On a question of van Aardt et al. on destroying all longest cycles

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Abstract. We describe an infinite family of 2-connected graphs, each of which has the property that the intersection of all longest cycles is empty. In particular, we present such graphs with circumference 10, 13, and 16. This settles a question of van Aardt et al. [*Discrete Appl. Math.* **186** (2015) 251–259] concerning the existence of such graphs for all but one case, namely circumference 11. We also present a 2-connected graph of circumference 11 in which all but one vertex are avoided by some longest cycle.

Key Words. Longest cycle, circumference, vertex deletion

MSC 2010. 05C38

1 Introduction

In 2015, van Aardt et al. [1, p. 254] asked the following.

Question. *Does there exist for any* $k \in \{10, 11, 13, 16\}$ *a 2-connected graph with circumference* k *that has no vertex meeting every* k*-cycle?*

We here answer this question affirmatively for $k \in \{10, 13, 16\}$, while the case $k = 11$ remains open. To this end, we adapt an approach described by the author in [5]. Special interest lies in the planar case and the minimum degree, motivated by work of Thomassen on hypohamiltonian graphs. (A graph is *hypohamiltonian* if it is non-hamiltonian, yet every vertex-deleted subgraph is hamiltonian—these constitute the extremal case when the graph's circumference is one less than its order.) In the seventies Chvátal had asked whether planar hypohamiltonian graphs exist and Thomassen showed this to be the case by describing an infinite family of such graphs. Thomassen also asked whether hypohamiltonian graphs of minimum degree at least 4 exist [4], but this remains open. Further references and a discussion of related problems on the intersection of all longest cycles can be found in [3].

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2 Result

Theorem 1. *For every pair of integers $\ell \geq 3$ and $p \geq 4$ there exists a 2-connected graph G of order ℓp and circumference $\ell(p-1)+1$ whose intersection of all longest cycles is empty.*

Proof. Consider the cartesian product of a cycle of length p and a path P_2 on two vertices. We shall denote the two copies of the cycle with $a_1 \ldots a_p$ and $b_1 \ldots b_p$ such that a_i and b_i are adjacent for all i. Insert on each copy of P_2 exactly $\ell - 2$ vertices, yielding p induced subgraphs isomorphic to the path on ℓ vertices. We denote such a path with end-vertices a_i and b_i by H_i and put $X_i = \{a_i, b_i\}$. If p is even, remove the edges $a_1 a_p$ and $b_1 b_p$ from the graph and add the edges $a_1 b_p$ and $a_p b_1$. (If p is odd, we do not perform this transformation.) We have obtained the graph G.

We now show that there exists a cycle C in G of length $\ell(p-1)+1$ omitting any vertex $v \in V(G)$. To avoid unnecessarily complicated indices, consider $v \in V(H_1) \setminus \{b_1\}$. Put $C =$

$$\left(\bigcup_{i \neq 1} V(H_i) \cup \{b_1\}, \bigcup_{i \neq 1} E(H_i) \cup \{a_j a_{j+1}\}_{j < p \text{ even}} \cup \{b_k b_{k+1}\}_{k \text{ odd}} \cup \begin{cases} \emptyset & \text{for odd } p \\ a_p b_1 & \text{for even } p \end{cases} \right)$$

where $a_{p+1} = a_1$ and $b_{p+1} = b_1$. For all other vertices v in G, a cycle of length $\ell(p-1)+1$ avoiding v can be constructed in the same fashion.

Let us now determine the circumference of G. We first observe that a cycle in G cannot traverse all H_i's: since X_i is a vertex-cut in G, the subgraph H_i can only be visited by entering through a_i and leaving through b_i. By construction, a cycle in G visiting all H_i's would have to traverse an even number of H_i's if p is odd and an odd number of H_i's if p is even, a contradiction. Thus, in a longest cycle of G, at least one H_i, say H_2, cannot be traversed. Since X_2 is a vertex-cut in G, we lose all vertices in $V(H_2) \setminus \{a_2, b_2\}$. Is there a cycle C' in G strictly longer than $\ell(p-1)+1$ visiting both a_2 and b_2 but no other vertex of H_2? We now show that the answer to this question is *no*.

There are two ways for C' to visit vertices in H_3: either (a) C' visits only a_3 and b_3 among vertices in H_3, or (b) C' traverses H_3, i.e. visits at least one vertex in $V(H_3) \setminus \{a_3, b_3\}$.

Case (a): C' misses at least $2(\ell-2)$ vertices, so C' is by $\ell - 3$ shorter than C (which is the cycle of length $\ell(p-1)+1$ constructed above). If $\ell = 3$, then the lengths of C' and C are equal. If $\ell \geq 4$, then C' is strictly shorter than C. Hence, for all $\ell \geq 3$, C' is not strictly longer than C.

Case (b): If C' does not traverse H_1 it misses all vertices in $V(H_1) \setminus \{a_1, b_1\}$ and we are in Case (a). If C' traverses H_1, then C' misses all vertices in $(V(H_2) \setminus \{a_2, b_2\}) \cup \bigcup_{i=4}^{p} V(H_i)$. The length of C' is $2\ell + 2$ and the length of C is at least $3\ell + 1$ since $p \geq 4$. As $\ell \geq 3$, we have that $3\ell + 1 > 2\ell + 2$, so again C' is shorter than C.

We conclude that the circumference of G is $\ell(p-1)+1$. We had already proven that for any vertex v in G there exists a cycle of length $\ell(p-1)+1$ avoiding v, so we have that the intersection of all longest cycles in G is empty. □

Proposition 1. *There exists a graph G which in addition to the properties given in Theorem 1 satisfies that either*

(i) *the minimum degree of G is $\ell - 1$, or*

(ii) *if p is odd, G is planar and its minimum degree is* $\begin{cases} 2 & \text{if } \ell = 3 \\ 3 & \text{if } 4 \leq \ell \leq 5 \\ 4 & \text{if } 6 \leq \ell \leq 11 \\ 5 & \text{else.} \end{cases}$

Proof. Observe that in the proof of Theorem 1, in each H_i we may add edges as we wish (but not between H_i and H_j for $i \neq j$) without altering the fact that the intersection of all longest cycles is empty. Doing so we obtain a graph G that contains p paths of order ℓ, to each of which a certain number of edges were added. We denote these paths plus edges with H'_i and the set of edges that were added to H_i in order to obtain H'_i with E_i. Each H'_i contains precisely two vertices that are the end-points of the underlying copy of P_ℓ (i.e. the path on ℓ vertices), namely a_i and b_i.

It is obvious that the minimum degree of G is $\ell - 1$ if we choose every H'_i to be a complete graph on ℓ vertices—this satisfies (i)—, and that G is planar if p is odd and we choose every H'_i to be P_ℓ^2, i.e. on each copy of P_ℓ we add an edge between all vertices at distance 2 (on the respective copy of P_ℓ). In this case we have for the minimum degree $\delta(G)$ of G that $\delta(G) = 2$ if $\ell = 3$ and $\delta(G) = 3$ if $\ell \geq 4$. For $\delta(G) = 4$ and $\ell \geq 6$, again consider P_ℓ^2. Observe that there are exactly two cubic vertices present, say w and w'. These are adjacent to an end-point of (the respective copy of) P_ℓ, say a and b, respectively. Then adding the edges wb and $w'a$ if ℓ is even or adding the edge ww' when ℓ is odd (here we need that $\ell \geq 6$) provides $\delta(G) = 4$ for $\ell \geq 6$. For $\delta(G) = 5$ consider [5, Fig. 3(B)] for $\ell = 12$. The remaining cases (i.e. $\ell > 12$) can be dealt with similarly and are left to the reader. Thus, we satisfy (ii). □

It is clear from the above proofs that any traceable graph is the subgraph of some 2-connected graph in which the intersection of all longest cycles is empty. A stronger result was recently proven by the author and Tudor Zamfirescu [6]: *any* graph is the subgraph of some hypohamiltonian graph. This settled an old problem of Chvátal.

Consider the following statement: if in a planar graph G with minimum degree at least 4 all vertex-deleted subgraphs contain a cycle of length k, then G contains a cycle of length greater than k. Thomassen showed that every planar hypohamiltonian graph contains a cubic vertex [4]—thus, the aforementioned statement is true for $k = n - 1$, where n is the order of G. Proposition 1 implies that it does not hold if $n = \ell p$ and $k = \ell(p-1) + 1$, where $p \geq 5$ is odd and $\ell \geq 6$.

With above theorem, we can now address the question of van Aardt et al. Although we cannot settle the $k = 11$ case, our solutions for $k \in \{10, 13, 16\}$ are optimal in the sense that no smaller graphs with the requested properties exist, since they cannot be hypohamiltonian [2].

Corollary 1. *For every* $k \in \{10, 13, 16\}$ *there exists a* 2*-connected graph of order* $k+2$ *and circumference* k *whose intersection of all longest cycles is empty.*

Proof. Denote the graphs constructed in the proof of Theorem 1 with $G_{p,\ell}$. To prove the statement for $k = 10, 13, 16$ it now suffices to consider $G_{4,3}$, $G_{5,3}$, and $G_{6,3}$, respectively, see Fig. 1. Note that $G_{p,\ell}$ is planar for odd p and bipartite when p is even while ℓ is odd. □

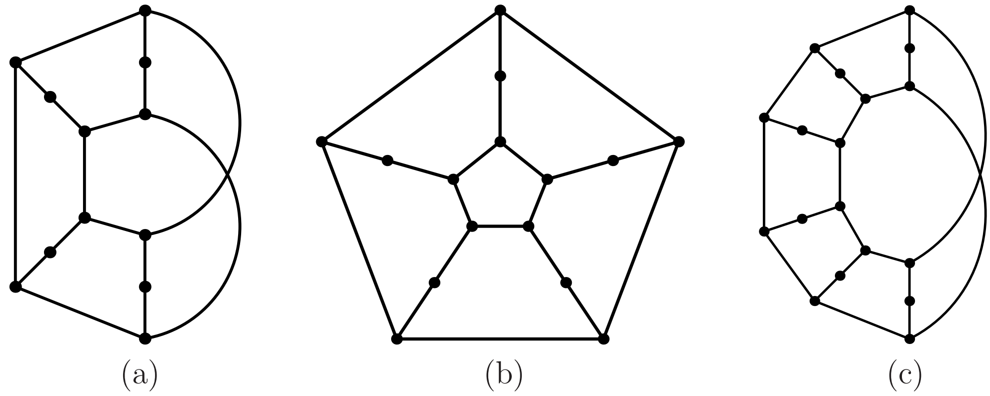

Fig. 1: (a) $G_{4,3}$, (b) $G_{5,3}$, and (c) $G_{6,3}$

We note that the graph $G_{4,3}$, shown in Fig. 1 (a), was already given by Tudor Zamfirescu (see [3, Fig. 19 (a)]) and $G_{5,3}$, depicted in Fig. 1 (b), belongs to Thomassen (see [3, Fig. 16]). Brinkmann and Van Cleemput (personal communication) showed with a computer that the former is, excluding Petersen's graph, the smallest 2-connected graph in which every vertex is missed by a longest cycle—a fact that can also be deduced from [1, Theorem 2.10]—, while the latter is the smallest planar 2-connected graph in which every vertex is missed by a longest cycle.

Let us briefly comment upon the remaining open case. Let G be a 2-connected graph of circumference 11 in which the intersection of all longest cycles is empty. Obviously, G must be non-hamiltonian, and G cannot be hypohamiltonian [2], so its order is at least 13 (as already observed in [1]). We end this note with a proposition concerning the circumference 11 case.

Proposition 2. *There exists a* 2*-connected graph of order* 13 *and circumference* 11 *in which all but one vertex are avoided by some longest cycle.*

Proof. Denote Petersen's graph with P and consider $v \in V(P)$. Let v_1, v_2, v_3 be the neighbours of v. Insert a vertex w_i on each edge vv_i. As we shall see, the resulting graph P' proves the statement. The order of P' is 13. Let C be a longest cycle in P'. Clearly C cannot visit all w_i's. Let $j \in \{1, 2, 3\}$. Assume C visits all vertices in $V(P') \setminus \{w_j\}$. This yields a hamiltonian cycle in P, a contradiction. So the circumference of P' is at most 11.

Let $u \in V(P) \setminus \{v\}$. Since P is hypohamiltonian, there is a hamiltonian cycle C_u in $P - u$. Seeing C_u as a cycle in P', it visits all vertices of P with the exception of u, cannot visit all w_i's, but clearly visits two of them (since C_u contains v), so the length of C_u is 11. Thus C_u is a longest cycle in P' avoiding u. Let $i \in \{1, 2, 3\}$ and C_{v_i} be a hamiltonian cycle in $P - v_i$. Then C_{v_i} must visit all vertices in $P - v_i$, so it visits v. Thus, seeing C_{v_i} as a cycle in P', it contains w_j and w_k, where $j \neq i \neq k$. Therefore C_{v_i} has length 11 and visits all vertices in P' excluding v_i and w_i.

Removing v from P' yields a graph with three vertices of degree 1 (namely w_1, w_2, w_3), so no cycle of length 11 exists in $P' - v$. □

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