Non-hamiltonian 1-tough triangulations with disjoint separating triangles

Jun FUJISAWA* and Carol T. ZAMFIRESCU†

Abstract. In this note, we consider triangulations of the plane. Ozeki and the second author asked whether there are non-hamiltonian 1-tough triangulations in which every two separating triangles are disjoint. We answer this question in the affirmative and strengthen a result of Nishizeki by proving that there are infinitely many non-hamiltonian 1-tough triangulations with pairwise disjoint separating triangles.

Key words. Triangulation, separating triangle, non-hamiltonian, 1-tough.

MSC 2010. 05C45, 05C42, 05C10.

1 Introduction

In this note, a *triangulation* shall be a plane 3-connected graph in which every face is a triangle. (Triangulations are also known as *maximal planar graphs*, since the addition of any edge renders the graph non-planar.) For a possibly disconnected graph G, denote by $c(G)$ the number of connected components of G. In a triangulation G, a triangle T is said to be *separating* if $c(G-T) > 1$. For triangles T and T' in G the *distance* between T and T' shall be the number of edges of a shortest path in G between $v \in V(T)$ and $v' \in V(T')$ for all possible combinations of v and v'.

Answering a question of Böhme, Harant, and Tkáč [2], Böhme and Harant [1] showed that for any non-negative integer d there exists a non-hamiltonian triangulation with seven separating triangles every two of which lie at distance at least d. Ozeki and the second author [8] proved that the result holds even if we replace 'seven' by 'six'. We note that no non-hamiltonian triangulation with fewer than six separating triangles is known, while Jackson and Yu [6] showed that every triangulation with at most three separating triangles is hamiltonian. (It was recently proven that this result's generalisation to polyhedral graphs—where 3-vertex-cuts replace separating triangles—is valid, as well [3].)

*Faculty of Business and Commerce, Keio University, Hiyoshi 4-1-1, Kohoku-Ku, Yokohama, Kanagawa 223-8521, Japan. E-mail address: fujisawa@fbc.keio.ac.jp

†Department of Applied Mathematics, Computer Science and Statistics, Ghent University, Krijgslaan 281 - S9, 9000 Ghent, Belgium and Department of Mathematics, Babeş-Bolyai University, Cluj-Napoca, Roumania. E-mail address: czamfirescu@gmail.com

Chvátal [4] introduced the *toughness* of a non-complete graph G as

$$t(G) = \min\left\{\frac{|X|}{c(G-X)} : X \subseteq V(G),\, c(G-X) > 1\right\}.$$

The toughness of a complete graph is convened to be ∞. A graph G is *t-tough* whenever $t \leq t(G)$. Chvátal observed that every hamiltonian graph is 1-tough [4]. In 1979 he raised the question whether l-toughness is a sufficient condition for a triangulation to be hamiltonian, and Nishizeki settled this by proving that there is a non-hamiltonian 1-tough triangulation [7]. (Dillencourt [5] showed that there exists a smaller such triangulation, namely one of order 15, and Tkáč [10] proved that there exists such a triangulation of order 13, and no smaller one. Tkáč's triangulation contains seven separating triangles.)

Recently, Ozeki and the second author asked whether there are non-hamiltonian 1-tough triangulations in which every two separating triangles are disjoint, see [8, Remark (a)]. We now answer this question in the affirmative and strengthen Nishizeki's result.

2 Result

Theorem. *There exist infinitely many non-hamiltonian* 1*-tough triangulations with pairwise disjoint separating triangles.*

For the proof of this theorem we will use the following lemma.

Lemma (Nishizeki [7])**.** *Let G be a graph and $S \subset V(G)$. If for a vertex v in G, the graph $G - v$ is* 1*-tough, and if $c(G - S) > |S|$, then v does not belong to S but all of its neighbours do.*

Proof of the Theorem. In the first part of the proof, we construct a triangulation G with the desired properties, and in the second part we present an infinite family. Consider the circular arrangement of five copies $H_1, \ldots, H_5$ of the graph H shown in Fig. 1 so that the respective copies of x_1v_1 and x_3v_7 are being identified. All 15 outer half-edges are connected to the vertex y (which does not lie in H). We obtain a plane graph G' in which all faces are triangles with exactly one exception, which is a decagon $D = x_1x_2 \ldots x_{10}$. Inside D, we insert the graph F depicted in Fig. 2 so that $G' \cap F = D$. We have obtained a triangulation G.

Visual inspection of Figs. 1 and 2 yields that the separating triangles of G, of which there are 20 in total, are pairwise disjoint. In G', the separating triangles are the respective copies of $v_1x_1v_3$ and $v_4x_2v_6$. In F, the separating triangles are bce and its symmetric counterparts. We leave to the reader the verification that these are indeed all separating triangles of G.

Suppose there exists a hamiltonian cycle $\mathfrak{h}$ in G. Denote the five copies of $H-x_3-v_7$ by H'_i such that $H'_i \subset H_i$. Because F has 40 black vertices (marked by black dots in Fig. 2) and 41 non-black vertices (in what follows called *white*) $\mathfrak{h}$ has exactly two edges between F and $G-F$, so in one of H'_i, w.l.o.g. H'_1, the cycle $\mathfrak{h}$ contains no edge incident with x_1, x_2 or y. Then there exists a path $\mathfrak{p} = \mathfrak{h} \cap H'_1$ which is a hamiltonian v_1v_6-path in $H'_1 - x_1 - x_2$ (vertices in H'_i carry the same name as their counterparts in H). It is

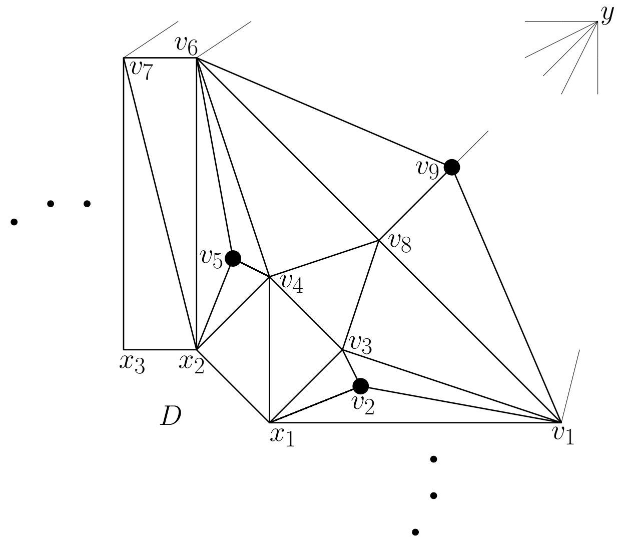

Fig. 1: The graph H, five copies of which, together with y, form G'.

clear that $\mathfrak{p}$ contains $v_1v_2v_3$ and $v_4v_5v_6$ as subpaths. But this implies that v_9 cannot be visited by $\mathfrak{p}$, a contradiction. Therefore G is non-hamiltonian.

We now show that G is indeed 1-tough. We follow a similar strategy as Nishizeki in [7] and first prove that for every vertex v in a certain set $W \subset V(G)$, the graph $G - v$ is hamiltonian, ergo 1-tough. The set W is composed of the copies of v_2, v_5, and v_9 in each copy H_i of H (marked with black dots in Fig. 1—henceforth, these vertices will be called *black*, and non-black vertices *white*). We define three types of path in H (using the notation from Fig. 1):

Type 1: $v_1v_9v_8v_3v_4v_5v_6$ (avoids v_2) or $v_1v_2v_3v_4v_8v_9v_6$ (avoids v_5) or $v_1v_2v_3v_8v_4v_5v_6$ (avoids v_9)

Type 2: $x_1v_2v_3v_1v_9v_8v_4v_5v_6$

Type 3: $v_1v_2v_3v_4v_5v_6v_8v_9$

We use these paths to show that $G - v$ is hamiltonian for every v in W. In what follows, in certain cases it may be necessary to consider symmetric versions of these paths. By symmetry, it suffices to show that $G - v_2$, $G - v_5$, and $G - v_9$ are hamiltonian. These cycles can be found by using Types 1–3 as depicted in Fig. 3. In F, we use the path shown in Fig. 2.

Assume that there exists a set $S \subset V(G)$ such that $c(G - S) > |S|$. By above argument, we can apply the Lemma and obtain that $W \cap S = \emptyset$ and for every vertex in W, all of its neighbours lie in S. Let $S_1 \subset S$ be the white vertices of G' (this includes y as well as $x_1, \ldots, x_{10}$), and $S_2 \subset S$ be located in $F - D$. Thus S is the disjoint union of S_1 and S_2. There are 36 white vertices in G' and we would obtain $|W| = 15$ components if these white vertices were to be removed from G'. Since F is hamiltonian, $F - (S \cap V(F))$ contains at most $|S \cap V(F)| = |S_2| + 10$ components. In $G - S$, we obtain at most $15 + |S_2| + 10 = |S_2| + 25$ components. Since

$$c(G - S) \leq |S_2| + 25 < |S_2| + 36 = |S|,$$

we have obtained a contradiction.

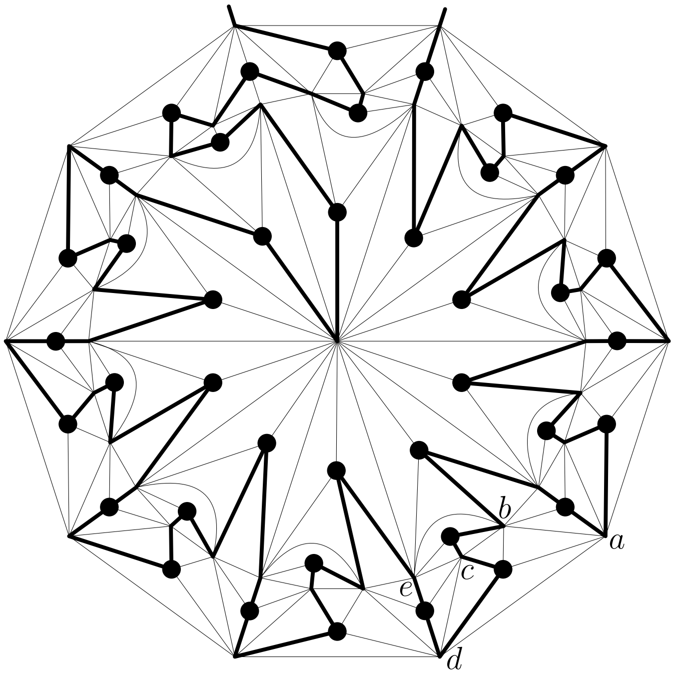

Fig. 2: A hamiltonian path in F.

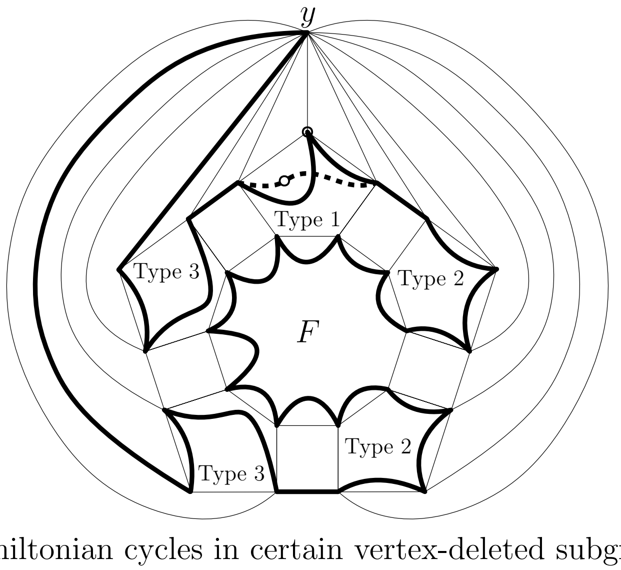

Fig. 3: Hamiltonian cycles in certain vertex-deleted subgraphs of G.
When avoiding v_9, we use the dotted path.

In this second part of the proof we show that there are infinitely many graphs with the properties described in the theorem's statement. Consider the graph from Fig. 4 from which the vertex w has been removed. We call this graph Q. Adding to Q a new vertex w and the edges wa, wb, wc, wd, we obtain a graph Q'. As Q' is planar and 4-connected, by a theorem of Thomas and Yu [9] there exists a hamiltonian cycle $\mathfrak{h}$ in $Q' - a - b$. Then $\mathfrak{h} - w$ yields a hamiltonian cd-path in $Q - a - b$. We now insert Q into the quadrilateral $abcd$ from Fig. 2 from which the interior vertex has been removed and the proof is complete. This shows that each member of this infinite family is non-hamiltonian and, by the same argument, 1-tough. □

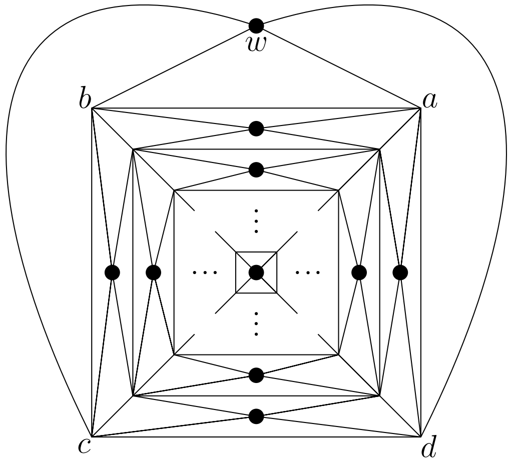

Fig. 4: The graph Q'.

An important problem in this field is the question whether there are non-hamiltonian $\frac{3}{2}$-tough triangulations. Unfortunately, we do not see how our method can be applied to attack this problem. The intriguing question of Böhme, Harant, and Tkáč (see [2, Remark 1]) whether non-hamiltonian triangulations with fewer than six separating triangles exist also remains open. We end this note with a problem of our own.

Question. *What is the minimum number of separating triangles in a non-hamiltonian* 1*-tough triangulation with pairwise disjoint separating triangles?*

Acknowledgements. Fujisawa's work is supported by the Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (B) 16H03952 and (C) 17K05349. Zamfirescu's research is supported by a Postdoctoral Fellowship of the Research Foundation Flanders (FWO).

References

[1] T. Böhme and J. Harant. On Hamiltonian cycles in 4- and 5-connected plane triangulations. *Discrete Math.* **191** (1998) 25–30.

[2] T. Böhme, J. Harant, and M. Tkáč. On the minimal number of separating 3-cycles in non-Hamiltonian maximal planar graphs. *Tatra Mt. Math. Publ.* **9** (1996) 97–102.

[3] G. Brinkmann and C. T. Zamfirescu. Polyhedra with few 3-cuts are hamiltonian. *Electron. J. Combin.* **26**, Iss. 1 (2019) Article Number P1.39.

[4] V. Chvátal. Tough graphs and hamiltonian circuits. *Discrete Math.* **5** (1973) 215–228.

[5] M. B. Dillencourt. An upper bound on the shortness exponent of 1-tough, maximal planar graphs. *Discrete Math.* **90** (1991) 93–97.

[6] B. Jackson and X. Yu. Hamilton cycles in plane triangulations. *J. Graph Theory* **41** (2002) 138–150.

[7] T. Nishizeki. A 1-tough nonhamiltonian maximal planar graph. *Discrete Math.* **30** (1980) 305–307.

[8] K. Ozeki and C. T. Zamfirescu. Non-hamiltonian triangulations with distant separating triangles. *Discrete Math.* **341** (2018) 1900–1902.

[9] R. Thomas and X. Yu. 4-Connected Projective-Planar Graphs are Hamiltonian. *J. Combin. Theory, Ser. B* **62** (1994) 114–132.

[10] M. Tkáč. On the shortness exponent of 1-tough, maximal planar graphs. *Discrete Math.* **154** (1996) 321–328.