# Non-hamiltonian graphs in which every edge-contracted subgraph is hamiltonian

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**Abstract.** A graph $G$ is *perihamiltonian* if $G$ itself is non-hamiltonian, yet every edge-contracted subgraph of $G$ is hamiltonian. These graphs form a superclass of the hypohamiltonian graphs. By applying a recent result of Wiener on path-critical graphs, we prove the existence of infinitely many perihamiltonian graphs of connectivity $k$ for any $k \geq 2$. We also show that every planar perihamiltonian graph of connectivity $k$ contains a vertex of degree $k$. This strengthens a theorem of Thomassen, and entails that if in a polyhedral graph of minimum degree at least 4 the set of vertices whose removal yields a non-hamiltonian graph is independent, the graph itself must be hamiltonian. Finally, while we here prove that there are infinitely many polyhedral perihamiltonian graphs containing no adjacent cubic vertices, whether an analogous result holds for the hypohamiltonian case remains open.

**Keywords.** Non-hamiltonian, perihamiltonian, hypohamiltonian

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## 1 Introduction

A graph is *hypohamiltonian* if it does not contain a hamiltonian cycle, but all of its vertex-deleted subgraphs do. For a survey, see Holton and Sheehan's [9], while for recent progress on the planar case we refer to [12]. In 1978, Thomassen [19] proved that a hypohamiltonian graph that is planar will always contain vertex of

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degree 3, which is equivalent to the statement that if each vertex-deleted subgraph of a planar graph $G$ in which every vertex has degree at least 4 is hamiltonian, then $G$ is hamiltonian, and constitutes a generalisation of Tutte's celebrated result that planar 4-connected graphs are hamiltonian [21]. The aforementioned theorem of Thomassen was recently strengthened in several directions by the last author [28], among which is the result that a planar hypohamiltonian graph must contain at least four vertices of degree 3. However, the following intriguing question of the first two authors remains open: Do planar hypohamiltonian graphs necessarily contain *adjacent* vertices of degree 3?

Motivated by this problem, we introduce a class of graphs defined as follows. A graph $G$ is *perihamiltonian* if $G$ itself is non-hamiltonian, yet every edge-contracted subgraph of $G$ is hamiltonian. We shall make frequent and tacit use of the fact that a non-hamiltonian graph $G$ is perihamiltonian if and only if for every edge $vw$ in $G$, the vertex-deleted subgraph $G - v$ or $G - w$ (possibly both) is hamiltonian. Thus, every hypohamiltonian graph is perihamiltonian. Clearly, perihamiltonian graphs are *traceable* (i.e. contain a hamiltonian path), but perhaps surprisingly not every vertex-deleted subgraph of a perihamiltonian graph need be traceable. Therefore, perihamiltonian graphs are not a subclass of the so-called *platypuses* [27], non-hamiltonian graphs in which every vertex-deleted subgraph is traceable: take for instance Herschel's graph, which is perihamiltonian but not a platypus. A non-traceable graph in which each vertex-deleted subgraph is traceable is called *hypotraceable*. The existence of such graphs [20] shows that there are non-perihamiltonian platypuses.

This article is organised as follows. We close this introductory section with further definitions and basic structural properties of perihamiltonian graphs. In Sections 2 and 3 we present results on perihamiltonian graphs of connectivity 2 and connectivity 3, respectively, followed by a fourth section on higher connectivity. Special attention is given to the planar case. In Section 5 we describe our computational results. The paper ends with Section 6, in which we present open problems on perihamiltonian graphs.

For a possibly disconnected graph $G$, we denote by $V(G)$ ($E(G)$; $\omega(G)$) its vertex set (edge set; number of connected components). Let now $G$ be a non-complete graph of connectivity $\kappa$. Then $G$ contains induced subgraphs $H_1, H_2$, both non-empty, with the property that $G = H_1 \cup H_2$ and $V(H_1) \cap V(H_2) = X$, with $X$ containing exactly $\kappa$ vertices. The subgraph $H_1$ (and $H_2$) shall be called a *$\kappa$-fragment* of $G$. (If $\kappa$ is clear from the context or irrelevant for the argument, we will suppress it and simply write *fragment*.) Moreover, $X$ is the set of *attachments* of $H_1$. We say that a $\kappa$-fragment $F$ is *trivial* if $|V(F)| = \kappa + 1$. A cut $X$ (in this paper, all cuts are vertex-cuts) of cardinality $\kappa$ will be called a *$\kappa$-cut*. The set of attachments of a fragment that is trivial forms a cut which we call *trivial*. A path with end-vertex $v$ is a *$v$-path*, and a $v$-path with end-vertex $w \neq v$ is a *$vw$-path*. For a graph $G$ we denote by $\overline{G}$ its complement, with $V_k(G)$ its set of vertices of degree $k$ (when a vertex $v$ has degree $k$, we say that $v$ is *$k$-valent*, and we will sometimes shorten this to $\deg(v) = k$), and for an edge $e$ in $G$ we write $G/e$ for the graph obtained when contracting $e$—that is, removing $e$, identifying its end-vertices, and preserving exactly one edge of any multi-edge formed by this identification. For a graph $G$, we denote by $\delta(G)$ its minimum degree. Planar 3-connected graphs will be called *polyhedral.*

Let $G$ be a 2-connected graph in which a longest cycle has length $|V(G)|-1$. Let the set $\mathrm{exc}(G)$ in $G$ be defined as the set of all vertices $w$ of $G$ satisfying the property that $G-w$ contains no hamiltonian cycle. We emphasise that given this definition, deleting any vertex from $V(G) \setminus \mathrm{exc}(G) = \mathrm{nexc}(G)$ will result in a hamiltonian graph, and that $\mathrm{nexc}(G)$ is non-empty. Following [26], we shall consider $G$ to be $|\mathrm{exc}(G)|$-*hypohamiltonian.* Furthermore, a vertex in $\mathrm{exc}(G)$ is *exceptional*, and a vertex in $\mathrm{nexc}(G)$ is *non-exceptional.* Throughout this article, figures will depict non-exceptional vertices as black and exceptional vertices as white. Denote the family of all $k$-hypohamiltonian graphs by $\mathcal{H}_k$. A graph from $\mathcal{H}_0$ is hypohamiltonian, and members of $\mathcal{H}_1$ are called *almost hypohamiltonian.*

$\bigcup_{k\geq 0} \mathcal{H}_k$ is a disjoint union and constitutes the family of all 2-connected graphs $G$ of circumference $|V(G)|-1$. Every perihamiltonian graph $G$ is $k$-hypohamiltonian for an appropriate $k$, and the $k$ exceptional vertices in $G$ form an independent set. In particular, every almost hypohamiltonian graph is perihamiltonian.

The maximum size of an independent set of a graph $G$ is denoted by $\alpha(G)$. Observe that the removal of a non-exceptional vertex (of which there always exists at least one) from an $n$-vertex perihamiltonian graph yields an $(n-1)$-cycle in which at most half the vertices are exceptional. Thus, we have:

**Lemma 1.** *Let $G$ be a $k$-hypohamiltonian graph. $G$ is perihamiltonian if and only if* $\mathrm{exc}(G)$ *is an independent set. In particular, for a perihamiltonian graph $G$ we have*

$$|\mathrm{exc}(G)| \leq \min\left\{\alpha(G), \frac{|V(G)|-1}{2}\right\}.$$

Note that a simple argument gives that in a perihamiltonian graph the independence number is always bounded from above by $(|V(G)|+1)/2$. We shall see in the next proposition that every bipartite perihamiltonian graph $G$ is extremal in the sense that it satisfies $|\mathrm{exc}(G)| = (|V(G)|-1)/2$. A graph $G$ is *homogeneously traceable* if for every vertex $v$ in $G$ there is a hamiltonian $v$-path of $G$. For $k \geq 2$, a *subdivided wheel* shall be the graph

$$(\{v_i, x_i, v_k\}_{i=0}^{k-1}, \{v_iv_k, v_ix_i, x_iv_{i+1}\}_{i=0}^{k-1}),$$

indices mod $k$, see Fig. 1 for $k=4$.

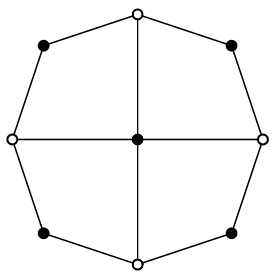

Figure 1: The subdivided wheel on 9 vertices, a planar bipartite perihamiltonian graph.

**Proposition 1.** *Let $G$ be a perihamiltonian graph. Then the following hold.*

*(i) $G$ contains for every non-exceptional vertex $x$ a hamiltonian $x$-path. In particular, $G$ is traceable. However, there exist infinitely many perihamiltonian graphs which are not homogeneously traceable.*

*(ii) If $G$ contains a triangle $T$, then every vertex in $T$ has degree at least 4.*

*(iii) If $G$ has a bipartition $(A, B)$ with $|B| \geq |A|$, then $A = \text{exc}(G), B = \text{nexc}(G)$, and $|B| - |A| = 1$. Furthermore, for every $\ell \geq 2$ the complete bipartite graph $K_{\ell,\ell+1}$ is perihamiltonian, and $K_{2,3}$ is the smallest perihamiltonian graph.*

*Proof.* (i) Let $y$ be a neighbour of $x$, $\mathfrak{h}$ a hamiltonian cycle in $G - x$, and $z$ a neighbour of $y$ on $\mathfrak{h}$. Then $(\mathfrak{h} - yz) + yx$ yields the desired path. For the second statement, consider subdivided wheels—that they are indeed perihamiltonian will be shown in the proof of Theorem 1.

(ii) $G$ is 2-connected, so $\delta(G) \geq 2$. Put $T = uvw$ and assume $\deg(v) = 2$. Then $\delta(G/uw) = 1$, so $G/uw$ cannot be hamiltonian. Now suppose $\deg(v) = 3$ and denote the third neighbour of $v$ by $x$. Then the degree of $v$ in $G/uw$ is 2, so a hamiltonian cycle $\mathfrak{h}$ in $G/uw$ must visit the edges $xv$ and $vy$, where $y$ is the vertex obtained when merging $u$ and $w$. However, it is now easy to modify $\mathfrak{h}$ into a hamiltonian cycle in $G$, a contradiction.

(iii) Assume $|B| \geq |A| + 2$ or $|B| = |A|$. Every edge $xy \in E(G)$ satisfies $x \in A$ and $y \in B$. Since neither $G - x$ nor $G - y$ is a balanced bipartite graph, neither $G - x$ nor $G - y$ is hamiltonian, so $G$ cannot be perihamiltonian by Lemma 1. We have shown that $|B| - |A| = 1$. For every $x \in A$, the graph $G - x$ has an unbalanced bipartition, so $G - x$ cannot be hamiltonian, whence $x \in \text{exc}(G)$. It follows that $A \subset \text{exc}(G)$. By Lemma 1, no two exceptional vertices are adjacent, so $\text{exc}(G) = A$. Every neighbour of an exceptional vertex must be non-exceptional, so $\text{nexc}(G) = B$.

If $G = K_{\ell,\ell+1}$, then it has a bipartition $(A, B)$ with $\ell = |B| = |A| + 1$. Consider an arbitrary vertex $y \in B$. Then $G - y = K_{\ell,\ell}$ is hamiltonian. It suffices to note that $K_{\ell,\ell+1}$ is non-hamiltonian, and we obtain that $G$ is perihamiltonian. It is easy to verify that there is no smaller perihamiltonian graph than $K_{2,3}$. □

Collier and Schmeichel [6] showed that in a hypohamiltonian graph, every vertex lying on a triangle has degree at least 4. Goedgebeur and the last author [8] proved that this holds for almost hypohamiltonian graphs, as well. Proposition 1 (ii) generalises both of these results. It is worth noting that hypohamiltonian (and thus perihamiltonian) graphs of girth 3 do exist, as proven by Thomassen [17]. In fact, hypohamiltonian graphs of every girth between 3 and 7 are known, an example of girth 7 being Coxeter's graph. We do not know whether hypohamiltonian graphs of larger girth exist, and even in the superclass of perihamiltonian graphs this is open.

Although hypohamiltonian graphs are necessarily 3-connected, the superclass of perihamiltonian graphs also contains graphs of connectivity 2, which we now investigate.

## 2 Connectivity 2

**Lemma 2.** *In a perihamiltonian graph $G$, every 2-valent vertex is non-exceptional, while every neighbour of a 2-valent vertex is exceptional, and in particular non-2-valent. Thus, every exceptional vertex in $G$ has degree at least 3 and $G$ does not contain adjacent 2-valent vertices. Furthermore, unless $G = K_{2,3}$, a vertex in $G$ has at most two 2-valent neighbours.*

*Proof.* Consider a 2-valent vertex $v$ in $G$ and let $w$ be a neighbour of $v$. Then $G - w$ is non-hamiltonian, and thus $w$ is exceptional. Hence $v$ must be non-exceptional by Lemma 1.

Assume there is a vertex $x$ in $G$ with at least four 2-valent neighbours. Each of these is non-exceptional, and all of their neighbours, including $x$, are exceptional. Consider one such 2-valent neighbour $y$ and remove it. We ought to obtain a hamiltonian cycle in $G - y$, but this graph contains a vertex (namely $x$) with at least three 2-valent neighbours, so it cannot be hamiltonian. Thus, $G - y$ is non-hamiltonian, so $y$ is exceptional. As $x$ is exceptional as well, $G$ is non-perihamiltonian, a contradiction. Now consider $G$ to be a perihamiltonian graph containing a vertex $w$ with exactly three 2-valent neighbours. As before, $w$ is exceptional, its 2-valent neighbours non-exceptional, and all of their neighbours exceptional. We can argue as above and obtain that $w$ must be cubic. First consider the case that the vertices at distance 2 from $w$ are pairwise distinct. In this graph, no additional (i.e. other than the three 2-valent vertices surrounding $w$) non-exceptional vertices may exist, as the removal of such a vertex would leave a graph with a vertex surrounded by more than two 2-valent vertices, so it cannot be hamiltonian. Any exceptional vertex may only neighbour a non-exceptional vertex, but these are 2-valent, so every vertex at distance 2 from $w$ has degree 1 in $G$, which gives a contradiction. In a very similar fashion we can show that if exactly two of the vertices at distance 2 from $w$ coincide, we also obtain a contradiction. Finally, if all three vertices at distance 2 from $w$ are identical we obtain $K_{2,3}$, which is in fact perihamiltonian and the only such graph containing a vertex which has more than two 2-valent neighbours. □

**Proposition 2.** *If $X = \{x, y\}$ is a 2-cut of a perihamiltonian graph $G$, then $xy$ is not an edge of $G$. Furthermore, we have $\omega(G - X) = 3$ iff $G = K_{2,3}$ and $\omega(G - X) = 2$ else. Lastly, every 2-cut in a perihamiltonian graph is trivial, so every perihamiltonian graph of connectivity 2 contains a 2-valent vertex.*

*Proof.* The first statement follows from the fact that any vertex contained in a 2-cut of $G$ is exceptional, but such vertices cannot be adjacent in a perihamiltonian graph. Alternatively, $\kappa(G/xy) = 1$, so $G/xy$ cannot be hamiltonian.

Assume there exists a 2-cut $X$ such that $\omega(G - X) \geq 4$. Then for any edge $e \in E(G)$, we have that $\omega(G/e - X) \geq 3$. However, this is a contradiction, since $G/e$ ought to be hamiltonian. Now assume that $X$ is a 2-cut of $G$ such that $\omega(G - X) = 3$ and that not all 2-fragments $F, F', F''$ with attachments $X$ are trivial. Again, in the non-trivial 2-fragment we can contract an edge and obtain a non-hamiltonian graph. So $F, F', F''$ all must be $P_3$ (the path on three vertices), i.e. $\omega(G - X) = 3$ if and only if $G = K_{2,3}$ as advertised.

Finally, let $G$ be a perihamiltonian graph containing a non-trivial 2-cut $X = \{x, y\}$. Let $F, F'$ be 2-fragments of $G$ with attachments $X$. Contracting an edge in

$F$ we obtain a hamiltonian $xy$-path in $F'$, and analogously we obtain a hamiltonian $xy$-path in $F$. Putting these paths together, we obtain a hamiltonian cycle in $G$, a contradiction. □

We have proven that if a perihamiltonian graph has connectivity 2, then it contains at least one vertex of degree 2, i.e. it has minimum degree 2. How many vertices of degree 2 may it contain? This is the question we now address.

First, observe that if we drop the connectivity 2 constraint, there may be no vertices of degree 2 at all, as exemplified by $K_{3,4}$ or any hypohamiltonian graph. We recall that for a graph $G$, we denote by $V_2(G)$ the set of all 2-valent vertices of $G$.

**Theorem 1.** *For a perihamiltonian graph $G \neq K_{2,3}$ of order $n$, we have that $|V_2(G)| \leq (n-1)/2$, and $G$ attains this bound iff $G$ is a subdivided wheel. Furthermore, if $G$ is a bipartite perihamiltonian graph, e.g. a subdivided wheel, then for any $e \in E(\overline{G})$ such that $G + e$ is bipartite, the graph $G + e$ is perihamiltonian.*

*Proof.* By Lemma 2, $G$ does not contain adjacent 2-valent vertices and no vertex in $G$ may have more than two 2-valent neighbours, so $|V_2(G)| \leq n/2$. Assume there exists a perihamiltonian graph $G'$ with $|V_2(G')| = |V(G')|/2$. Then $G'[V_2(G')]$ is a 2-factor consisting of at least two cycles, each made up of alternating exceptional and non-exceptional (2-valent) vertices. But then half of the vertices in $G'$ must be exceptional, contradicting Lemma 1.

Let $G$ be a subdivided wheel of order $n$. We now show that $G$ is perihamiltonian. $G[V_2(G)]$ is an $(n-1)$-cycle, so the (unique) vertex $v$, where $\{v\} = V(G) \setminus V(G[V_2(G)])$, is non-exceptional. $G$ has a bipartition

$$(A = V_2(G) \mathbin{\dot\cup} \{v\}, B = V(G) \setminus A),$$

where $|A| = |B| + 1$, so $G$ is non-hamiltonian. By Lemma 2, every neighbour of a vertex of degree 2 is exceptional. It is easy to see that every 2-valent vertex in $G$ is avoided by an $(n-1)$-cycle and thus, is non-exceptional. As $B = \mathrm{exc}(G)$ is an independent set, $G$ is perihamiltonian.

Now let us show that there is no perihamiltonian graph $G$ of order $n$ with $|V_2(G)| = (n-1)/2 = k$ other than the subdivided wheel. The graph $G$ has exactly $k + 1$ vertices of degree at least 3, which we call *white*. All 2-valent vertices in $G$, which we call *black*, are non-exceptional. By Lemma 2 no two black vertices in $G$ are adjacent.

We now study the (possibly disconnected) graph $H$, which is defined as $G$ from which we have removed all edges between white vertices. Assume among the components of $H$ is a cycle $C$ and a non-trivial (i.e. $\neq K_1$) component $C'$. The cycle $C$ must be of even length, and have alternating black and white vertices. There must be at least one black vertex $v \in V(C')$ (in particular, $v$ lies outside of $C$), and since black vertices are non-exceptional, $G - v$ is hamiltonian. But since $C' - v$ is non-empty and every second vertex of $C$ is 2-valent, we have a contradiction. Two cases remain:

Case 1. $H$ is a path $v_1 \ldots v_n$. Necessarily, $H$ consists of alternating black and white vertices, and has white end-vertices. As $v_2$ is black, it is non-exceptional, so $G - v_2$ has a hamiltonian cycle $\mathfrak{h}$. This cycle must contain the path $v_3 \ldots v_n$, as well as the edges $v_1v_3$ and $v_1v_n$. But then $G$ is hamiltonian—a contradiction.

CASE 2. $H$ consists of two disjoint components, namely an $(n-1)$-cycle $C$ and $K_1$. The cycle $C$ consists of alternating black and white vertices, so the (unique) vertex $v$ in $H - C$ must be white. The white vertices on $C$ have degree at least 3, so each such vertex must have at least one additional incident edge (in $G$), which is not yet present in $H$. Furthermore, each such vertex is exceptional, so they are pairwise non-adjacent. Thus, they must all be adjacent to $v$.

The only possible graph obtained in this manner is the subdivided wheel.

Let $G$ be a non-complete bipartite perihamiltonian graph and $e \in E(\overline{G})$ such that $G + e$ is bipartite. We now show that $G + e$ is perihamiltonian. $G + e$ is non-hamiltonian since the unbalanced bipartition guaranteeing the non-hamiltonicity of $G$ persists in $G + e$. The same argument yields that no exceptional vertex in $G$ becomes non-exceptional through the addition of the edge $e$. Clearly, every vertex which was non-exceptional in $G$ is non-exceptional in $G + e$. □

Starting from a subdivided wheel and adding edges one-by-one as indicated in Theorem 1, we obtain:

**Corollary 1.** *For every odd $n > 5$ and every $k \in \{0, \dots, (n-1)/2\}$, there exists a perihamiltonian graph of order $n$ having exactly $k$ 2-valent vertices.*

**Corollary 2.** *The number of bipartite $n$-vertex perihamiltonian graphs is $\Omega(n^2)$.*

**Theorem 2.** *For all $n \geq 5$ except for $n \in \{6, 8, 10\}$ there is a planar perihamiltonian graph of order $n$. Moreover, there is no perihamiltonian graph of order $\leq 4, 6$ or $8$, while on $10$ vertices there are exactly two such graphs, namely the Petersen graph $P$ and $P$ minus an edge.*

*Proof.* For odd $n \geq 5$ consider the subdivided wheel on $n$ vertices. For $n = 5$ this is $K_{2,3}$, which turns out to be the only perihamiltonian graph of order 5 (and it is planar).

For even $n \geq 10$, we first note that there are exactly two perihamiltonian graphs of order 10: the Petersen graph, and the Petersen graph minus an edge. Both of these graphs are non-planar. Consider the $(2k-1)$-gonal prism

$$\Pi_{2k-1} = (\{x_i, x'_i\}_{i=0}^{2k-2}, \{x_i x_{i+1}, x'_i x'_{i+1}, x_i x'_i\}_{i=0}^{2k-2}),$$

addition mod $2k - 1$.

In the remainder of the proof, let $k \geq 3$. For $n = 4k$ let $H_{4k}$ be the graph resulting from $\Pi_{2k-1}$ by subdividing the edges $x_1 x'_1$ and $x_3 x'_3$ by new vertices $x''_1$ and $x''_3$, respectively, see the left-hand side of Fig. 2 for $k = 3$. $H_{4k}$ is of order $4k$ and the straightforward proof that it is perihamiltonian is left to the reader.

For $n = 4k + 2$ with $k \geq 3$ let $H_{4k+2}$ be the graph resulting from $\Pi_{2k-1}$ by subdividing the edges $x'_0 x'_1, x'_1 x'_2$, and $x_3 x'_3$ by new vertices $x''_0, x''_2$, and $x''_3$, respectively, and by connecting $x''_0$ and $x''_2$ by the path $x''_0 x''_1 x''_2$. See the right-hand side of Fig. 2 for an illustration of the case $k = 3$. $H_{4k+2}$ is of order $4k + 2$ and, as above, the fact that it is perihamiltonian is not difficult to verify and therefore omitted here.

We used a computational approach to check that the theorem's second statement holds. The details are described in Section 5. □

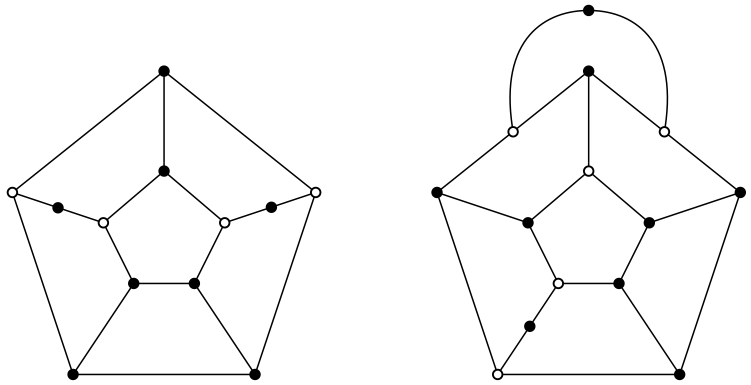

Figure 2: The graphs $H_{12}$ and $H_{14}$ from the proof of Theorem 2, which are planar perihamiltonian graphs of order 12 and 14, respectively.

# 3 Connectivity 3

**Lemma 3.** *Let $G$ be a perihamiltonian graph of connectivity 3, and let $X$ be a 3-cut in $G$. If $X$ contains a non-exceptional vertex $v$, then $\omega(G - X) = 2$. Furthermore, in general, no two non-exceptional vertices of $X$ are adjacent. In particular, $G[X] \neq K_3$.*

*Proof.* Put $X = \{x, y, z\}$. Assume that $\omega(G - X) \geq 3$. Since $v$ is non-exceptional we have that $G - v$ is hamiltonian. But then $G - v$ has a 2-cut $X' = X \setminus \{v\}$ such that $\omega(G - X') \geq 3$, a contradiction.

Suppose $x$ and $y$ are adjacent and non-exceptional in $G$ and let $F$ and $F'$ be the 3-fragments with attachments $X$. Then $F$ contains a hamiltonian $yz$-path and $F'$ contains a hamiltonian $xz$-path. But these paths together with the edge $xy$ form a hamiltonian cycle in $G$, a contradiction. Now assume $G[X] = K_3$, so $xy, yz, zx \in E(G)$. By Lemma 1, $\text{exc}(G)$ is an independent set, so at most one of $x, y, z$, say $y$, lies in $\text{exc}(G)$. Thus $x$ and $z$ are non-exceptional, and we are led to a contradiction as above. □

The first result of Lemma 3 is best possible in the sense that if all vertices in a 3-cut of a perihamiltonian graph are allowed to be exceptional, then the removal of this 3-cut can yield more than two components: consider $K_{3,4}$.

We now discuss Thomassen's gluing procedure for 3-fragments of hypohamiltonian graphs [19] and an extension thereof. Let $F, F'$ be 3-fragments of (not necessarily planar) graphs of connectivity 3, and let $F$ have attachments $x_1, x_2, x_3$ and $F'$ have attachments $x'_1, x'_2, x'_3$. Identifying $x_i$ with $x'_i$ for all $i$, we obtain the graph $(F, \{x_1, x_2, x_3\}) \vdots (F', \{x'_1, x'_2, x'_3\})$. When the vertices that are being identified (always using a bijection) are clear from context, we simply write $F \vdots F'$. Thomassen proved that if $F$ and $F'$ are 3-fragments of hypohamiltonian graphs, not both trivial, then $F \vdots F'$ is hypohamiltonian [19]. A generalisation of this statement can be found in [28]. We now give a perihamiltonian version of this result:

**Theorem 3.** *Let $G_1$ and $G_2$ be disjoint 3-connected perihamiltonian graphs and $i \in \{1,2\}$. Furthermore, let there be a 3-fragment $F_i$ in $G_i$ with attachments $X_i$ such that either $\mathrm{exc}(G_i) \cap X_i = \emptyset$, or $\mathrm{exc}(G_i) \cap X_i = \{w_i\}$ and $E(G_i[X_i]) \neq \emptyset$. In either case, $\Gamma = (F_1, X_1) \vdots (F_2, X_2)$ is perihamiltonian, where in the latter case $w_1$ is identified with $w_2$. If not both $F_1$ and $F_2$ are trivial, then $\Gamma$ is 3-connected, and if $G_1$ and $G_2$ are planar, then so is $\Gamma$.*

*Proof.* Consider first the case $|\mathrm{exc}(G_i) \cap X_i| = 0$. As in the proof of [28, Lemma 2] one deduces that, seeing $F_1$ and $F_2$ as subgraphs of $\Gamma$, we have $\mathrm{exc}(\Gamma) = \mathrm{exc}(F_1) \cup \mathrm{exc}(F_2)$ and $\mathrm{exc}(F_1) \cap \mathrm{exc}(F_2) = \emptyset$. Since $G_1$ and $G_2$ are perihamiltonian, no two exceptional vertices in $F_1$ and $F_2$ are adjacent. Thus $\mathrm{exc}(\Gamma)$ forms an independent set in $\Gamma$, so $\Gamma$ is perihamiltonian.

Now assume $\mathrm{exc}(G_i) \cap X_i = \{w_i\}$ and $E(G_i[X_i]) \neq \emptyset$. Although the following reasoning to a large extent mimics the proof of [29, Lemma 4], certain arguments differ, and these we explain in detail below.

Let $X_i = \{x_{i1}, w_i, x_{i2}\}$. By Lemma 3, $\omega(G_i - X_i) = 2$. In $G_i$, consider the 3-fragment $J_i \neq F_i$ with attachments $X_i$. We shall denote by $x_j \in V(\Gamma)$, $j \in \{1,2\}$, the vertex one obtains by identifying $x_{1j}$ with $x_{2j}$, and $w$ the vertex obtained by identifying $w_1$ and $w_2$.

Claim. *There is no hamiltonian $x_{i1}x_{i2}$-path in $F_i - w_i$.*

*Proof of the Claim.* Assume there is such a path $\mathfrak{p}$. As $E(G_i[X_i]) \neq \emptyset$, due to Lemma 3, $x_1w$ or $x_2w$ lie in $E(\Gamma)$, say the former. Since $x_{i1}$ is non-exceptional in $G_i$, there exists a hamiltonian path $\mathfrak{p}'$ in $J_i - x_{i1}$ with end-vertices $w_i, x_{i2}$. $\mathfrak{p} \cup \mathfrak{p}' + x_1w$ is a hamiltonian cycle in $G_i$, a contradiction.

Due to the claim, $\Gamma - w$ cannot be hamiltonian. Assume $\Gamma$ contains a hamiltonian cycle $\mathfrak{h}$. Without loss of generality let $F_1 \cap \mathfrak{h} = \mathfrak{p}$ be connected. Since $x_1$ and $x_2$ are non-exceptional, the end-vertices of $\mathfrak{p}$ must be $x_1$ and $x_2$ (otherwise we obtain a contradiction with the non-hamiltonicity of $G_1$). Then $(F_2 - w) \cap \mathfrak{h}$ is a hamiltonian $x_{21}x_{22}$-path, contradicting the claim.

As the graph $G_i - x_{ij}$ is hamiltonian, we have a hamiltonian $x_{ik}w_i$-path $\mathfrak{p}_{ij}$ in $F_i - x_{ij}$, where $k \neq j$, for all $i, j$. Then $\mathfrak{p}_{1i} \cup \mathfrak{p}_{2i}$ implies that $x_1$ and $x_2$ are non-exceptional in $\Gamma$. The remainder of the proof can be dealt with exactly as in the proof of [29, Lemma 4] (choosing instead of an arbitrary vertex a non-exceptional one), so it is omitted.

We have proven that no non-exceptional vertex of $F_1$ or $F_2$ is exceptional in $\Gamma$. This completes the proof. □

Theorem 3 is false without requiring "$E(G_i[X_i]) \neq \emptyset$": in [8] an almost hypohamiltonian graph is given (reproduced here in Fig. 3) with exceptional vertex $y$. Now consider two copies of the 3-fragment with attachments $\{x, y, z\}$ (these form a non-trivial cut) which has ten vertices. By identifying, using a bijection, the respective attachments of these two fragments, we get a graph $G$ containing a vertex $y'$ of degree 2 such that $G - y'$ is non-hamiltonian (this is easily inferred from the non-hamiltonicity of the Petersen graph). Since $y'$ is 2-valent, the removal of one of its neighbours also yields a non-hamiltonian graph, so $G$ cannot be perihamiltonian by Lemma 1.

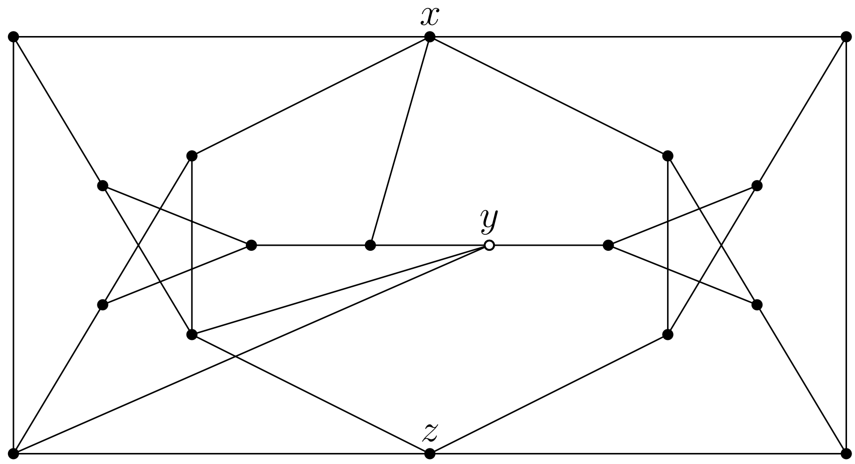


Figure 3: An almost hypohamiltonian graph; its exceptional vertex is $y$.

In Theorem 3, $F_1$ or $F_2$ (possibly both) may be trivial: if both are trivial, we obtain $K_{2,3}$, which is perihamiltonian, and if without loss of generality $F_1$ is trivial, we simply have to consider $F_2 \,\vdots\, F_1 = F_2 \,\vdots\, K_{1,3}$, i.e. add a new vertex $w$ to $F_2$ together with the three edges between $w$ and the attachments of $F_2$.

We also remark that every perihamiltonian graph containing a cubic exceptional vertex $w$ contains at least one non-trivial 3-fragment satisfying above conditions, namely the non-trivial one with attachments $N(w)$.

If all vertices in the 3-cuts we glue are exceptional, gluing may fail to produce a perihamiltonian graph: take $K_{3,4}$ and $x$ a vertex in the bigger partition. Then $K_{3,4} - x$ is a 3-fragment of a perihamiltonian graph. Glue two of these and we get $K_{3,6}$, a non-perihamiltonian graph. Even restricting ourselves to the polyhedral case, gluing may fail to yield a perihamiltonian graph: consider the graph shown in Fig. 5 and remove the vertex $v_2$. We obtain a 3-fragment $F$. Gluing $F$ with a copy thereof can be done such that the resulting graph is bipartite and polyhedral, with bipartite sets of order 30 and 27, which by Proposition 1 (iii) cannot be perihamiltonian.

Thomassen [20] used the following operation to show that there exist infinitely many planar cubic hypohamiltonian graphs, thereby solving a problem of Chvátal. Consider a graph $G$ to contain a 4-cycle $C = v_1v_2v_3v_4$. Then $\mathrm{Th}(G_C)$ shall be the graph we obtain by removing the edges $v_1v_2$ and $v_3v_4$ from $G$, adding a 4-cycle $v_1'v_2'v_3'v_4'$ disjoint from $G$, as well as the edges $v_iv_i'$, $1 \leq i \leq 4$. Whenever we use the notation $\mathrm{Th}(G_C)$ without giving $C$ explicitly (i.e. in the form $v_1v_2v_3v_4$), then our statement concerns any one of the two graphs resulting from the application of Th to $G$ and a cycle $C$ therein. The operation Th preserves planarity, 3-connectedness, and 3-regularity. We shall also make use of the operation $\mathrm{Cu}(G_C) = \mathrm{Th}(G_C) + v_1v_2 + v_3v_4$. The following result is essentially due to Thomassen, who gives it (without proof) in [20]. A detailed proof for the planar case can be found in [25].

**Proposition 3** (Thomassen [20]). *If a graph $G$ is hypohamiltonian and contains a 4-cycle $C$ whose vertices are cubic, then $\mathrm{Th}(G_C)$ is hypohamiltonian, as well.*

We now prove a perihamiltonian analogue:

**Lemma 4.** *Let $G$ be a perihamiltonian graph that contains a 4-cycle $C = v_1v_2v_3v_4$.*

*(i) If $v_1$ is exceptional and has degree at most 3, then $\mathrm{Cu}(G_C)$ is perihamiltonian.*

*(ii) If $v_1$ is non-exceptional and the degree of $v_3$ is at most 3, then $\mathrm{Th}(G_C)$ is perihamiltonian.*

*(iii) If $v_1$ and $v_3$ are exceptional vertices, while at least one of $v_2$ and $v_4$ has degree at most 3, then $\mathrm{Cu}(G_C)$ is perihamiltonian.*

*Proof.* (i) It is easily verified that if $\mathrm{Cu}(G_C)$ is hamiltonian, then any hamiltonian cycle in $\mathrm{Cu}(G_C)$ can be translated to a hamiltonian cycle in $G$ by replacing the parts visiting $v'_1, v'_2, v'_3, v'_4$ by edges of $C$. Thus, $\mathrm{Cu}(G_C)$ is non-hamiltonian. We consider $G$ to be a subgraph of $\mathrm{Cu}(G_C)$.

First we will show that each non-exceptional vertex $v$ in $V(G) \setminus V(C)$ remains non-exceptional in $\mathrm{Cu}(G_C)$. Let $\mathfrak{h}$ be a hamiltonian cycle in $G - v$. Since $v_1$ is cubic, we have that at least one edge of $v_1v_2$ or $v_1v_4$ is contained in $\mathfrak{h}$. In either case we can replace that edge by a path visiting the new vertices $v'_1, v'_2, v'_3, v'_4$ in order to obtain a hamiltonian cycle in $\mathrm{Cu}(G_C) - v$.

Since $v_1$ is exceptional in $G$, we have that both $v_2$ and $v_4$ are non-exceptional in $G$. We will now show that they remain non-exceptional in $\mathrm{Cu}(G_C)$. The proof for both cases is very similar, so we only give the proof for $v_2$. Let $\mathfrak{h}$ be a hamiltonian cycle in $G - v_2$. Since $v_1$ is cubic, we have that $v_1v_4$ is contained in $\mathfrak{h}$, so $(\mathfrak{h} - v_1v_4) \cup v_1v'_1v'_2v'_3v'_4v_4$ is a hamiltonian cycle in $\mathrm{Cu}(G_C) - v_2$.

Next we show that if $v_3$ is non-exceptional, it remains non-exceptional. Assume that $G - v_3$ is hamiltonian and let $\mathfrak{h}$ be a hamiltonian cycle in $G - v_3$. Since $v_1$ is cubic, we have that at least one edge of $v_1v_2$ or $v_1v_4$ is contained in $\mathfrak{h}$. In either case we can replace that edge by a path visiting the new vertices $v'_1, v'_2, v'_3, v'_4$ in order to obtain a hamiltonian cycle in $\mathrm{Cu}(G_C) - v$.

Finishing the proof of (i), we show that $v'_1$ and $v'_3$ are non-exceptional. The vertex $v_2$ is non-exceptional in $G$. Let $\mathfrak{h}$ be a hamiltonian cycle in $G - v_2$. Since $v_1$ is cubic, we have that $v_1v_4$ is contained in $\mathfrak{h}$. The hamiltonian cycle $(\mathfrak{h} - v_1v_4) \cup v_1v_2v'_2v'_3v'_4v_4$ is a hamiltonian cycle in $\mathrm{Cu}(G_C) - v'_1$ and the hamiltonian cycle $(\mathfrak{h} - v_1v_4) \cup v_1v_2v'_2v'_1v'_4v_4$ is a hamiltonian cycle in $\mathrm{Cu}(G_C) - v'_3$.

(ii) As $\mathrm{Cu}(G_C)$ is non-hamiltonian, so is $\mathrm{Th}(G_C)$. We consider $G - \{v_1v_2, v_3v_4\}$ to be a subgraph of $\mathrm{Th}(G_C)$. Since $v_1$ is non-exceptional and $v_3$ has degree at most 3, there exists a hamiltonian $v_2v_4$-path $\mathfrak{p}$ in $G - \{v_1, v_3\}$. Seeing $\mathfrak{p}$ as lying in $\mathrm{Th}(G_C)$, we may add to $\mathfrak{p}$ the paths $v_2v_3v'_3v'_2v'_1v'_4v_4$, $v_2v'_2v'_3v'_4v'_1v_1v_4$, $v_2v_3v'_3v'_4v'_1v_1v_4$, or $v_2v_3v'_3v'_2v'_1v_1v_4$. Thus, we find hamiltonian cycles in $\mathrm{Th}(G_C) - v$ for $v = v_1, v_3, v'_2, v'_4$, respectively.

Note that it cannot occur that $v_2$ was non-exceptional in $G$ but it (more precisely, its image under Th) has become exceptional in $\mathrm{Th}(G_C)$: Let $\mathfrak{h}$ be a hamiltonian cycle in $G - v_2$. Since the degrees of $v_3$ and $v_4$ are at most 3, we have $v_3v_4 \in E(\mathfrak{h})$. Then $(\mathfrak{h} - v_3v_4) \cup v_3v'_3v'_2v'_1v'_4v_4$ gives a hamiltonian cycle in $\mathrm{Th}(G_C) - v_2$, so $v_2$ is non-exceptional in $\mathrm{Th}(G_C)$, too. The reasoning for $v_4$ is analogous. More generally, very similar arguments yield that if a vertex is exceptional in $G$, then so it is in $\mathrm{Th}(G_C)$, and if a vertex is non-exceptional in $G$, then so it is in $\mathrm{Th}(G_C)$—the details are left to the reader and use the same rerouting idea as described above.

Since $G$ is perihamiltonian, the existence of the aforementioned hamiltonian cycles implies that $\text{Th}(G_C)$ is perihamiltonian.

(iii) The proof is very similar to the proofs of (i) and (ii), and therefore omitted. □

**Theorem 4.** *For $n$ odd, there exists a polyhedral perihamiltonian graph of order $n$ if and only if $n \geq 11$. For $n$ even, there exists a polyhedral perihamiltonian graph of order $n$ if $n = 24$, or $n \geq 28$, and there does not exist a polyhedral perihamiltonian graph of order $n$ if $n < 20$.*

*Proof.* It is well-known that Herschel's 11-vertex graph is the smallest non-hamiltonian polyhedral graph, and it is straightforward to verify that Herschel's graph is a 5-hypohamiltonian graph whose set of exceptional vertices is independent. Thus, by Lemma 1, it is perihamiltonian. Applying Lemma 4, we obtain perihamiltonian polyhedral graphs of order $11 + 4k$ for every $k \geq 0$.

Take a hexagonal prism, denoting a facial hexagon with $v_1 \dots v_6$. Add a new vertex $v$ and the edges $vv_1$, $vv_3$ and $vv_5$. We obtain a non-hamiltonian polyhedron on 13 vertices. This graph is perihamiltonian. Applying Lemma 4, we obtain perihamiltonian polyhedral graphs of order $13 + 4k$ for every $k \geq 0$.

Consider the perihamiltonian polyhedron on 24 vertices from Fig. 4. This graph was discovered as a platypus graph by Neyt in [15]. This graph contains exactly one quadrangle which satisfies the properties of Lemma 4 (ii) so applying this lemma we obtain perihamiltonian polyhedral graphs of order $24 + 4k$ for every $k \geq 0$. We can also find a fragment containing 23 vertices in this graph that satisfies the properties of Theorem 3 such that the quadrangle is completely disjoint form the 3-cut. Herschel's 11-vertex graph contains a fragment on 10 vertices that satisfies the properties of Theorem 3. Combining these two fragments using that theorem, we obtain a perihamiltonian polyhedron on 30 vertices which still contains a quadrangle which satisfies the properties of Lemma 4 (ii). Applying this lemma we obtain perihamiltonian polyhedral graphs of order $30 + 4k$ for every $k \geq 0$.

The non-existence of even orders up to 18 is handled computationally in Section 5. □

Thomassen [20] showed that there exist planar graphs which are not induced subgraphs of any planar hypohamiltonian graph. We now extend this result to the family of perihamiltonian graphs.

**Proposition 4.** *There exists no perihamiltonian graph containing a triangulation of the plane on at least 4 vertices as an induced subgraph. Thus, there exist no perihamiltonian triangulations of the plane. Furthermore, there exist infinitely many planar graphs which are not induced subgraphs of any planar perihamiltonian graph.*

*Proof.* Let $H \neq K_3$ be a triangulation of the plane and an induced subgraph of a planar perihamiltonian graph $G$. $G = H$ is impossible: on the one hand, in a perihamiltonian graph $G$, any three pairwise adjacent vertices must be non-exceptional, but on the other hand, by Whitney's theorem [22] stating that planar 4-connected triangulations are hamiltonian, a non-hamiltonian triangulation contains at least one separating triangle—Lemma 3 states that every vertex of such a triangle must be exceptional, a contradiction.

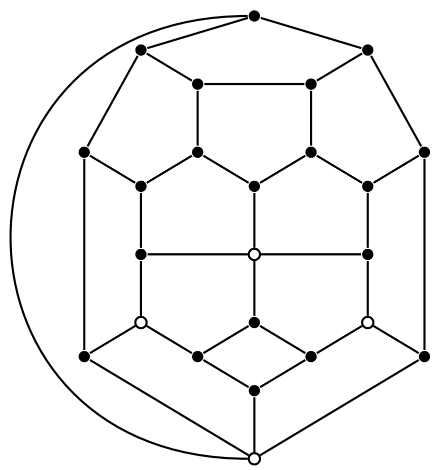

Figure 4: A perihamiltonian polyhedron on 24 vertices. This graph was discovered as a platypus graph by Neyt in [15].

So $G \neq H$. Then $H$ is a 3-fragment of $G$ and the attachments of $H$ form a 3-cut $X$ in $G$. But since $H$ is a triangulation, $G[X]$ forms a separating triangle in $G$, contradicting Lemma 3. □

On the other hand, if we do not restrict ourselves to the plane, it was proven in [30] that given an arbitrary graph $H$ one can find a suitable hypohamiltonian graph $G$ (which, of course, is also perihamiltonian) such that $H$ is an induced subgraph of $G$. This settles an old problem of Chvátal.

We end this section with the principal results of this article, emphasising that the following theorems are motivated by Thomassen's result stating that a planar hypohamiltonian graph must contain a vertex that is cubic [19], and the question whether this is true for the significantly larger class of perihamiltonian graphs. There are two main ingredients in the proof of Thomassen's result: on the one hand, that every polyhedral graph with at most one 3-cut is hamiltonian, an extension of Tutte's classic theorem that planar 4-connected graphs are hamiltonian [21], and on the other hand a gluing procedure for 3-fragments of hypohamiltonian graphs. The former was recently strengthened to allow up to three 3-cuts while still being able to infer hamiltonicity [4], and an extension of the latter to perihamiltonian graphs follows shortly. But first, we answer affirmatively the question from this paragraph's first sentence and thus extend Thomassen's theorem, noting that our strategy is entirely different than Thomassen's, but instead follows the same lines as a proof given in [29].

**Theorem 5.** *Every planar perihamiltonian graph of connectivity $\kappa$ contains a vertex of degree $\kappa$.*

*Proof.* Let $G$ be a planar perihamiltonian graph. Combining the definition of perihamiltonian graphs with Tutte's result [21] stating that planar 4-connected graphs are hamiltonian, $G$ has connectivity 2 or 3.

The case when $G$ has connectivity 2 follows immediately from Proposition 2. We now may assume $G$ to be of connectivity 3. It was proven in [4] that a polyhedral graph of connectivity 3 must contain a 3-cut $X = \{x, y, z\}$ such that for at least one of the 3-fragments $F, F'$ with attachments $X$ (there are exactly two such fragments since $K_{3,3}$ is non-planar), say $F$, the graph $\overline{F} = F + xy + yz + zx$ is either $K_4$ or 4-connected. Since $G$ has minimum degree at least 4, $\overline{F} \neq K_4$, whence $\overline{F}$ must be 4-connected.

It is now easy to see that $\overline{F}$ contains a non-exceptional vertex $v$ due to the fact that, as $G$ is perihamiltonian, no two exceptional vertices in $G$ may be adjacent. Since $X$ is a 3-cut, the hamiltonicity of $G - v$ yields, ignoring analogous cases, that there exists either a hamiltonian $yz$-path $\mathfrak{p}'$ in $F' - x$ (for $v = x$) or $F'$ (for $v \notin X$).

If there is a hamiltonian $yz$-path $\mathfrak{p}'$ in $F' - x$, we use a special case of [4, Lemma 14], from which follows that $\overline{F}$ contains a hamiltonian $yz$-path $\mathfrak{p}$ such that $E(\mathfrak{p}) \cap \{xy, yz, zx\} = \emptyset$. Then $\mathfrak{p} \cup \mathfrak{p}'$ is a hamiltonian cycle in $G$, a contradiction. If there is a hamiltonian $yz$-path $\mathfrak{p}'$ in $F'$, we apply Sanders' theorem stating that in a planar 4-connected graph there exists a hamiltonian cycle through any pair of edges [16]. Thus, we obtain a hamiltonian cycle in $\overline{F}$ using the edges $xy, zx$ and thus a hamiltonian $yz$-path $\mathfrak{p}$ in $\overline{F} - x$. Since none of the edges $xy, yz, zx$ lie in $\mathfrak{p}$, the path $\mathfrak{p}$ lies in $F - x$. As above, $\mathfrak{p} \cup \mathfrak{p}'$ is a hamiltonian cycle in $G$, which contradicts the perihamiltonicity of $G$. □

We note that Theorem 5 cannot be strengthened to the statement "every planar perihamiltonian graph contains a 2-valent vertex" as planar cubic perihamiltonian graphs exist (see e.g. [20]).

However, by Proposition 2 we know that every perihamiltonian graph of connectivity 2—planar or not—contains a 2-valent vertex. Is it true that every perihamiltonian graph of connectivity 3 must contain a cubic vertex? Related to this, for the subclass of hypohamiltonian graphs Thomassen's 1978 questions whether there are hypohamiltonian graphs with no cubic vertices, or whether 4-connected such graphs exist [19], remain open.

Going back to our introductory question whether every planar hypohamiltonian graph contains a pair of *adjacent* cubic vertices, we now show that this is not true for planar perihamiltonian graphs.

**Theorem 6.** *There exists a polyhedral perihamiltonian graph containing exactly ten cubic vertices, no two of which are adjacent. Furthermore, there exist infinitely many polyhedral perihamiltonian graphs in which no two cubic vertices are adjacent, and infinitely many polyhedral perihamiltonian graphs with a constant number of cubic vertices.*

*Proof.* For the first statement, let $G$ be the polyhedral bipartite graph depicted in Fig. 5. Denote the vertex bipartition given in Fig. 5 by $(A, B)$, where $A$ shall be the set of white vertices and $B$ the set of black vertices. As $|A| + 1 = |B|$, the graph $G$ is non-hamiltonian, and every vertex in $A$ is exceptional. We leave to the reader the straightforward verification that every vertex in $B$ is non-exceptional, whence, $G$ is perihamiltonian.

Let $x$ be a white cubic vertex in $G$, for instance as chosen in Fig. 5. Put $G' = G - x$. Consider two copies of the 3-fragment $G'$ and identify its attachments (which are, by construction, non-exceptional) as described in Theorem 3. We obtain a perihamiltonian graph with no adjacent cubic vertices. From this graph in the same manner a suitable 3-fragment can be obtained (again by choosing a cubic exceptional vertex and deleting it) which we glue to $G'$, etc. This proves the second statement.

For the final statement, we apply Lemma 4 to $G$ and a suitable quadrilateral therein (e.g. the quadrilateral marked $C = v_1v_2v_3v_4$ in Fig. 5) to construct an infinite family of polyhedral perihamiltonian graphs containing a constant number of cubic vertices—for $C$ as chosen here we obtain the constant 12. □

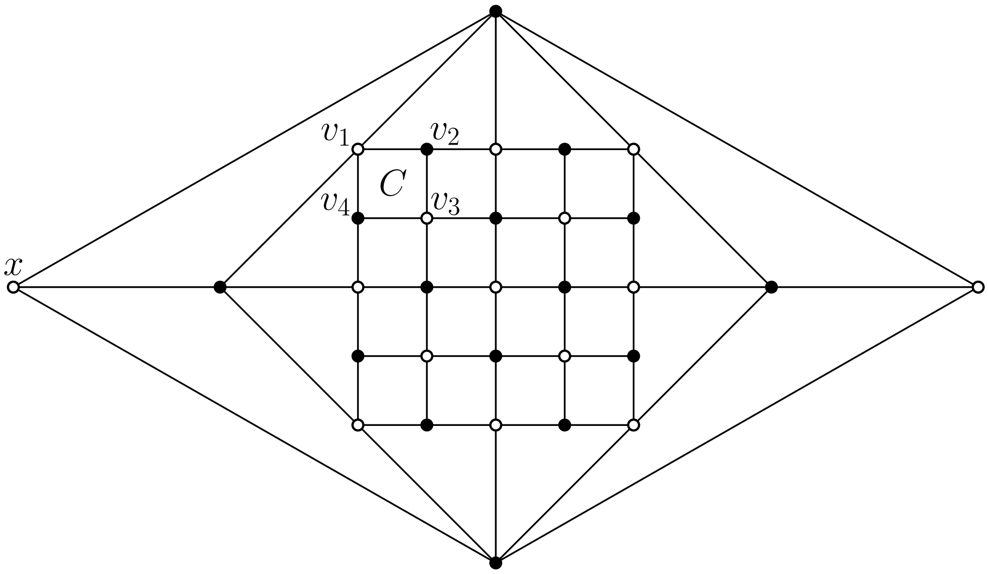


Figure 5: A polyhedral perihamiltonian graph containing exactly ten cubic vertices, no two of which are adjacent.

Having established Theorems 5 and 6, we would very much like to know how many cubic vertices a polyhedral perihamiltonian graph *must* contain—it is at least one and at most eight (Herschel's graph). It is known that a planar hypohamiltonian graph contains at least four and at most thirty cubic vertices [28].

# 4 Higher connectivity

Although Thomassen's question whether 4-connected hypohamiltonian graphs exist remains open [19], as indicated in Proposition 1 the complete bipartite graphs $K_{k,k+1}$ yield structurally simple examples of $k$-connected perihamiltonian graphs. We now address the natural question whether there are other such graphs. Following Chvátal [5], we say that a pair of vertices $(a, b)$ of a graph $G$ is *good* if there is a hamiltonian $ab$-path in $G$. A pair of pairs $((a,b),(c,d))$ of vertices of $G$ is said to be *good* if there exists a spanning subgraph of $G$ consisting of two disjoint paths, one between $a$ and $b$ and one between $c$ and $d$. Chvátal used in [5] so-called flip-flops, a generalisation of which (due to Hsu and Lin [11]) we now introduce: The quintuple $(H,a,b,c,d)$ is a *J-cell* if $H$ is a graph and $a,b,c,d \in V(H)$ such that

1. The pairs $(a,d),(b,c)$ are good in $H$.
2. None of the pairs $(a,b),(a,c),(b,d),(c,d),((a,b),(c,d)),((a,c),(b,d))$ are good in $H$.
3. For each $v \in V(H)$ there is a good pair in $H-v$ among $(a,b),(a,c),(b,d),(c,d)$, $((a,b),(c,d)),((a,c),(b,d))$.

J-cells can be obtained by deleting two adjacent cubic vertices of a hypohamiltonian graph—we remark that the smallest J-cell is obtained in this manner from the Petersen graph—, as was observed by Horton, who used the five copies of the J-cell obtained from the Petersen graph to construct the first (and smallest known) example of a 3-connected hypotraceable graph [10]. Thomassen generalised Horton's construction [18]. Wiener [23, 24] further generalised this construction as follows.

Let $F_i = (H_i, a_i, b_i, c_i, d_i)$ be pairwise disjoint J-cells for $i = 1, \ldots, k$ and put

$$G_k = \left( \bigcup_{i=1}^{k} V(H_i), \ \bigcup_{i=1}^{k} E(H_i) \cup \bigcup_{i=1}^{k-1} b_i a_{i+1} \cup \bigcup_{i=1}^{k-1} c_i d_{i+1} \cup b_k a_1 \cup c_k d_1 \right).$$

The graph $G_k$ is 3-connected for all $k \geq 4$. For a (possibly disconnected) graph $G$, we denote by $\mu(G)$ the minimum number of vertex-disjoint paths that cover the vertices of $G$ (a path may consist of just one vertex), and $G$ is *$k$-path-critical* if for any $v \in V(G)$ we have $\mu(G-v)+1 = \mu(G) = k$. (Hypotraceable and 2-path-critical graphs coincide.) We need the following result.

**Theorem 7** (Wiener [23]). *For every $k \geq 0$, the graph $G_{4k+5}$ is $(k+2)$-path-critical.*

For $k = 1$ the following lemma was already used in [26]—we now generalise it.

**Lemma 5.** *For every $k \geq 1$, the join $G'$ of $kK_1$ and a $(k+1)$-path-critical graph $G$ is perihamiltonian.*

*Proof.* We see $G$ and the $k$ copies of $K_1$, which we write as a $k$-component graph $H$, as subgraphs of $G'$. Let $w \in V(H)$. Assume $G'$ $(G'-w)$ contains a hamiltonian cycle $\mathfrak{h}$ $(\mathfrak{h}_w)$. By construction, $\omega(G \cap \mathfrak{h}) \leq k$ and $\omega(G \cap \mathfrak{h}_w) \leq k-1$, but $\mu(G) = k+1$, so $G'$ and $G'-w$ are non-hamiltonian. Now let $v \in V(G)$. Denote by $(a_i, b_i)_{i=1}^{k}$ the respective end-vertices of the $k$ paths $P_i$ which together span $G-v$, and put $V(H) = \{v_1, \ldots, v_k\}$. We write $a_i \ldots b_i$ for the path $P_i$ traversed from $a_i$ to $b_i$. Then

$$v_1 a_1 \ldots b_1 v_2 a_2 \ldots b_2 v_3 a_3 \ldots b_3 v_4 \ldots v_k a_k \ldots b_k v_1$$

is a hamiltonian cycle in $G'-v$.

We have established that $\mathrm{exc}(G') = V(H)$. Since the vertices of $H = kK_1$ are pairwise non-adjacent, $G'$ is perihamiltonian. $\square$

**Theorem 8.** *For every $k \geq 2$, there exist infinitely many $k$-connected perihamiltonian graphs.*

*Proof.* For $k \in \{2, 3\}$, this follows from the previous sections. It remains to show the statement for $k \geq 4$. Let $\ell \geq 4$ and construct $G_{4\ell+5}$ as defined above, i.e. by removing adjacent cubic vertices from hypohamiltonian graphs—there are infinitely many such graphs, see e.g. [1]. The graph $G_{4\ell+5}$ is 3-connected and, by Theorem 7, $(\ell+2)$-path-critical. Let $t \geq 1$. It is clear that $G_{4\ell+5} + tK_1$ is $(t+3)$-connected. By Lemma 5, $G_{4\ell+5} + tK_1$ is perihamiltonian. □

# 5 Computational results

Determining whether a graph is perihamiltonian is computationally hard. Therefore it is unlikely to find an algorithm which can solve all instances efficiently. Our goal was to develop a program which performs well for most instances we encounter. It should be clear that instead of actually checking all edge-contracted graphs for being hamiltonian, the better approach is to use Lemma 1. A naive implementation might determine for each vertex whether it is exceptional or not and then verify whether the set of exceptional vertices forms an independent set. Especially when a vertex is exceptional, it might take a long time to determine that. Therefore our algorithm receives non-hamiltonian graphs and starts by determining the status of a single vertex. If this vertex is exceptional, then all of its neighbours are tested. If any of the neighbours are exceptional, then the graph is not perihamiltonian. If the vertex is non-exceptional the algorithm proceeds by selecting a new vertex which has an undetermined status and has at least one neighbour with an undetermined status. If no such vertex exists, then the graph is perihamiltonian. The order in which the vertices is tested has been determined heuristically by trying several approaches and choosing the one that performs best for smallish graphs.

Our implementation has been tested against an independent implementation for all graphs on up to 10 vertices and we had complete agreement in all cases. It was also tested on larger graphs for which the perihamiltonicity had been determined theoretically.

In Table 1 we give an overview of all perihamiltonian graphs on up to 12 vertices. The graphs themselves can be downloaded from *House of Graphs* [2] at `http://hog.grinvin.org/Perihamiltonian`. The graphs were generated using `geng` [13], and the perihamiltonicity was verified using the algorithm above combined with a straightforward branch-and-bound algorithm which filters the non-hamiltonian graphs and for which the implementation has already extensively been tested against independent implementations. If we look at the girth of these graphs, we see that the smallest perihamiltonian graphs with girth 3 have 12 vertices and there are six such graphs (one is shown in Fig. 6). The smallest perihamiltonian graph with girth 4 has five vertices and is the subdivided wheel on five vertices, i.e. $K_{2,3}$. The smallest perihamiltonian graphs with girth 5 have ten vertices and are the Petersen graph and the Petersen graph minus one edge. The smallest perihamiltonian graph with girth 6 (resp. 7) has at most 25 (resp. 28) vertices: the former bound is shown by the existence of a hypohamiltonian graph of girth 6 and order 25, see [7], while the latter bound follows from Coxeter's graph.

| $n$ | Connectivity | | | | Total | Time |
|---|---|---|---|---|---|---|
| | 2 | 3 | 4 | 5 | | |
| 3 | | | | | 0 | 0.0 seconds |
| 4 | | | | | 0 | 0.0 seconds |
| 5 | 1 | | | | 1 | 0.0 seconds |
| 6 | | | | | 0 | 0.0 seconds |
| 7 | 3 | 1 | | | 4 | 0.0 seconds |
| 8 | | | | | 0 | 0.1 seconds |
| 9 | 32 | 10 | 1 | | 43 | 0.9 seconds |
| 10 | 1 | 1 | | | 2 | 39.2 seconds |
| 11 | 1305 | 410 | 14 | 1 | 1730 | 1.2 hours |
| 12 | 25 | | | | 25 | 10.7 days |

Table 1: Number of perihamiltonian graphs on $n$ vertices. The running time is total CPU time on a cluster of Intel E5-2680v3 (Haswell-EP @ 2.5 GHz) CPUs.

Since the Petersen graph is perihamiltonian, and it is one of the two smallest perihamiltonian graphs with even order, we have that the Petersen graph is the smallest cubic perihamiltonian graph. We used `genreg` [14] to generate $k$-regular graphs for $k > 3$ and test them for being perihamiltonian using the same setup as described above. All quartic graphs on up to 20 vertices are non-perihamiltonian, and all quintic graphs on up to 18 vertices are non-perihamiltonian.

For the generation of polyhedral graphs we used `plantri` [3] combined with a custom plugin to modify the generation to exclude some graphs which are guaranteed to not be perihamiltonian. More specifically, this plugin removed all graphs containing a facial triangle incident to at least one cubic vertex (see Proposition 1 (ii)). We specifically focused on this property since it is very compatible with the generation algorithm that `plantri` uses for general plane graphs and therefore allowed us to efficiently and significantly bound the number of polyhedral graphs that were generated. Using this combination of `plantri` and plugin we determined all polyhedral perihamiltonian graphs on up to 19 vertices. The counts and the time needed to generate these graphs are summarized in Table 2. The graphs themselves can be downloaded from *House of Graphs* [2] at `http://hog.grinvin.org/Perihamiltonian`.

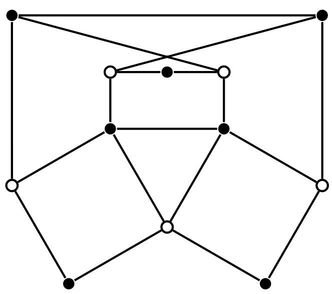

Figure 6: One of the six smallest perihamiltonian graphs with girth 3.

| $n$ | Count | Time |
|---|---|---|
| 11 | 1 | 0.3 seconds |
| 12 | 0 | 2.4 seconds |
| 13 | 5 | 31.4 seconds |
| 14 | 0 | 7.8 minutes |
| 15 | 40 | 2.0 hours |
| 16 | 0 | 1.3 days |
| 17 | 476 | 21.0 days |
| 18 | 0 | 340.3 days |
| 19 | 6808 | 15.3 years |

Table 2: Number of polyhedral perihamiltonian graphs on $n$ vertices. The running time is total CPU time on a cluster of Intel E5-2680v3 (Haswell-EP @ 2.5 GHz) CPUs.

The minimum number of cubic vertices in a perihamiltonian polyhedron with less than 20 vertices is eight, and none of the polyhedral perihamiltonian graphs with less than 20 vertices has independent cubic vertices. Besides this setup, we also developed plugins to only generate all polyhedral graphs with independent cubic vertices and all polyhedral graphs with less than eight cubic vertices. The generation of these graphs was faster than the generation above, but nevertheless the growth was such that neither of these was able to generate all polyhedral graphs with 20 vertices in their respective class.

# 6 Open Problems

We end this paper with three open problems on perihamiltonian graphs.

**1.** *Is there a perihamiltonian graph of girth greater than* 7*? Coxeter's graph provides an example of girth* 7*.*

**2.** *Do* $k$*-regular perihamiltonian graphs exist for* $k > 3$*?*

**3.** *What is the minimum number of cubic vertices a planar* 3*-connected perihamiltonian graph must contain?*

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## References


[1] J. A. Bondy. Variations on the Hamiltonian Theme. *Canad. Math. Bull.* **15** (1972) 57–62.

[2] G. Brinkmann, K. Coolsaet, J. Goedgebeur, and H. Mélot. House of graphs: a database of interesting graphs. *Discrete Appl. Math.* **161** (2013) 311–314. Available at http://hog.grinvin.org.

[3] G. Brinkmann and B. D. McKay. Fast generation of planar graphs. *MATCH Commun. Math. Comput. Chem.* **42** (2007) 909–924. see http://cs.anu.edu.au/∼bdm/index.html.

[4] G. Brinkmann and C. T. Zamfirescu. Polyhedra with few 3-cuts are hamiltonian. *Electron. J. Combin.* **26** (2019) #P39.

[5] V. Chvátal. Flip-flops in hypohamiltonian graphs. *Canad. Math. Bull.* **16** (1973) 33–41.

[6] J. B. Collier and E. F. Schmeichel. Systematic searches for hypohamiltonian graphs. *Networks* **8** (1978) 193–200.

[7] J. Goedgebeur and C. T. Zamfirescu. Improved bounds for hypohamiltonian graphs. *Ars Math. Contemp.* **13** (2017) 235–257.

[8] J. Goedgebeur and C. T. Zamfirescu. On almost hypohamiltonian graphs. *Discrete Math. Theoret. Comput. Sci.* **21** (2019) #5.

[9] D. A. Holton and J. Sheehan. The Petersen Graph, Chapter 7: Hypohamiltonian graphs, Cambridge University Press, New York, 1993.

[10] J. D. Horton. A Hypotraceable Graph. Research Report CORR 73–4, Dept. Combin. and Optim., Univ. Waterloo, 1973.

[11] L.-H. Hsu and C.-K. Lin. Graph Theory and Interconnection Networks. CRC Press, Boca Raton, 2008.

[12] M. Jooyandeh, B. D. McKay, P. R. J. Östergård, V. H. Pettersson, and C. T. Zamfirescu. Planar hypohamiltonian graphs on 40 vertices. *J. Graph Theory* **84** (2017) 121–133.

[13] B. D. McKay and A. Piperno. Practical Graph Isomorphism II, *J. Symbolic Computation* **60** (2013) 94–112.

[14] M. Meringer. Fast Generation of Regular Graphs and Construction of Cages. *J. Graph Theory* **30**, 137-146, 1999.

[15] A. Neyt. Platypus Graphs: Structure and Generation. MSc Thesis, Ghent University, 2017 (in Dutch).

[16] D. P. Sanders. On paths in planar graphs. *J. Graph Theory* **24** (1997) 341–345.

[17] C. Thomassen. On hypohamiltonian graphs. *Discrete Math.* **10** (1974) 383–390.

[18] C. Thomassen. Planar and infinite hypohamiltonian and hypotraceable graphs. *Discrete Math.* **14** (1976) 377–389.

[19] C. Thomassen. Hypohamiltonian graphs and digraphs. In: Proc. Internat. Conf. Theory and Appl. of Graphs, Kalamazoo, 1976, *LNCS* **642**, Springer, Berlin (1978) 557–571.

[20] C. Thomassen. Planar cubic hypohamiltonian and hypotraceable graphs. *J. Combin. Theory, Ser. B* **30** (1981) 36–44.

[21] W. T. Tutte. A theorem on planar graphs. *Trans. Amer. Math. Soc.* **82** (1956) 99–116.

[22] H. Whitney. A theorem on graphs. *Ann. Math.* **32** (1931) 378–390.

[23] G. Wiener. On non-traceable, non-hypotraceable, arachnoid graphs. *Electron. Notes Discrete Math.* **49** (2015) 621–627. *Proc. EuroComb* 2015 (eds.: J. Nešetřil, O. Serra, J. A. Telle).

[24] G. Wiener. Leaf-Critical and Leaf-Stable Graphs. *J. Graph Theory* **84** (2017) 443–459.

[25] G. Wiener and M. Araya. On planar hypohamiltonian graphs. *J. Graph Theory* **67** (2011) 55–68.

[26] C. T. Zamfirescu. Hypohamiltonian and almost hypohamiltonian graphs. *J. Graph Theory* **79** (2015) 63–81.

[27] C. T. Zamfirescu. On non-hamiltonian graphs for which every vertex-deleted subgraph is traceable. *J. Graph Theory* **86** (2017) 223–243.

[28] C. T. Zamfirescu. Cubic vertices in planar hypohamiltonian graphs. *J. Graph Theory* **90** (2019) 189–207.

[29] C. T. Zamfirescu. On the hamiltonicity of a planar graph and its vertex-deleted subgraphs. Submitted.

[30] C. T. Zamfirescu and T. I. Zamfirescu. Every graph occurs as an induced subgraph of some hypohamiltonian graph. *J. Graph Theory* **88** (2018) 551–557.