Non-hamiltonian 1-tough triangulations with disjoint separating triangles

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Abstract. In this note, we consider triangulations of the plane. Ozeki and the second author asked whether there are non-hamiltonian 1-tough triangulations in which every two separating triangles are disjoint. We answer this question in the affirmative and strengthen a result of Nishizeki by proving that there are infinitely many non-hamiltonian 1-tough triangulations with pairwise disjoint separating triangles.

Key words. Triangulation, separating triangle, non-hamiltonian, 1-tough.

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1 Introduction

In this note, a triangulation shall be a plane 3-connected graph in which every face is a triangle. (Triangulations are also known as maximal planar graphs, since the addition of any edge renders the graph non-planar.) For a possibly disconnected graph $G$, denote by $c(G)$ the number of connected components of $G$. In a triangulation $G$, a triangle $T$ is said to be separating if $c(G - T) > 1$. For triangles $T$ and $T'$ in $G$ the distance between $T$ and $T'$ shall be the number of edges of a shortest path in $G$ between $v \in V(T)$ and $v' \in V(T')$ for all possible combinations of $v$ and $v'$.

Answering a question of Böhme, Harant, and Tkáč [2], Böhme and Harant [1] showed that for any non-negative integer $d$ there exists a non-hamiltonian triangulation with seven separating triangles every two of which lie at distance at least $d$. Ozeki and the second author [8] proved that the result holds even if we replace ‘seven’ by ‘six’. We note that no non-hamiltonian triangulation with fewer than six separating triangles is known, while Jackson and Yu [6] showed that every triangulation with at most three separating triangles is hamiltonian. (It was recently proven that this result’s generalisation to polyhedral graphs—where 3-vertex-cuts replace separating triangles—is valid, as well [3].)

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Chvátal [4] introduced the toughness of a non-complete graph $G$ as
\[ t(G) = \min \left\{ \frac{|X|}{c(G - X)} : X \subseteq V(G), c(G - X) > 1 \right\}. \]

The toughness of a complete graph is convened to be $\infty$. A graph $G$ is $t$-tough whenever $t \leq t(G)$. Chvátal observed that every hamiltonian graph is 1-tough [4]. In 1979 he raised the question whether $l$-toughness is a sufficient condition for a triangulation to be hamiltonian, and Nishizeki settled this by proving that there is a non-hamiltonian 1-tough triangulation [7]. (Dillencourt [5] showed that there exists a smaller such triangulation, namely one of order 15, and Tkáč [10] proved that there exists such a triangulation of order 13, and no smaller one. Tkáč’s triangulation contains seven separating triangles.)

Recently, Ozeki and the second author asked whether there are non-hamiltonian 1-tough triangulations in which every two separating triangles are disjoint, see [8, Remark (a)]. We now answer this question in the affirmative and strengthen Nishizeki’s result.

2 Result

**Theorem.** There exist infinitely many non-hamiltonian 1-tough triangulations with pairwise disjoint separating triangles.

For the proof of this theorem we will use the following lemma.

**Lemma (Nishizeki [7]).** Let $G$ be a graph and $S \subseteq V(G)$. If for a vertex $v$ in $G$, the graph $G - v$ is 1-tough, and if $c(G - S) > |S|$, then $v$ does not belong to $S$ but all of its neighbours do.

**Proof of the Theorem.** In the first part of the proof, we construct a triangulation $G$ with the desired properties, and in the second part we present an infinite family. Consider the circular arrangement of five copies $H_1, \ldots, H_5$ of the graph $H$ shown in Fig. 1 so that the respective copies of $v_1v_3$ and $x_3v_7$ are being identified. All 15 outer half-edges are connected to the vertex $y$ (which does not lie in $H$). We obtain a plane graph $G'$ in which all faces are triangles with exactly one exception, which is a decagon $D = x_1x_2 \ldots x_{10}$. Inside $D$, we insert the graph $F$ depicted in Fig. 2 so that $G' \cap F = D$. We have obtained a triangulation $G$.

Visual inspection of Figs. 1 and 2 yields that the separating triangles of $G$, of which there are 20 in total, are pairwise disjoint. In $G'$, the separating triangles are the respective copies of $v_1x_1v_3$ and $v_4x_2v_6$. In $F$, the separating triangles are $bce$ and its symmetric counterparts. We leave to the reader the verification that these are indeed all separating triangles of $G$.

Suppose there exists a hamiltonian cycle $h$ in $G$. Denote the five copies of $H - x_3 - v_7$ by $H'_i$ such that $H'_i \subset H_i$. Because $F$ has 40 black vertices (marked by black dots in Fig. 2) and 41 non-black vertices (in what follows called white) $h$ has exactly two edges between $F$ and $G - F$, so in one of $H'_i$, w.l.o.g. $H'_1$, the cycle $h$ contains no edge incident with $x_1, x_2$ or $y$. Then there exists a path $p = h \cap H'_1$ which is a hamiltonian $v_1v_6$-path in $H'_1 - x_1 - x_2$ (vertices in $H'_i$ carry the same name as their counterparts in $H$). It is
clear that $p$ contains $v_1v_2v_3$ and $v_4v_5v_6$ as subpaths. But this implies that $v_9$ cannot be visited by $p$, a contradiction. Therefore $G$ is non-hamiltonian.

We now show that $G$ is indeed 1-tough. We follow a similar strategy as Nishizeki in [7] and first prove that for every vertex $v$ in a certain set $W \subset V(G)$, the graph $G - v$ is hamiltonian, ergo 1-tough. The set $W$ is composed of the copies of $v_2$, $v_5$, and $v_9$ in each copy $H_i$ of $H$ (marked with black dots in Fig. 1—henceforth, these vertices will be called black, and non-black vertices white). We define three types of path in $H$ (using the notation from Fig. 1):

Type 1: $v_1v_9v_8v_3v_4v_5v_6$ (avoids $v_2$) or $v_1v_2v_3v_8v_9v_6$ (avoids $v_5$) or $v_1v_2v_3v_4v_5v_6$ (avoids $v_9$)

Type 2: $x_1v_2v_3v_1v_9v_8v_4v_5v_6$

Type 3: $v_1v_2v_3v_4v_5v_6v_9$

We use these paths to show that $G - v$ is hamiltonian for every $v$ in $W$. In what follows, in certain cases it may be necessary to consider symmetric versions of these paths. By symmetry, it suffices to show that $G - v_2$, $G - v_5$, and $G - v_9$ are hamiltonian. These cycles can be found by using Types 1–3 as depicted in Fig. 3. In $F$, we use the path shown in Fig. 2.

Assume that there exists a set $S \subset V(G)$ such that $c(G - S) > |S|$. By above argument, we can apply the Lemma and obtain that $W \cap S = \emptyset$ and for every vertex in $W$, all of its neighbours lie in $S$. Let $S_1 \subset S$ be the white vertices of $G'$ (this includes $y$ as well as $x_1, \ldots, x_{10}$), and $S_2 \subset S$ be located in $F - D$. Thus $S$ is the disjoint union of $S_1$ and $S_2$. There are 36 white vertices in $G'$ and we would obtain $|W| = 15$ components if these white vertices were to be removed from $G'$. Since $F$ is hamiltonian, $F - (S \cap V(F))$ contains at most $|S \cap V(F)| = |S_2| + 10$ components. In $G - S$, we obtain at most $15 + |S_2| + 10 = |S_2| + 25$ components. Since

$$c(G - S) \leq |S_2| + 25 < |S_2| + 36 = |S|,$$

we have obtained a contradiction.
In this second part of the proof we show that there are infinitely many graphs with the properties described in the theorem’s statement. Consider the graph from Fig. 4 from which the vertex $w$ has been removed. We call this graph $Q$. Adding to $Q$ a new vertex $w$ and the edges $wa, wb, wc, wd$, we obtain a graph $Q'$. As $Q'$ is planar and 4-connected, by a theorem of Thomas and Yu [9] there exists a hamiltonian cycle $h$ in $Q' - a - b$. Then $h - w$ yields a hamiltonian $cd$-path in $Q - a - b$. We now insert $Q$ into the quadrilateral $abcd$ from Fig. 2 from which the interior vertex has been removed and the proof is complete. This shows that each member of this infinite family is non-hamiltonian and, by the same argument, 1-tough.
An important problem in this field is the question whether there are non-hamiltonian $\frac{3}{2}$-tough triangulations. Unfortunately, we do not see how our method can be applied to attack this problem. The intriguing question of Böhme, Harant, and Tkáč (see [2, Remark 1]) whether non-hamiltonian triangulations with fewer than six separating triangles exist also remains open. We end this note with a problem of our own.

**Question.** What is the minimum number of separating triangles in a non-hamiltonian 1-tough triangulation with pairwise disjoint separating triangles?

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**References**


