatically in Fig. 4—contradicts Lemma 4. Hence $G_9^\square$ is non-hamiltonian.

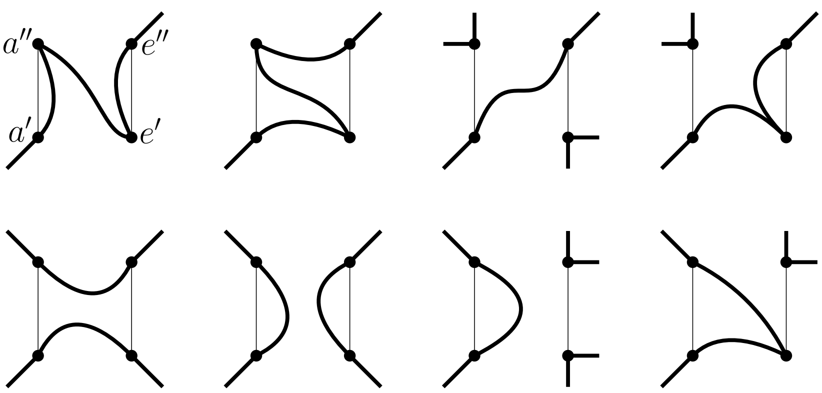

Fig. 4: The eight essentially different ways in which $\mathfrak{h}$ might span $U_i^\square$. Each diagram in the top row contradicts Lemma 4 (i), while for the second row, the first diagram contradicts Lemma 4 (ii), the second diagram Lemma 4 (iii), and the last two diagrams contradict Lemma 4 (iv).

Next, we show that infinitely many graphs satisfying (ii) can be obtained from G_9. We choose the prism of the 2ℓ-cycle, a simple bipartite 3-polytope, as H as well as Q. With this choice, for $\ell \to \infty$, the graphs G_9 satisfy the last property of (ii). We show that G_9 contains no odd-length face with length less than k with this choice of H and Q. Since the prism of the 2ℓ-cycle has faces of length four and a face of length 2ℓ, we can ignore the faces in H and Q. The faces not containing any edge of the outer face of H and Q have even length. For each copy of H (resp. Q) in G_9, there are three vertices adjacent to a vertex in $V(G_9) \setminus (V(H) \cup V(Q))$. When applying Lemma 1, we need to choose the two vertices of Q adjacent to $V(G_9) \setminus (V(Q) \cup \{c\})$ carefully: we select the three vertices from H (resp. Q) so that between any two among them, the distance in the outer face of H (resp. Q) is large enough.

For part (iii), we argue as follows. We consider a graph constructed similarly to G_9, but instead of nine copies we consider ℓ copies, and instead of U we use the graph $\widehat{U}$ shown in Fig 5. Denote the unbounded face of $\widehat{U}$ by F. We call every vertex of F strictly below (strictly above) the dotted line in Fig. 5 a(n) *lower vertex* (*upper vertex*). Connect all 2-valent upper vertices of every copy of $\widehat{U}$ with x and all 2-valent lower vertices of every copy of $\widehat{U}$ with y.

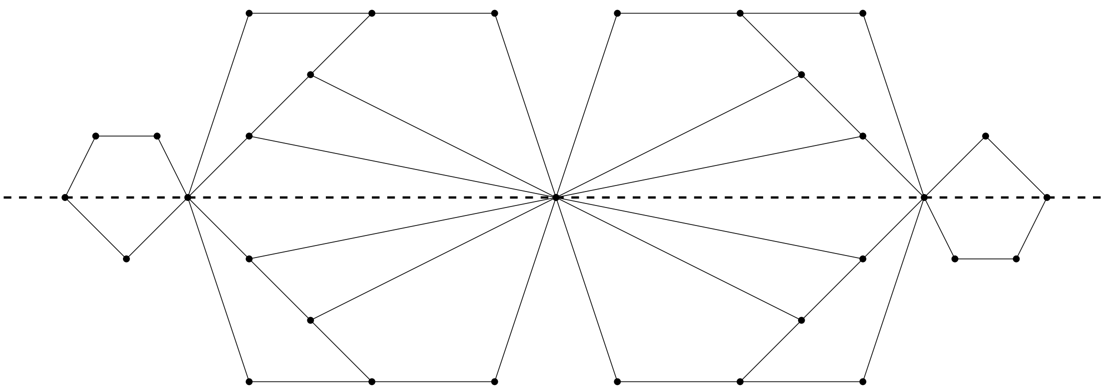

Fig. 5: The graph $\widehat{U}$.

Denote this graph by Z_ℓ. This graph is clearly planar, and the proof of its 3-connectedness is very similar to that of Lemma 5 and therefore omitted. We now consider the situation when $\ell \to \infty$. Let $\mathfrak{c}$ be a longest cycle in $Z_\ell^\square$. Arguing as in the first paragraph of this proof, there exist at least $\ell - 8 \to \infty$ copies of $\widehat{U}^\square$ in $Z_\ell^\square$ that have no vertex adjacent to either x', x'', y', y'' in $\mathfrak{c}$. With arguments very similar to what has already been discussed, one can show that in each such copy the cycle $\mathfrak{c}$ misses at least one vertex, so

$$\rho(\mathcal{P}_3^\square) \leq \lim_{\ell\to\infty} \frac{\mathrm{circ}(Z_\ell^\square)}{|V(Z_\ell^\square)|} \leq \lim_{\ell\to\infty} \frac{60 \cdot 8 + 59(\ell - 8) + 4}{60\ell + 4} = \frac{59}{60}.$$

□

3 Discussion

1. Determine (or improve the bounds) for

$$\rho(\mathcal{P}_3^\square) \qquad \text{and} \qquad \Delta_0 := \min_{\Pi \in \mathcal{P}_3^\square \cap \mathcal{N}} \Delta(\Pi),$$

where $\mathcal{N}$ is the family of all non-hamiltonian graphs. By Paulraja's aforementioned result and our theorem we have $5 \leq \Delta_0 \leq 37$. Perhaps $\rho(\mathcal{P}_3^\square) = 0$ is true. If so, an estimation of $\sigma(\mathcal{P}_3^\square)$ would be interesting.

2. We have described here, like Špacapan, an infinite family of non-hamiltonian 4-polytopes. 4-polytopes are by Balinski's theorem necessarily 4-connected. By a theorem of Tutte, planar 4-connected graphs are hamiltonian, so their prisms are certainly hamiltonian. But are there non-hamiltonian 5-connected 4-polytopes? Mohar [11] asks: Is there an integer k such that every k-connected 4-polytopal graph is hamiltonian? (Of course, the question makes sense for higher-dimensional polytopes as well.) In contrast to 3-polytopes, 4-polytopes with arbitrarily large connectivity exist [5], for instance the family of all complete graphs of order at least 5.

3. As Malkevitch recalls in [10], Barnette showed that every 3-polytope has a spanning tree of maximum degree 3 but for 4-polytopes there is no uniform value of k which will guarantee that a 4-polytope has a spanning tree of maximum degree k. For prisms over 3-polytopes however it follows immediately from Barnette's result that these always contain a spanning tree of maximum degree 3. (Improving this to maximum degree 2 is impossible by results from [18].)

4. One can describe non-hamiltonian 4-polytopes via toughness-based arguments, and Špacapan [18] has shown that even the prisms over 3-polytopes may be non-hamiltonian. Our question now is whether there is a 3-polytope G whose prism is non-hamiltonian, yet has *many* hamiltonian vertex-deleted subgraphs. It is not difficult to construct non-hamiltonian 4-polytopes of order n with $\frac{n+1}{2}$ hamiltonian vertex-deleted subgraphs, but it is unclear how to move passed this threshold. This question is motivated by a paper of Thomassen, who asked in 1978 whether 4-connected *hypohamiltonian* graphs—i.e. non-hamiltonian graphs in which every vertex-deleted subgraph is hamiltonian—exist [20].

5. Rosenfeld discusses in [16] a series of interesting problems on the subject, many of which remain open.

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