On minimum leaf spanning trees and a criticality notion

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Abstract. The minimum leaf number of a connected non-hamiltonian graph G is the number of leaves of a spanning tree of G with the fewest leaves among all spanning trees of G. Based on this quantity, Wiener introduced leaf-stable and leaf-critical graphs, concepts which generalise hypotraceability and hypohamiltonicity. In this article, we present new methods to construct leaf-stable and leaf-critical graphs and study their properties. Furthermore, we improve several bounds related to these families of graphs. These extend previous results of Horton, Thomassen, and Wiener.

Key Words. Spanning tree, minimum leaf number, leaf-stable, leaf-critical

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1 Introduction

Let G be a graph and $\mathcal{T}(G)$ the set of all spanning trees of G. Denote by $\ell(T)$ the number of leaves of a tree T. The *minimum leaf number* $\mathrm{ml}(G)$ of G is defined as

$$\mathrm{ml}(G) = \begin{cases} \infty & \text{if } G \text{ is not connected,} \\ \min\limits_{T \in \mathcal{T}(G)} \ell(T) & \text{if } G \text{ is connected but not hamiltonian,} \\ 1 & \text{if } G \text{ is hamiltonian.} \end{cases}$$

Wiener [13] introduced the following. Consider an integer $\ell \geq 2$. A connected graph G with $\mathrm{ml}(G) = \ell$ is called *ℓ-leaf-critical* if $\mathrm{ml}(G - v) = \ell - 1$ for every

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$v \in V(G)$, and *ℓ-leaf-stable* if $\mathrm{ml}(G - v) = \ell$ for every $v \in V(G)$. A graph is *hypohamiltonian* (*hypotraceable*) if it is non-hamiltonian (non-traceable), yet all of its vertex-deleted subgraphs are hamiltonian (traceable)—for an overview we refer to Holton and Sheehan's survey [4], and for recent results, see [6]. The family of all 2-leaf-critical graphs (3-leaf-critical graphs) and the family of all hypohamiltonian (hypotraceable) graphs coincide.

Wiener showed that ℓ-leaf-stable and ℓ-leaf-critical graphs exist for every $\ell \geq 2$. He also studied these graphs under the additional condition of planarity. Among the applications of these results is the affirmative answer to the question of Gargano et al. [1, p. 93] whether non-traceable non-hypotraceable arachnoid graphs (defined in [1]) exist, see [13].

Determining the minimum leaf number $\mathrm{ml}(G)$ plays an important role in designing efficient networks, but this minimisation problem is obviously NP-hard, since $\mathrm{ml}(G) = 1$ if and only if G is hamiltonian and $\mathrm{ml}(G) = 2$ if and only if G is traceable. Actually, Lu and Ravi [7] showed that the problem does not even have a constant factor approximation, unless $\mathrm{P} = \mathrm{NP}$. On the other hand, approximation algorithms to maximise the number of non-leaves of spanning trees exist, see e.g. [9].

In [13], Wiener asked for constructions of ℓ-leaf-critical graphs of connectivity 2 for $\ell \geq 4$, and for determining (or at least bounding the order of) the smallest ℓ-leaf-critical graphs. Note that there exist 3-leaf-critical graphs of connectivity 2, see [10]. Applying a result of Thomassen [11], we answer the first request in Section 2 (see Theorem 1). This is summarised in Proposition 2 in Section 4.

We also give new constructions of ℓ-leaf-stable graphs, which will be discussed in Section 3 (see Theorems 4 and 5). This improves the known bounds on smallest ℓ-leaf-stable graphs with a specified connectivity, which are further discussed in Proposition 3 in Section 4.

We will call a path starting at a vertex v a *v-path*, and a v-path ending at a vertex $w \neq v$ a *vw-path*. For a graph G and a subgraph H of G, $d_H(v)$ denotes the degree of v in H. If S is a set, we write $|S|$ for its cardinality. Consider a graph G of connectivity k which is not a complete graph. Then G contains a k-vertex-cut A. Denote with $C_1, ..., C_p$ the connected components of $G - A$, where $p \geq 2$. We say that $F_i = G[V(C_i) \cup A]$ is a *k-fragment of* G, and that A is the set of *vertices of attachment* of F_i. When we simply speak of a k-fragment, we refer to a graph which can be obtained as the k-fragment of some graph of connectivity k. A k-fragment is *trivial* if it has exactly $k + 1$ vertices.

2 Construction of k-leaf-critical graphs

We now present a method to construct k-leaf-critical graphs using fragments of 3-leaf-critical graphs for any $k \geq 3$. For the proof of the next theorem we need a result of Thomassen which we now recall:

Lemma 1 (Thomassen [11]). *Let $i \in \{1, 2\}$ and $G = F_1 \cup F_2$ be a graph with $V(F_1) \cap V(F_2) = \{x, y\}$ and $F_i - V(F_{3-i}) \neq \emptyset$. Suppose that G is hypotraceable. Then F_i has no hamiltonian path starting at x or y, and if $z \in V(F_i)$, then $F_i - z$ has a hamiltonian path starting at x or y. Conversely, if these properties are satisfied, then G is hypotraceable.*

Theorem 1. *Let $k \geq 2$ and F_i be pairwise disjoint 2-fragments of 3-leaf-critical graphs with vertices of attachment $\{x_i, y_i\}$ for $0 \leq i \leq k-1$, respectively. Identifying y_i with x_{i+1}, indices taken mod k, we obtain a $(k+1)$-leaf-critical graph G.*

Proof. We first recall certain facts concerning 2-fragments of 3-leaf-critical (i.e. hypotraceable) graphs—their proofs follow from Thomassen's characterisation of such fragments, see Lemma 1. It is easy to see that 3-leaf-critical graphs have minimum degree at least 3, thus these fragments must be non-trivial. Let F be a 2-fragment of a 3-leaf-critical graph with vertices of attachment x and y. Then the following hold.

(a) F does not contain a hamiltonian x-path or y-path.

(b) There exists a hamiltonian x-path or y-path in $F - v$ for every $v \in V(F)$. In particular:

(c) $F - x$ and $F - y$ contain a hamiltonian y-path and x-path, respectively.

(d) F has a spanning tree with exactly three leaves, namely x, y, v, where $v \in V(F) \setminus \{x, y\}$.

Let $i \in \{0, ..., k-1\}$ be fixed and arbitrary. Indices are to be taken mod k and we shall treat F_i as a subgraph of G. Let z_i be the vertex in G obtained when identifying y_i with x_{i+1}, and let T be a spanning tree of G.

Claim 1. *For every $i = 0, 1, \ldots, k-1$, F_i contains a leaf of T different from z_i and z_{i-1}.*

Proof of Claim 1. Let T_i be the subgraph of T spanned by the vertex set of F_i and c_i the number of components of T_i. It is obvious that T_i is a forest containing c_i trees and that vertices of T_i different from z_i and z_{i-1} have the same degree in T and T_i. Now we distinguish two cases.

Case 1. $c_i \geq 2$. If there is no isolated vertex in T_i, then $\ell(T_i) \geq 4$. Since vertices of T_i different from z_i and z_{i-1} have the same degree in T and T_i, there must be at least two leaves of T in $F_i - \{z_i, z_{i-1}\}$. If T_i contains an isolated vertex, then it can only be z_i or z_{i-1}, but not both, therefore a component different from the isolated vertex contains a leaf of T different from z_i and z_{i-1}.

Case 2. $c_i = 1$. Now T_i is a tree. If it is a path, then z_i and z_{i-1} cannot be the end-vertices of the path because of fact (a), thus the end-vertices are leaves of T different from z_i and z_{i-1}. If T_i is not a path then it has at least three leaves, one of which must be a leaf of T different from z_i and z_{i-1}. ■

It follows directly from the above claim that $\ell(T) \geq k$. Let us now prove that $\ell(T) \geq k+1$. If $\ell(T) = k$, then each F_i contains just one leaf of T different from z_i and z_{i-1} and by the proof of Claim 1 it follows that each T_i is either a tree with three leaves such that both z_i and z_{i-1} are leaves, or T_i contains an isolated vertex which is either z_i or z_{i-1}. It is obvious that the latter situation must occur for exactly one i: if it would not occur at all, then T would contain a cycle and if it would occur for at least two i's, then T would not be connected. Now if T_j is the subgraph that contains the isolated vertex, which is (say) z_j, then z_j is also a leaf of T, thus $\ell(T) \geq k+1$, a contradiction from which $\text{ml}(G) \geq k+1$ immediately follows. A spanning tree of G with exactly $k+1$ leaves is easy to describe: use fact (d) in $F_0, ..., F_{k-2}$ and fact (c) in F_{k-1}.

We now prove that $\text{ml}(G-v) = k$ for every $v \in V(G)$. If $v = z_i$ for some i, then let us use fact (d) for F_j with $j \notin \{i, i+1\}$ and fact (c) for F_i and F_{i+1}. Gluing together the trees and paths guaranteed by the facts we obtain a spanning tree of $G - z_i$ with k leaves.

Consider now $v \in V(F_i) \setminus \{z_{i-1}, z_i\}$. By fact (b), there exists a hamiltonian z_i-path or a hamiltonian z_{i-1}-path in $F_i - v$ (suppose w.l.o.g. that it is a z_{i-1}-path). Now let us use fact (d) for F_j with $j \notin \{i, i+1\}$ and fact (c) for F_{i+1} (there is a hamiltonian z_{i+1}-path in $F_{i+1} - z_i$). Once more we join the trees and paths guaranteed by the facts and obtain a spanning tree of $G - v$ with k leaves, proving that $\text{ml}(G-v) \leq k$ for every $v \in V(G)$. Since $\text{ml}(G) = k+1$, it is easy to see that $\text{ml}(G-v) \geq \text{ml}(G) - 1 = k$. Therefore, $\text{ml}(G-v) = k$ for every $v \in V(G)$ finishing the proof of the $(k+1)$-leaf-criticality of G. □

For Theorem 1 to be useful, we need 2-fragments of 3-leaf-critical graphs—fortunately, Thomassen [10] showed that 3-leaf-critical graphs of connectivity 2 exist. Theorem 1 thus provides ℓ-leaf-critical graphs of connectivity 2 for any $\ell \geq 3$. We shall summarise in Section 4 bounds obtainable through Theorem 1.

We have seen that 2-fragments of 3-leaf-critical graphs can be used to obtain k-leaf-critical graphs for any $k \geq 3$. We end this section with a brief discussion of 2-fragments of leaf-critical graphs (here and in the remainder of this section, we suppress the prefix "k-" in k-leaf-critical, as the exact value of k shall play no role in the arguments) motivated by Thomassen's characterisation of 2-fragments of hypotraceable graphs [11].

Theorem 2. *Every* 1*-fragment of a* 2*-fragment of a leaf-critical graph is also a* 2*-fragment of a leaf-critical graph.*

For the proof of this theorem we need the following immediate corollary of [13, Lemmas 4.4 and 4.6] characterising 2-fragments of leaf-critical graphs.

Theorem 3. *Let F_1 and F_2 be disjoint* 2*-fragments of leaf-critical (possibly different) graphs with vertices of attachment x, y and w, z, respectively, and let G be the graph obtained from the union of F_1 and F_2 by identifying x with w and y with z. Then G is a leaf-critical graph.*

Proof of Theorem 2. Let F be a 2-fragment of a leaf-critical graph with vertices of attachment x, y and a cut-vertex z. Let F' be a copy of F and x', y', z' the copies of x, y, z, respectively. Let now G be the graph obtained from the union of F and F' by identifying x with x' and y with y' and let us denote the vertices obtained by x'' and y'', respectively. Then G is a leaf-critical graph by Theorem 3. Since leaf-critical graphs are 2-connected, $F - z$ has just two components, one of which contains x and the other one contains y. It is obvious that $\{x'', z\}$ and $\{y'', z\}$ are vertex-cuts of G showing that indeed the 1-fragments of F are 2-fragments of a leaf-critical graph, namely G. □

We end this section by mentioning a result of Wiener [13], who proved that gluing together a 2-fragment of an ℓ-leaf-critical graph and a 2-fragment of a k-leaf-critical graph, we obtain an $(\ell + k - 3)$-leaf-critical graph. (Note that non-3-connected j-leaf-critical graphs exist only if $j \geq 3$.)

3 Construction of k-leaf-stable graphs

3.1 Connectivity 2

Wiener showed in [13, Theorem 7.1] that if G is a 3-leaf-critical graph with a 2-vertex-cut $\{x, y\}$, then $xy \notin E(G)$ and $G + xy$ is 2-leaf-stable. We now present a k-leaf-stable analogue of this result, but with an edge-connectivity requirement imposed on all 2-fragments. For its proof we need a definition and the following lemma. A graph G is called *almost hypohamiltonian* if G is non-hamiltonian and G contains a vertex w such that $G - w$ is non-hamiltonian, yet $G - v$ is hamiltonian for every $v \in V(G) \setminus \{w\}$.

Lemma 2. *Let F be a 2-fragment of a 3-leaf-critical graph with vertices of attachment x and y, and let $v \in V(F)$. If F contains a non-trivial 2-edge-cut then there exists no hamiltonian xy-path in $F - v$.*

Proof. Due to Theorem 3.1 of [14] we know that by deleting the edges of a non-trivial 2-edge-cut of a 2-fragment F of a hypotraceable (that is, 3-leaf-critical) graph, we obtain two components F_1 and F_2 that are both either vertex-deleted hypohamiltonian or almost hypohamiltonian graphs. Moreover, in the proof of the theorem the following are also shown:

1. The vertices x and y are in different components F_i (let us suppose w.l.o.g. that $x \in V(F_1)$, $y \in V(F_2)$).

2. If the edges of the 2-edge-cut are a_1a_2 and b_1b_2 such that $a_i, b_i \in V(F_i)$ for $i = 1, 2$, then the vertices a_1, a_2, b_1, b_2 are pairwise different and there is no hamiltonian path in F_1 whose end-vertices lie in $\{x, a_1, b_1\}$ and there is no hamiltonian path in F_2 whose end-vertices lie in $\{y, a_2, b_2\}$.

Now let us suppose to the contrary that there exists a hamiltonian xy-path P in $F - v$, where we may assume that $v \in V(F_1) \setminus \{x\}$. Then $P \cap F_2$ must be a hamiltonian path between y and one of a_2 and b_2, a contradiction. □

Theorem 4. *Let G be a graph obtained as in Theorem 1 and z_i be the vertex in G obtained when identifying y_i with x_{i+1}. If each F_i has edge-connectivity 2, then $G + z_iz_j$ is k-leaf-stable for any i, j with $i \neq j$.*

Proof. All terminology is as given in the statement and proof of Theorem 1. Let $i, j \in \{0, ..., k-1\}$ be arbitrary but fixed with $i \neq j$. By Theorem 1, G is $(k+1)$-leaf-critical, so $\text{ml}(G) = k + 1$ and $\text{ml}(G - v) = k$ for all $v \in V(G)$. Put $G' = G + z_iz_j$. In the following we consider G to be a subgraph of G'.

On one hand, as in G, in G' every F_i must contain a leaf of a spanning tree of G', so $\text{ml}(G') \geq k$ (this can be seen in the same fashion as in the proof of Theorem 1).

On the other hand, it is easy to construct a spanning tree of G' with exactly k leaves: this tree contains the edge $z_i z_j$ and uses fact (c) in F_i and F_j, and fact (d) in all other 2-fragments. Thus $\mathrm{ml}(G') = k$.

Consider $v \in V(G')$. Clearly, $v \in V(F_t)$ for some t. Since F_t has edge-connectivity 2, by Lemma 2 there is no hamiltonian $z_t z_{t+1}$-path in $F_t - v$. So each F_i contains at least one leaf of any spanning tree of $G' - v$, in other words $\mathrm{ml}(G' - v) \geq k$. Finally, $\mathrm{ml}(G - v) = k$ implies that $\mathrm{ml}(G' - v) \leq k$, so $\mathrm{ml}(G' - v) = k$. □

As we will see later in Proposition 3 in Section 4, there are 2-fragments of 3-leaf-critical graphs with edge-connectivity 2. Thus we can construct k-leaf-stable graphs by Theorem 4.

It is worth mentioning that when we "glue"—i.e. identify the cut-vertices using a bijection—two leaf-critical fragments we obtain a leaf-critical graph, see [13, Lemma 4.6]. However, this is not true for leaf-stable graphs.

Motivated by Thomassen's 1978 question whether hypohamiltonian graphs with minimum degree at least 4 exist [12], minimum degrees have played an important role in various classes of non-hamiltonian graphs with rich hamiltonian properties. An example of such a result is [15, Theorem 4.3(ii)]: For every $d \geq 2$ there exists a non-hamiltonian graph G with minimum degree d in which every vertex-deleted subgraph is traceable. In other words, $\mathrm{ml}(G) \geq 2$ and $\mathrm{ml}(G - v) \leq 2$ for all $v \in V(G)$. This can be achieved by considering the cartesian product of a triangle and P_2, and replacing each of the three copies of P_2 by P_4 when $d = 2$ and K_{d+1} if $d > 2$ (where the end-vertices of the copy of P_2 are replaced by the end-vertices of the copy of P_4, or two arbitrary vertices in the complete graph). Note that all of these graphs have connectivity 2. We leave to the reader the straightforward verification that, in fact, these graphs are 2-leaf-stable, i.e. $\mathrm{ml}(G) = \mathrm{ml}(G - v) = 2$ for all $v \in V(G)$. Thus, there exist for every $d \geq 2$ graphs that are 2-leaf-stable and have minimum degree d. We propose the following relaxation of Thomassen's question mentioned above: Do 3-connected leaf-stable or leaf-critical graphs with minimum degree at least 4 exist?

3.2 Connectivity 3

In this section, we construct for each $k \geq 3$ infinitely many k-leaf-stable graphs which, for appropriate input graphs, have connectivity 3. In order to proceed, we need some terminology. Let H be a graph with a cubic vertex x satisfying the following three conditions:

(H1) H is non-hamiltonian.

(H2) For every $v \in N(x)$ the graph $H - v$ is hamiltonian.

(H3) For any edge e incident with x there is a hamiltonian x-path in H using e.

We say that such a graph H is *good* and call the vertex x *special*. For example, consider the Petersen graph with any vertex acting as a special vertex.

Consider a cubic graph G and let H be a graph containing a cubic vertex x. We denote by $G \cdot H_x$ the graph obtained when the following operation is performed for every vertex $v \in V(G)$: we remove v, take a copy H^v of H (disjoint from G) and the copy x^v of x in each H^v, and join in $G - v$ and $H^v - x^v$, using a bijection, each vertex in $N_G(v)$ with each vertex in $N_{H^v}(x^v)$ by an edge. By the construction, we can regard the edge set $E(G)$ as a subset of the edge set of $G \cdot H_x$; an edge uv in G corresponds to the edge connecting a vertex in $N_{H^u}(x^u)$ and a vertex in $N_{H^v}(x^v)$. We illustrate this operation in Fig. 1.

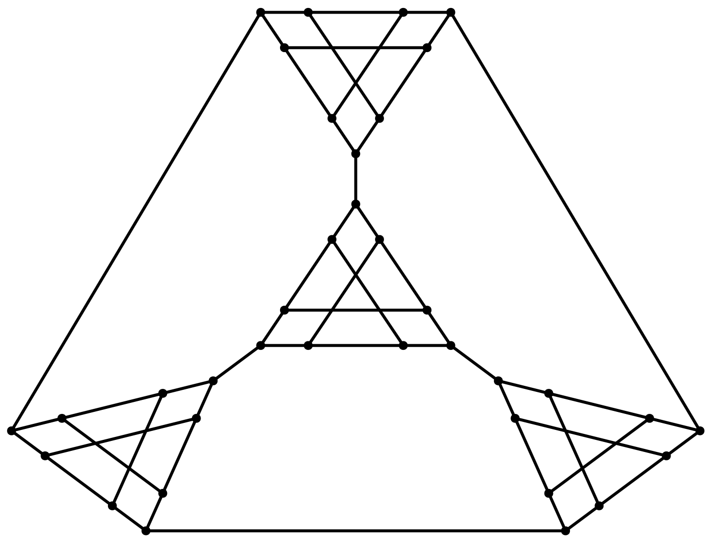

Fig. 1: $K_4 \cdot P_x$, where P is the Petersen graph and x is an arbitrary vertex of P. This particular example was already studied in the seventies [17].

For a tree T and for a positive integer i, let $V_i(T)$ be the set of vertices of degree exactly i in T. Thus, we have $\ell(T) = |V_1(T)|$.

Theorem 5. *Let G be a 2-edge-connected cubic graph and H good with special vertex x. Then $G \cdot H_x$ is $(|V(G)|/2 + 1)$-leaf-stable.*

Proof. First we show that $\mathrm{ml}(G \cdot H_x) \leq |V(G)|/2 + 1$. Let T be a spanning tree of G. We consider the set $\mathcal{P}$ of all non-trivial paths among the components of $G - E(T)$ (and ignore isolated vertices and cycles). Since T is a spanning tree of the cubic graph G, we have $|V(G)| = \ell(T) + |V_2(T)| + |V_3(T)|$ and $\ell(T) = |V_3(T)| + 2$, which implies $|V(G)| = 2\ell(T) + |V_2(T)| - 2$. Since any end-vertex of $P \in \mathcal{P}$ belongs to $V_2(T)$ and any vertex in $V_2(T)$ is an end-vertex of some path $P \in \mathcal{P}$, we also have $|V_2(T)| = 2|\mathcal{P}|$. These imply

$$\ell(T) + |\mathcal{P}| = \frac{|V(G)| - |V_2(T)| + 2}{2} + \frac{|V_2(T)|}{2} = \frac{|V(G)|}{2} + 1. \tag{1}$$

Consider $v \in V(G)$. We denote by H^v the copy of H replacing the vertex v and, abusing notation, by x^v the copy of x in H^v. Put $N_{H^v}(x^v) = \{x_1^v, x_2^v, x_3^v\}$. The following claim follows from the fact that H is good.

Claim 2. *All of the following hold.*

(C1) *$H^v - x^v$ contains a spanning tree whose leaves are exactly x_1^v, x_2^v, x_3^v.*

(C2) *For every $i \in \{1, 2, 3\}$, $H^v - x^v$ contains a hamiltonian x_i^v-path.*

(C3) *For every pairwise distinct* $i, j, k \in \{1, 2, 3\}$*,* $H^v - x^v$ *contains a spanning forest with exactly two components, one of which is an* $x_i^v x_j^v$*-path and the other one is an* x_k^v*-path.*

Proof of Claim 2. If $V(H^v) = \{x^v, x_1^v, x_2^v, x_3^v\}$, then condition (H2) immediately shows that H^v is isomorphic to K_4, contradicting condition (H1).

Thus, there exists a vertex y in $H^v - \{x^v, x_1^v, x_2^v, x_3^v\}$ with an edge $x_i^v y$ for some $i \in \{1, 2, 3\}$, say $i = 1$. By condition (H2), $H^v - x^v - x_1^v$ contains a hamiltonian $x_2^v x_3^v$-path and then adding x_1^v with the edge $x_1^v y$ gives a spanning tree desired in (C1). Condition (H3) with specifying the edge $x^v x_i^v$ as e and deleting x^v give a hamiltonian x_i^v-path in $H^v - x^v$. Hence (C2) is satisfied. (C3) is a direct corollary of (C1). ■

In T, we will distinguish the following four types of vertices v in G: (i) $d_T(v) = 3$, (ii) $d_T(v) = 2$, (iii) $d_T(v) = 1$ and $v \notin V(P)$ for all $P \in \mathcal{P}$ (in this case v belongs to a cycle in $G - E(T)$), (iv) $d_T(v) = 1$ and there exists a $P \in \mathcal{P}$ such that $d_P(v) = 2$.

We now define a spanning forest $\mathfrak{T}^v$ of $H^v - x^v$ for each vertex v. First consider a vertex v of type (i), (iii) and (iv).

(i) $d_T(v) = 3$. Due to (C1), $H^v - x^v$ contains a spanning tree $\mathfrak{T}^v$ whose leaves are exactly x_1^v, x_2^v, x_3^v.

(iii) $d_T(v) = 1$ and $v \notin V(P)$ for all $P \in \mathcal{P}$. Let $i \in \{1, 2, 3\}$ such that x_i^v is an end-vertex of the edge corresponding to the edge in T incident to v. Due to (C2), there is a hamiltonian path $\mathfrak{T}^v$ in $H^v - x^v$ with end-vertex x_i^v.

(iv) $d_T(v) = 1$ and there exists a $P \in \mathcal{P}$ such that $d_P(v) = 2$. Let $k \in \{1, 2, 3\}$ such that x_k^v is an end-vertex of the edge corresponding to the edge in T incident to v, and let $\{i, j\} = \{1, 2, 3\} - \{k\}$. Due to (C3), there is a spanning forest $\mathfrak{T}^v$ in $H^v - x^v$ with exactly two components, one of which is an $x_i^v x_j^v$-path and the other one is an x_k^v-path.

Finally we deal with type (ii) vertices v of G. Note that for each $P \in \mathcal{P}$, there are two such vertices, each corresponding to an end-vertex of P. Let $v(P)$ and $w(P)$ be the end-vertices of P. For $v(P)$, let $k \in \{1, 2, 3\}$ such that $x_k^{v(P)}$ is an end-vertex of the edge corresponding to the one in P incident to $v(P)$, and let $\{i, j\} = \{1, 2, 3\} - \{k\}$. Due to (C3), there is a spanning forest $\mathfrak{T}^{v(P)}$ in $H^{v(P)} - x^{v(P)}$ with exactly two components, one of which is an $x_i^{v(P)} x_j^{v(P)}$-path and the other one is an $x_k^{v(P)}$-path. On the other hand, for $w(P)$, due to (C1), $H^{w(P)} - x^{w(P)}$ contains a spanning tree $\mathfrak{T}^{w(P)}$ whose leaves are exactly $x_1^{w(P)}, x_2^{w(P)}, x_3^{w(P)}$.

Let $\mathfrak{T}$ be the subgraph of $G \cdot H_x$ obtained from $T \cup \bigcup_{P \in \mathcal{P}} P$ by replacing each vertex v with $\mathfrak{T}^v$. By the construction, it is not difficult to see that $\mathfrak{T}$ contains all vertices in $G \cdot H_x$ and no cycle. Now we count the number of edges in $\mathfrak{T}$. For each $v \in V(G)$, we have

$$|E(\mathfrak{T}^v)| = \begin{cases} |V(H)| - 2 & \text{if } v \text{ is of either type (i), or type (iii),} \\ & \text{or type (ii) and } v = w(P) \text{ for some } P \in \mathcal{P}, \\ |V(H)| - 3 & \text{if } v \text{ is of either type (ii) and } v = v(P) \text{ for some } P \in \mathcal{P}, \\ & \text{or type (iv).} \end{cases}$$

There are exactly $|\mathcal{P}|$ vertices v of type (ii) with $v = v(P)$ for some $P \in \mathcal{P}$. Note that all inner vertices in $P \in \mathcal{P}$ are of type (iv), and hence there are exactly

$\sum_{P\in\mathcal{P}}\big(|V(P)|-2\big)$ vertices of type (iv). Thus, we obtain

$$\begin{aligned}|E(\mathfrak{T})| &= |E(T)| + \sum_{P\in\mathcal{P}} |E(P)| + \sum_{v\in V(G)} |E(\mathfrak{T}^v)| \\ &= |V(G)| - 1 + \sum_{P\in\mathcal{P}} \big(|V(P)|-1\big) + |V(G)|\Big(|V(H)|-2\Big) \\ &\qquad\qquad -|\mathcal{P}| - \sum_{P\in\mathcal{P}}\big(|V(P)|-2\big) \\ &= |V(G)|\Big(|V(H)|-1\Big) - 1 = \big|V\big(G\cdot H_x\big)\big| - 1.\end{aligned}$$

Since $\mathfrak{T}$ contains no cycle, it must be a spanning tree of $G \cdot H_x$.

Furthermore, any leaf of $\mathfrak{T}$ is either an end-vertex in $\mathfrak{T}^v$ other than x_i^v for some vertex v of type (iii), or the vertex x_k^v for some vertex v of type (iv), or the vertex $x_k^{v(P)}$ for some $P \in \mathcal{P}$. Therefore, it follows from equality (1) that

$$\ell(\mathfrak{T}) = \ell(T) + |\mathcal{P}| = \frac{|V(G)|}{2} + 1.$$

Therefore, we have $\mathrm{ml}(G\cdot H_x) \le |V(G)|/2 + 1$.

Next we prove $\mathrm{ml}(G\cdot H_x) \ge |V(G)|/2+1$. Let $\mathfrak{T}$ be a spanning tree of $G\cdot H_x$. A vertex v in G is said to be *full* if $\mathfrak{T}\cap(H^v - x^v)$ is connected and $|E_G(v)\cap E(\mathfrak{T})| = 3$, where $E_G(v)$ is the set of edges incident with v in G. Otherwise v is *non-full*. The following claim plays a crucial role in the proof.

Claim 3. *For any non-full vertex v, the copy $H^v - x^v$ corresponding to v contains at least one leaf of $\mathfrak{T}$.*

Proof of Claim 3. If $\mathfrak{T}\cap(H^v - x^v)$ is disconnected or $|E_G(v)\cap E(\mathfrak{T})| = 1$, then clearly $H^v - x^v$ contains at least one leaf of $\mathfrak{T}$. Suppose that $\mathfrak{T}\cap(H^v - x^v)$ is connected and $|E_G(v)\cap E(\mathfrak{T})| = 2$. In this case, if $H^v - x^v$ does not contain a leaf of $\mathfrak{T}$, then $\mathfrak{T}\cap(H^v - x^v)$ is a hamiltonian path in $H^v - x^v$ connecting two vertices in $\{x_1^v, x_2^v, x_3^v\}$. However, adding x^v to $\mathfrak{T}\cap(H^v - x^v)$ through two edges incident with x^v, we obtain a hamiltonian cycle of a copy of H, contradicting condition (H1). Thus, $\mathfrak{T}\cap(H^v - x^v)$ contains a leaf of $\mathfrak{T}$. ■

Now, we show that $\mathfrak{T}$ contains at least $|V(G)|/2 + 1$ leaves. By Claim 3, if there are at least $|V(G)|/2 + 1$ non-full vertices in G, then we are done. Thus, we may assume that there are at most $|V(G)|/2$ non-full vertices in G, which implies that there are at least $|V(G)|/2$ full vertices in G. Since for each full vertex v in G, the graph $H^v - x^v$ contains a vertex of degree at least three in $\mathfrak{T}$, we see that $|V_3(\mathfrak{T})| \ge |V(G)|/2$. Therefore, we have

$$\ell(\mathfrak{T}) = |V_3(\mathfrak{T})| + 2 \ge |V(G)|/2 + 2,$$

and we are also done.

Finally, we prove that $\mathrm{ml}(G\cdot H_x - w) = |V(G)|/2+1$ for any vertex w in $G\cdot H_x$. Let w be a vertex in $G\cdot H_x$, and let u be the vertex in G such that w is a vertex in $H^u - x^u$.

Since G is 2-edge-connected and cubic, G contains a spanning tree T_u such that u is a leaf of T_u. (For example, take a depth-first-search from u.) Then by the same argument as we have shown $\mathrm{ml}(G \cdot H_x) \le |V(G)|/2+1$, starting from the spanning tree T_u, we can find a spanning tree of $G \cdot H_x$ with at most $|V(G)|/2+1$ leaves. Therefore, we have $\mathrm{ml}(G \cdot H_x - w) \le |V(G)|/2+1$. Thus, it suffices to show that $\mathrm{ml}(G \cdot H_x - w) \ge |V(G)|/2+1$.

Let $\mathfrak{T}'$ be a spanning tree of $G \cdot H_x - w$, and define a *full* vertex and a *non-full* vertex in G with respect to $\mathfrak{T}'$. In this case, we obtain the following claim. We omit its proof since it is the same as the proof of Claim 3.

Claim 4. *For any non-full vertex v such that $v \ne u$, the copy $H^v - x^v$ corresponding to v contains at least one leaf of $\mathfrak{T}'$.*

Now we show that $\mathfrak{T}'$ contains at least $|V(G)|/2+1$ leaves. By Claim 4, if there are at least $|V(G)|/2+2$ non-full vertices in G, then we are done. Therefore, we may assume that there are at most $|V(G)|/2+1$ non-full vertices in G, which implies that there are at least $|V(G)|/2-1$ full vertices in G. Since for each full vertex v in G, the graph H_x^v contains a vertex of degree at least three in $\mathfrak{T}'$, we see that $|V_3(\mathfrak{T}')| \ge |V(G)|/2-1$. Therefore, we have

$$\ell(\mathfrak{T}') = |V_3(\mathfrak{T}')| + 2 \ge |V(G)|/2+1,$$

and we are done. This completes the proof of Theorem 5. □

With the appropriate good graphs, Theorem 5 can be used to obtain ℓ-leaf-stable graphs satisfying various additional properties. For example, since the Petersen graph is good, it gives a 3-edge-connected ℓ-leaf-stable cubic graph of order $18\ell-18$ for $\ell \ge 3$. In fact, any hypohamiltonian graph satisfies conditions (H1)–(H3), so we can construct infinitely many 3-edge-connected ℓ-leaf-stable cubic graphs. We also note here that multigraphs (i.e. graphs in which two vertices may be connected by more than one edge) may be used as good graphs.

Neyt [8] found the 24-vertex graph H' given in Fig. 2. H' is a non-hamiltonian graph in which all vertex-deleted subgraphs are traceable. Considering the vertex x, specified in Fig. 2, as its special vertex guarantees the goodness of H'; we leave to the reader the straightforward verification of conditions (H2)–(H3).

As described above, we use Theorem 5 and Petersen's graph (as H) to obtain a ℓ-leaf-stable graph of order $18(\ell-1)$ for $\ell \ge 3$. We can also construct an ℓ-leaf-stable planar graph of order $46(\ell-1)$ by Theorem 5 with $H = H'$, i.e. the 24-vertex graph depicted in Fig. 2. In either case any bridgeless cubic graph can be chosen as G.

Corollary 1. *For each $\ell \ge 3$, there are ℓ-leaf-stable graphs of order $18(\ell-1)$ and ℓ-leaf-stable planar graphs of order $46(\ell-1)$.*

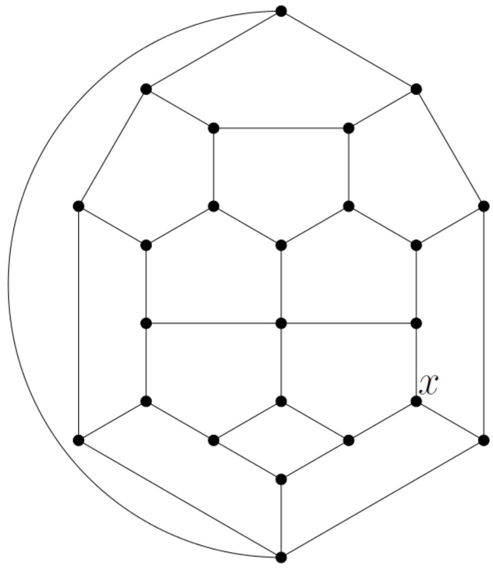

Fig. 2: A good polyhedral graph on 24 vertices due to Neyt [8].

Motivated by the usefulness of good graphs, we end this section with a structural result concerning good graphs and their toughness. In the following $\omega(G)$ denotes the number of connected components of a possibly disconnected graph G.

Proposition 1. *Every good graph is* 1*-tough.*

Proof. Consider G to be a good graph, x its special vertex, and v a neighbour of x. Let us assume that G is not 1-tough, i.e. there exists an $A \subseteq V(G)$ such that $\omega(G - A) > |A|$. By (H2), $G - v$ is hamiltonian, so it is also 1-tough, therefore $\omega(G - v - A) \leq |A|$. As $G - v - A$ is obviously the same as $G - A - v$, this means that $G - A - v$ has fewer components than $G - A$. This is possible only if v is an isolated vertex of $G - A$, which implies that $x \in A$, since v and x are adjacent in G. Let now $A' := A - x$. Observe that as (H1) holds, G has a hamiltonian x-path, so $G - x$ is traceable, which implies $\omega(G - x - X) \leq |X| + 1$ for any $X \subseteq V(G) \setminus \{x\}$. Substituting $X = A'$ we obtain

$$\omega(G - A) = \omega(G - x - A') \leq |A'| + 1 = |A|,$$

a contradiction. □

4 Small leaf-stable and leaf-critical graphs

As mentioned earlier, Wiener [13] expressed interest in determining the orders of the smallest ℓ-leaf-stable and ℓ-leaf-critical graphs. Let S^ℓ_κ (R^ℓ_κ) be the order of the smallest ℓ-leaf stable (ℓ-leaf critical) graph of connectivity κ. $\overline{S}^\ell_\kappa$ and $\overline{R}^\ell_\kappa$ denote the respective numbers for the planar case. Whenever for certain κ and ℓ no such numbers exist, we set them to be ∞. We here give a summary of the known bounds on the aforementioned numbers, including our new ones, but remark that nothing is known for $\kappa \geq 4$. In particular, Thomassen's question whether 4-connected hypohamiltonian graphs exist [12], i.e. $R^2_{\geq 4} = ?$, remains open.

Thomassen [10] showed that $R^3_2 \leq 34$, Wiener [14] proved that $\overline{R}^3_2 \leq 138$ and $R^3_3 \leq 40$ is due to Horton [5]. We can generalise these as follows.

Proposition 2. *For $\ell \geq 3$, we have*

$$R_2^\ell \leq 17(\ell-1), \quad \overline{R}_2^\ell \leq 69(\ell-1), \quad R_3^2 = 10, \quad 23 \leq \overline{R}_3^2 \leq 40,$$
$$R_3^\ell \leq 16\ell - 8, \quad \textit{and} \quad \overline{R}_3^\ell \leq 76\ell - 38.$$

Proof. Every 2-leaf-critical graph is 3-connected, so $R_2^2 = \overline{R}_2^2 = \infty$. The first two inequalities follow from Theorem 1 applied to Thomassen's 18-vertex 2-fragment A of a 3-leaf-critical graph [10] and the 70-vertex planar analogue B (constructed from two copies of a planar 36-vertex almost hypohamiltonian graph with a cubic exceptional vertex, discovered independently by Wiener [14], and Goedgebeur and Zamfirescu [3]), respectively—no smaller such fragments are known.

$R_3^2 = 10$ is given by Petersen's graph and the well-known fact that it is the smallest 2-leaf-critical graph.

The lower and upper bound for the order of the smallest planar 2-leaf-critical graph was established in [2] and [6], respectively.

The final two inequalities are based on Wiener's [13, Theorem 3.8]. For the non-planar case we use the Petersen graph, while for the planar case we use the smallest known planar 2-leaf-critical graph [6], which has order 40. □

We also give a counterpart of Proposition 2 for the leaf-stable case.

Proposition 3. *For $\ell \geq 3$, we have*

$$S_2^2 = \overline{S}_2^2 = 12, \quad S_2^\ell \leq 17\ell, \quad \overline{S}_2^\ell \leq 69\ell,$$
$$S_3^\ell \leq \min\{18(\ell-1), 16\ell\}, \quad \textit{and} \quad \overline{S}_3^\ell \leq 46(\ell-1).$$

Proof. The equalities follow from computational results of Van Cleemput and Zamfirescu [15]. Both $S_2^2 \leq 12$ and $\overline{S}_2^2 \leq 12$ are given by the same (planar) graph, obtained by adding in the cartesian product of K_3 and P_2 on each copy of P_2 two extra vertices.

The first and the second inequality are obtained by applying Theorem 4 to the fragments A and B, defined in the proof of Proposition 2, respectively.

The 16ℓ bound of the third inequality is given in the article [13] of Wiener, while the remaining two bounds are given by Corollary 1. □

We end this paper with a problem motivated by work of Thomassen [12]: He proved that every planar 2-leaf-critical graph contains a cubic vertex. Zamfirescu [15] showed that there exist planar 2-leaf-stable graphs with no cubic vertices. Contrasting this, in [16] he proved—using a result of Wiener—that planar 3-leaf-critical graphs of connectivity 2 in which every 2-fragment has edge-connectivity 2 must contain a cubic vertex. However, the general case is open, and the same holds for planar 3-leaf-stable graphs.

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References

[1] L. Gargano, M. Hammar, P. Hell, L. Stacho, and U. Vaccaro. Spanning spiders and light-splitting switches. *Discrete Math.* **285** (2004) 83–95.

[2] J. Goedgebeur and C. T. Zamfirescu. Improved bounds for hypohamiltonian graphs. *Ars Math. Contemp.* **13** (2017) 235–257.

[3] J. Goedgebeur and C. T. Zamfirescu. On almost hypohamiltonian graphs. *Discrete Math. Theoret. Comput. Sci.* **21** (2019) #5.

[4] D. A. Holton and J. Sheehan. The Petersen Graph, Chapter 7: Hypohamiltonian graphs, Cambridge University Press, New York, 1993.

[5] J. D. Horton. A Hypotraceable Graph. Research Report CORR 73–4, Dept. Combin. and Optim., Univ. Waterloo, 1973.

[6] M. Jooyandeh, B. D. McKay, P. R. J. Östergård, V. H. Pettersson, and C. T. Zamfirescu. Planar hypohamiltonian graphs on 40 vertices. *J. Graph Theory* **84** (2017) 121–133.

[7] H.-I. Lu and R. Ravi. The Power of Local Optimization: Approximation Algorithms for Maximum-leaf Spanning Tree (DRAFT), CS-96-05, Department of Computer Science, Brown University, Providence, Rhode Island, 1996.

[8] A. Neyt. Platypus Graphs: Structure and Generation. M.Sc. Thesis, Ghent University, 2017.

[9] G. Salamon and G. Wiener. On finding spanning trees with few leaves. *Inf. Proc. Lett.* **105** (2008) 164–169.

[10] C. Thomassen. Hypohamiltonian and hypotraceable graphs. *Discrete Math.* **9** (1974) 91–96.

[11] C. Thomassen. Planar and infinite hypohamiltonian and hypotraceable graphs. *Discrete Math.* **14** (1976) 377–389.

[12] C. Thomassen. Hypohamiltonian graphs and digraphs. Theory and Applications of Graphs, Lecture Notes in Mathematics **642**, Springer, Berlin (1978) 557–571.

[13] G. Wiener. Leaf-critical and leaf-stable graphs. *J. Graph Theory* **84** (2017) 443–459.

[14] G. Wiener. New constructions of hypohamiltonian and hypotraceable graphs. *J. Graph Theory* **87** (2018) 526–535.

[15] C. T. Zamfirescu. On non-hamiltonian graphs for which every vertex-deleted subgraph is traceable. *J. Graph Theory* **86** (2017) 223–243.

[16] C. T. Zamfirescu. Cubic vertices in planar hypohamiltonian graphs. *J. Graph Theory* **90** (2019) 189–207.

[17] T. Zamfirescu. On longest paths and circuits in graphs. *Math. Scand.* **38** (1976) 211–239.