Cubic vertices in planar hypohamiltonian graphs

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Abstract. Thomassen showed in 1978 that every planar hypohamiltonian graph contains a cubic vertex. Equivalently, a planar graph with minimum degree at least 4 in which every vertex-deleted subgraph is hamiltonian, must be itself hamiltonian. By applying work of Brinkmann and the author, we extend this result in three directions. We prove that (i) every planar hypohamiltonian graph contains at least four cubic vertices, (ii) every planar almost hypohamiltonian graph contains a cubic vertex which is not the exceptional vertex (solving a problem of the author raised in [*J. Graph Theory* **79** (2015) 63–81]), and (iii) every hypohamiltonian graph with crossing number 1 contains a cubic vertex. Furthermore, we settle a recent question of Thomassen by proving that asymptotically the ratio of the minimum number of cubic vertices to the order of a planar hypohamiltonian graph vanishes.

Key Words. Hypohamiltonian; planar; 3-connected; 3-cut.

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1 Introduction

Throughout this paper all graphs are undirected, finite, connected, and contain neither loops nor multiple edges. A graph G is called *hypohamiltonian* if G is non-hamiltonian, but for every vertex v in G, the graph $G - v$ is hamiltonian. See the 1993 survey [15] by Holton and Sheehan for an overview of results. Hypohamiltonian graphs have a wide range of applications in problems on longest paths and longest cycles [21]—one can for instance prove that Gallai's question [10] whether in every graph there is a vertex in the intersection of all longest paths has a negative answer, and that there exist

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3-connected graphs in which any pair of vertices is avoided by a longest cycle [4, 37]. Recently, Ozeki and Vrána [20] applied hypohamiltonicity to show that infinitely many 2-hamiltonian but not 2-edge-hamiltonian-connected graphs exist, while Wiener uses hypohamiltonian graphs to study a criticality notion concerning the number of leaves in spanning trees [31]. Ties between hypohamiltonian graphs and snarks have been studied extensively, see for instance [6]. Recently it was shown that every hypohamiltonian snark has a 5-flow [18]. The Four Colour Theorem is equivalent to the statement that every snark is non-planar; here, we will be interested in the structural properties of *planar* hypohamiltonian graphs. After Chvátal had asked whether such graphs exist [8] and Grünbaum conjectured that they do not [13, p. 37], an infinite family was described by Thomassen [25].

For results treating the planar case and which are not included in [15], we refer to the works of Aldred, Bau, Holton, and McKay [3], the author and Zamfirescu [35, 36], Araya and Wiener [4, 33], Jooyandeh, McKay, Östergård, Pettersson, and the author [16], McKay [19], Goedgebeur and the author [11, 12], and Wiener [32]. For the situation in directed graphs, see for instance [1], which answers affirmatively Thomassen's [26, Question 9] from 1976 whether planar hypohamiltonian oriented graphs exist.

A graph G of connectivity k which is not a complete graph contains two non-empty induced subgraphs G_1, G_2 such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = A$, where $|A| = k$. We say that G_1 is a *k-fragment* of G and that A is the set of *vertices of attachment* of G_1. A k-fragment is *trivial* if it contains exactly $k + 1$ vertices.

In the light of Steinitz' Theorem [22], we call planar 3-connected graphs *polyhedra*. In a 3-connected graph, a vertex cut-set X of cardinality 3 will be called a 3*-cut*. X is called *trivial* if it coincides with the set of vertices of attachment of a trivial 3-fragment. 3-connected graphs in which every 3-cut is trivial are called *essentially* 4*-connected*. A path with end-vertices v and w is a *vw-path*. A path, cycle, or face (in a plane graph) with k vertices will be called a *k-path*, *-cycle*, or *-face*, respectively, and, seen as a subgraph of a graph G, it will be called *cubic* if all of its vertices are cubic in G.

Consider a 2-connected graph G of circumference $|V(G)| - 1$ and let $W \subset V(G)$ be the (possibly empty) set of all vertices such that for every $w \in W$ the graph $G - w$ is non-hamiltonian. Thus, for all $v \in V(G) \setminus W$, the graph $G - v$ is hamiltonian, and we have $|W| \leq |V(G)| - 1$. We say that G is *$|W|$-hypohamiltonian*. (Note that van Aardt et al. define in [2] an r-hypohamiltonian *di*graph differently.) A vertex from W is called *exceptional*. 0-hypohamiltonian graphs coincide with hypohamiltonian graphs and 1-hypohamiltonian graphs are called *almost hypohamiltonian*. Hypohamiltonian and almost hypohamiltonian graphs are 3-connected, but 2-hypohamiltonian graphs of connectivity 2 exist. A graph G is *k-hamiltonian* if $G - S$ is hamiltonian for every $S \subset V(G)$ with $|S| \leq k$.

In the remainder of this article, we shall tacitly use the following fact. Let G be a k-hypohamiltonian graph of connectivity 3, and X a 3-cut in G containing a vertex v which is not an exceptional vertex of G. Then $G - X$ has exactly two components—this follows directly from the hamiltonicity of $G - v$. Note that this holds for arbitrary 3-cuts in polyhedra as well, since $K_{3,3}$ is non-planar. When we say that a graph is the "smallest" graph satisfying a particular condition, we mean that there exist no graphs of smaller order with that property.

2 Planar hypohamiltonian graphs

Using a result of Tutte [29], Thomassen showed the following.

Theorem 1 (Thomassen [26])**.** *Every planar hypohamiltonian graph contains a cubic vertex.*

As Thomassen writes in [28] we can reformulate this result in a perhaps more appealing way: if G is a planar graph with minimum degree at least 4 and every vertex-deleted subgraph of G is hamiltonian, then G itself must be hamiltonian.

For the first extension of this result, we need the following ingredients, the first of which is a strengthening of the classical result of Tutte that every 4-connected polyhedron is hamiltonian [29].

Theorem 2 (Brinkmann and Zamfirescu [7])**.** *Polyhedra containing at most three* 3*-cuts are hamiltonian.*

Lemma 1 (Thomassen [26])**.** *Let G be a* 3*-connected graph with $n \geq 6$ vertices. If G contains more than one non-trivial* 3*-cut, then G contains a non-trivial* 3*-fragment with fewer than $\frac{1}{2}(n+3)$ vertices.*

The following lemma generalises the author's [34, Theorem 3] and [34, Theorem 6], as well as Thomassen's [26, Corollary 1], and in essence belongs to him.

Lemma 2. *Let $i \in \{1,2\}$. Consider the disjoint graphs G_1 and G_2 such that G_i is k_i-hypohamiltonian, with W_i as set of exceptional vertices. Let H_i be a non-trivial* 3*-fragment of G_i, with the set X_i of vertices of attachment of H_i disjoint from W_i. Put $U_i = W_i \cap V(H_i)$. Then the graph Γ obtained from $H_1 \cup H_2$ by identifying X_1 with X_2 using a bijection is a $(|U_1| + |U_2|)$-hypohamiltonian graph. If G_1 and G_2 are planar, then Γ is planar, as well.*

Proof. Let $X_i = \{x_{i1}, x_{i2}, x_{i3}\}$, and denote by J_i the 3-fragment in G_i which is not H_i and whose set of vertices of attachment is X_i. We call $x_j \in V(\Gamma)$ the vertex obtained when identifying x_{1j} with x_{2j}.

Assume Γ is hamiltonian. Treating H_i as a subgraph of Γ, for appropriate i, j, k there exists a hamiltonian $x_j x_k$-path in H_i. Since $X_i \cap W_i = \emptyset$, we have a hamiltonian $x_{ij} x_{ik}$-path in $J_i - x_{i\ell}$, where $j \neq \ell \neq k$. The union of these paths yields a hamiltonian cycle in G_i, a contradiction.

Since $G_i - x_{ij}$ is hamiltonian, there exist hamiltonian $x_{ik} x_{i\ell}$-paths $\mathfrak{p}_{ij}$ in $H_i - x_{ij}$, where $k \neq j \neq \ell$, for all i, j. Then $\mathfrak{p}_{1j} \cup \mathfrak{p}_{2j}$ gives a hamiltonian cycle in $\Gamma - x_j$. Consider $v \in V(H_1) \setminus (X_1 \cup U_1)$. Since G_1 is k_1-hypohamiltonian and $v \notin U_1$, there exists a hamiltonian $x_{1j} x_{1k}$-path $\mathfrak{p}$ in $H_1 - v$ for appropriate j, k. (Note that $x_{1\ell}$, $j \neq \ell \neq k$, necessarily lies in $\mathfrak{p}$. Assume it does not, and consider the hamiltonian cycle $\mathfrak{h}$ in $G_1 - v$ which contains $\mathfrak{p}$. $\mathfrak{h} \cap J_1$ is a hamiltonian $x_{1j} x_{1k}$-path in J_1. But then $(\mathfrak{h} \cap J_1) \cup \mathfrak{p}_{1\ell}$ gives a hamiltonian cycle in G_1, a contradiction.) Then $\mathfrak{p} \cup \mathfrak{p}_{2\ell}$, $j \neq \ell \neq k$, provides the desired hamiltonian cycle in $\Gamma - v$. The same argument yields a hamiltonian cycle in $\Gamma - v'$ for $v' \in V(H_2) \setminus (X_2 \cup U_2)$.

In consequence, Γ does not contain multi-edges: if there exists a pair of vertices in Γ that are joined by multiple edges, it must be of the form x_j, x_k, where $j \neq k$.

We have shown that $\Gamma - x_j$ and $\Gamma - x_k$ are hamiltonian, so there exists a hamiltonian $x_k x_\ell$-path in $H_1 - x_j$ and a hamiltonian $x_j x_\ell$-path in $H_2 - x_k$. These paths together with the edge $x_j x_k$ form a hamiltonian cycle in Γ, in contradiction to what was proven above.

Finally, consider $u \in U_1$, where we see U_1 as also lying in Γ. Assume $\Gamma - u$ contains a hamiltonian cycle $\mathfrak{h}'$. Then $\mathfrak{p}' = \mathfrak{h}' \cap H_1$ is a hamiltonian $x_{1j}x_{1k}$-path in $H_1 - u$ for appropriate j, k. (As before, we can show that $x_{1\ell} \in V(\mathfrak{p}')$.) $\mathfrak{p}'$ together with the hamiltonian $x_{1j}x_{1k}$-path in $J_1 - x_{1\ell}$, $j \neq \ell \neq k$, we obtain that $G_1 - u$ is hamiltonian—a contradiction, since u is exceptional in G_1. For a vertex in U_2, the proof is the same. □

Occasionally, we shall refer to the identification from Lemma 2 as "gluing" two 3-fragments. A direct corollary of Lemmas 1 and 2 is that a smallest planar hypohamiltonian graph contains at most one non-trivial 3-cut. In Theorem 5 we present a stronger result. As a further corollary we obtain a strengthening of [11, Proposition 2.6 (ii)], for which we need the following lemma (which strengthens an observation of Collier and Schmeichel [9]). In [25], no proof is given—we have seen a proof in the penultimate paragraph of the proof of Lemma 2.

Lemma 3 (Thomassen [25])**.** *Let G be a hypohamiltonian graph containing a (trivial or non-trivial) 3-fragment H. Then no two vertices of attachment of H are adjacent. In particular, every vertex of a triangle in a hypohamiltonian graph has degree at least 4.*

Lemma 4. *Let G be a hypohamiltonian graph containing a non-trivial 3-fragment H. Then every vertex of attachment of H has at least two neighbours in H which are not vertices of attachment of H. Consequently, a vertex lying in a non-trivial 3-cut of a hypohamiltonian graph has degree at least 4.*

Proof. Assume H has among its vertices of attachment a vertex which has at most one neighbour in H. We apply Lemma 2 to two copies of H. By Lemma 3, we obtain a hypohamiltonian graph with a vertex of degree at most 2, which is impossible. □

In a graph G, an edge-cut M of G is called *trivial* (*bitrivial*) if one of the components of $G - M$ is K_1 (K_2).

Corollary 1.

(i) *In a planar hypohamiltonian graph, every vertex neighbouring only cubic vertices is incident with a face of size at least 5.*

(ii) *A 4-edge-cut in a hypohamiltonian graph is either trivial, bitrivial, or consists of four independent edges.*

(iii) *Cubic hypohamiltonian graphs are essentially 4-connected. Thus, a cubic hypohamiltonian graph G contains exactly $|V(G)|$ 3-cuts.*

Proof. (i) Let v be a vertex in a planar hypohamiltonian graph G such that every vertex in $N(v)$ is cubic. Thus, by Lemma 3, v is not incident with a triangle. Assume that v is incident exclusively with 4-faces $Q_0, ..., Q_{k-1}$, where Q_i shall share an edge with Q_{i+1}, i mod. k. Denote the vertex in Q_i which is not in $N[v]$ with v_i, and the vertex in $N(v) \cap N(v_i) \cap N(v_{i+1})$ with w_i.

Let $\mathfrak{h}$ be a hamiltonian cycle in $G - v_0$. Since w_0 and w_{k-1} are cubic, the path $v_{k-1}w_{k-1}vw_0v_1$ lies in $\mathfrak{h}$. As the vertices $w_1, ..., w_{k-2}$ are cubic, as well, we have that $\mathfrak{h} = v_{k-1}w_{k-1}vw_0v_1w_1v_2w_2...v_{k-2}w_{k-2}$. This implies that $V(G) = V(\mathfrak{h}) \cup \{v_0\}$. But then G is bipartite, and since hypohamiltonian graphs cannot be bipartite, we have obtained a contradiction.

(ii) This follows from Lemma 4 and the fact that hypohamiltonian graphs are 3-connected.

(iii) The first statement follows directly from Lemma 4, and the second statement from the first statement. □

Note that Lemmas 3 and 4 do not hold for almost hypohamiltonian graphs, as can be seen in Fig. 1. The three vertices we have emphasised in Fig. 1 provide a 3-cut X proving our point. We may choose instead of w a neighbour of w, which is not in X—this shows that the 3-cut need not contain the exceptional vertex to break Lemma 4. However, in this case Lemma 3 does hold. In fact, we have already seen in the proof of Lemma 2 that in a k-hypohamiltonian graph, two non-exceptional vertices in a 3-cut cannot be adjacent. We will look at similar arguments in Lemma 6, in which we prove useful structural properties of 3-cuts in almost hypohamiltonian graphs. Also note that a triangle in an almost hypohamiltonian graph, as in the hypohamiltonian case, cannot contain cubic vertices [12].

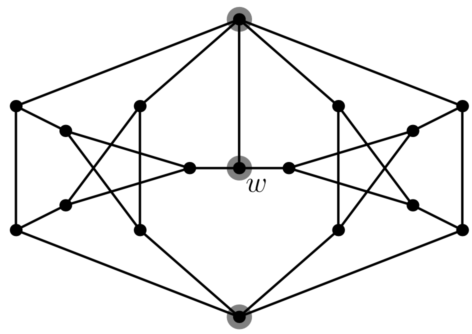

Fig. 1: The smallest almost hypohamiltonian graph. (That it is indeed the smallest such graph was proven in [12].) Its exceptional vertex is labeled w.

We now prove the first of three strengthenings of Theorem 1.

Theorem 3. *Every planar hypohamiltonian graph contains at least four cubic vertices.*

Proof. In this first part of the proof, assume there exists a planar hypohamiltonian graph G containing exactly one or two cubic vertices—due to Thomassen's Theorem 1, we need not treat the case that G contains no cubic vertices. Furthermore, we may assume that G is the smallest such graph. By Theorem 2, G contains at least two non-trivial 3-cuts.

Let A_H be a non-trivial 3-cut in G, and H be a 3-fragment of G of minimum order whose set of vertices of attachment is A_H. Note that by Lemma 4, the cubic vertices present in G cannot lie in A_H. Take two copies of H and identify their respective vertices of attachment using a bijection. By Lemma 2, the resulting graph G' is planar (as H is planar) and hypohamiltonian. By Lemma 4, each of the three vertices which the two copies of H have in common has degree at least 4 in G'. Every other vertex has the same degree in G' as in G. By Lemma 1, G' has fewer vertices than G. Put $\widehat{H} = V(H) \setminus A_H$.

(i) If $\widehat{H}$ contains no cubic vertices, then G' is a planar hypohamiltonian graph with no cubic vertices, in contradiction with Theorem 1.

(ii) If $\widehat{H}$ contains exactly one cubic vertex, then G' is a planar hypohamiltonian graph with exactly two cubic vertices and smaller than G, a contradiction.

(iii) If $\widehat{H}$ contains exactly two cubic vertices, let $H' \neq H$ be the other 3-fragment of G whose set of vertices of attachment is A_H. (H' is non-trivial as A_H is non-trivial.) Since G now contains exactly two cubic vertices, and these vertices lie in $\widehat{H}$, H' contains no vertices which are cubic in G. By gluing two copies of H' we obtain a contradiction to Theorem 1.

We have shown that a planar hypohamiltonian graph contains at least three cubic vertices. In this second part of the proof, assume G to be a planar hypohamiltonian graph containing exactly three cubic vertices. By Theorem 2, G contains a non-trivial 3-cut X. By Lemma 4, X does not contain cubic vertices. One of the 3-fragments whose set of vertices of attachment is X contains at most one cubic vertex. We call this 3-fragment H. Applying Lemma 2 to two copies of H, we obtain a planar hypohamiltonian graph with at most two cubic vertices, in contradiction to the first part of the proof. □

Corollary 2. *Let H be a non-trivial 3-fragment of a planar hypohamiltonian graph G containing exactly k vertices that are cubic in G. Then there are exactly $k+1$ cubic vertices in G if the set of vertices of attachment A_H of H forms a trivial 3-cut, and at least $k+2$ cubic vertices in G if A_H is a non-trivial 3-cut.*

Proof. Let $H' \neq H$ be the other 3-fragment whose set of vertices of attachment is A_H. If A_H is trivial, then so is H', and we obtain that G has $k+1$ cubic vertices. If A_H is non-trivial, then so is H'. Assume G contains at most $k+1$ cubic vertices. Since H has k cubic vertices and A_H contains no cubic vertices by Lemma 4, H' has at most one cubic vertex. We apply Lemma 2 to two copies of H' and obtain a planar hypohamiltonian graph which has at most two cubic vertices—in contradiction with Theorem 3. □

It was recently shown [16] that planar hypohamiltonian graphs containing 30 cubic vertices exist, see Fig. 2. No planar hypohamiltonian graph with fewer than 30 cubic vertices is known (and no hypohamiltonian graph with fewer than ten cubic vertices is known—it is an old question of Thomassen whether hypohamiltonian graphs of minimum degree at least 4 exist [26, Problem 4]). Narrowing the gap (between 4 and 30) would be a very welcome contribution.

During the 2016 Ghent Graph Theory Workshop on Longest Paths and Longest Cycles, Thomassen raised the question whether not a certain proportion of vertices must be cubic in a *planar* hypohamiltonian graph. We now prove that this is not the case. To this end, consider the following operation due to Thomassen [27]. Let G be a graph containing a 4-cycle $v_1v_2v_3v_4 = C$, and consider vertices $v_1', v_2', v_3', v_4' \notin V(G)$.

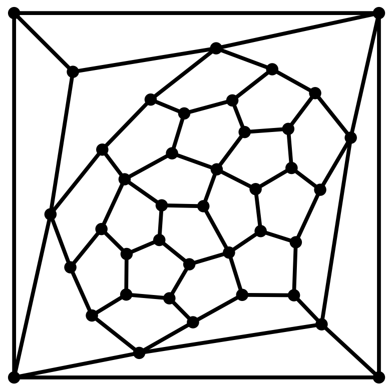

Fig. 2: A plane hypohamiltonian graph containing 30 cubic vertices.

We denote by $\mathrm{Th}(G_C)$ the graph obtained from G by deleting the edges v_1v_2, v_3v_4 and adding a new 4-cycle $v_1'v_2'v_3'v_4'$ and the edges v_iv_i', $1 \leq i \leq 4$ (see Fig. 3). If C is an unspecified 4-cycle in a graph G, when we speak of "the graph $\mathrm{Th}(G_C)$" we refer to (an arbitrary but fixed) one of the two (possibly isomorphic) graphs obtained when applying the operation Th.

Thomassen mentioned in [27] the following (he gives no proof; a detailed proof for the planar case can be found in [33], and planarity plays no role in the argument): if G is a hypohamiltonian graph containing a cubic 4-cycle C, then $\mathrm{Th}(G_C)$ is hypohamiltonian.

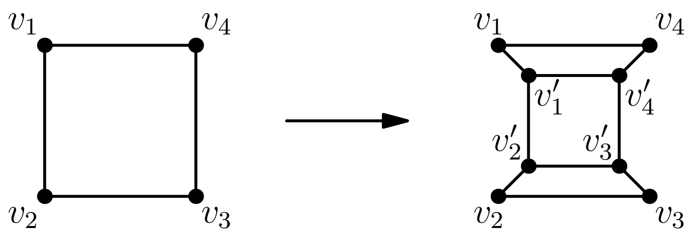

Fig. 3: The operation Th.

The following observation is crucial.

Lemma 5. *If G is a hypohamiltonian graph containing a cubic 4-cycle $C = v_1v_2v_3v_4$, then $G' = \mathrm{Th}(G_C) + v_1v_2 + v_3v_4$ is hypohamiltonian.*

Proof. With Thomassen's aforementioned result in mind, we only need to show that G' is non-hamiltonian. Assume G' does have a hamiltonian cycle $\mathfrak{h}$. Since $\mathrm{Th}(G_C)$ is non-hamiltonian, $\mathfrak{h}$ uses w.l.o.g. v_1v_2. Put $R = G'[\{v_i, v_i'\}_{i=1}^4]$. We now treat all essentially different situations. If $v_1v_2 + v_3v_3'v_2'v_1'v_4'v_4 \subset R \cap \mathfrak{h}$, we replace $v_3v_3'v_2'v_1'v_4'v_4$ with v_3v_4 and obtain a hamiltonian cycle in G, a contradiction. In case $R \cap \mathfrak{h} = v_3v_3'v_2'v_2v_1v_1'v_4'v_4$ or $v_3v_3'v_4'v_1'v_2'v_2v_1v_4$, replacing this path with $v_3v_2v_1v_4$ we obtain a contradiction. Lastly, if $R \cap \mathfrak{h} = v_1v_2v_2'v_1'v_4'v_3'v_3v_4$, we replace this path with $v_1v_2v_3v_4$, and once more we have a contradiction. □

Let $\overline{\mathcal{H}}_n$ be the family of all planar hypohamiltonian graphs of order n, and $V_3(G)$ the set of all cubic vertices in a graph G.

Theorem 4.

$$\frac{\min_{G \in \overline{\mathcal{H}}_n} |V_3(G)|}{n} \to 0 \quad as \quad n \to \infty.$$

Proof. Consider a planar hypohamiltonian graph containing a cubic 4-face, e.g. the (cubic) 70-vertex graph Γ of Araya and Wiener [33]. Applying k times the operation described in Lemma 5 to Γ, we obtain a graph of order $70 + 4k$ containing exactly 70 cubic vertices for all k. $\square$

By Euler's formula, in a planar hypohamiltonian graph of order n and girth 5, at least $\frac{2n+20}{3}$ vertices are cubic. So for girth 5, asymptotically the ratio of the minimum number of cubic vertices to the order of the graph is 2/3, in contrast to Theorem 4.

In [16], Jooyandeh et al. constructed the unique smallest planar hypohamiltonian graph of girth 5. It has 45 vertices, of which five have degree 4 and the remaining 40 are cubic.

We now investigate the application of the operation Th in the inverse direction.

Theorem 5. *A smallest planar hypohamiltonian graph G does not contain the graph shown on the right-hand side of Fig. 3 as an induced subgraph if v_1', v_2', v_3', v_4', and at least one of the vertices v_1, v_2, v_3, v_4 are cubic in G. The assertion also holds in the family of all planar cubic hypohamiltonian graphs. It also holds if v_1v_2 or v_3v_4 (possibly both) lie in G.*

Proof. Let G' be a smallest planar hypohamiltonian graph, and assume G' satisfies the degree conditions stipulated above but also contains the graph from the right-hand side of Fig. 3 as an induced subgraph. W.l.o.g. let v_1 be cubic. Removing from G' the vertices v_1', v_2', v_3', v_4' and adding the edges v_1v_2, v_3v_4, we obtain the graph G. We treat $G - \{v_1v_2, v_3v_4\}$ as a subgraph of G'. We denote the cycle $v_1v_2v_3v_4$ as C. Note that $\text{Th}(G_C) = G'$, but we here only know that v_1 is cubic—we shall see that this suffices for our present goals.

Assume G contains a hamiltonian cycle $\mathfrak{h}$. If v_1v_2 does not lie in $\mathfrak{h}$, then $v_1v_4 \in E(\mathfrak{h})$, as v_1 is cubic. Replacing it with $v_1v_1'v_2'v_3'v_4'v_4$ yields a hamiltonian cycle in G', a contradiction. If $v_1v_2 \in E(\mathfrak{h})$, then replace v_1v_2 with $v_1v_1'v_4'v_3'v_2'v_2$, and we obtain once again a contradiction. Hence G is non-hamiltonian.

Let $v \in V(G)$. Denote by $\mathfrak{h}_v$ the hamiltonian cycle in $G' - v$. As above, we verify that for every way $\mathfrak{h}_v$ traverses $G'[V(C) \cup \{v_1', v_2', v_3', v_4'\}]$, we can modify $\mathfrak{h}_v$ in such a manner that it provides a hamiltonian cycle in $G - v$. We have shown that G is hypohamiltonian—a contradiction, since G is smaller than G'.

Let G' be defined as above, but now additionally containing the edge v_1v_2. If v_1 is cubic in G', by removing v_4 from G' a non-hamiltonian graph is obtained (this follows from the fact that v_1 and v_4' are 2-valent in $G' - v_4$), contradicting the hypohamiltonicity of G'. If v_3 is cubic, removing from G' the vertices v_1', v_2', v_3', v_4' and adding the edge v_3v_4, we obtain the graph G. We can follow the above proof starting from the second paragraph (C again denoting the cycle $v_1v_2v_3v_4$, and taking into account that now v_3 is cubic, but v_1 possibly not) and obtain the statement. Very similar arguments can be used if v_3v_4 is an edge G', or if both v_1v_2 and v_3v_4 lie in G'. $\square$

Contrasting Theorem 5, in [36] the author and Zamfirescu recently settled affirmatively Chvátal's 1973 question [8] whether any graph can occur as the subgraph of some hypohamiltonian graph. In [36] it is also shown that for any outerplanar graph G—a graph is *outerplanar* if it possesses a planar embedding in which every vertex belongs to the unbounded face—, there exists a planar hypohamiltonian graph containing G as an induced subgraph. However, we cannot include all planar graphs due to an elegant argument of Thomassen, who proves in [27] that by a classical theorem of Whitney, a planar triangulation greater than a triangle cannot be an induced subgraph of any planar hypohamiltonian graph. At this point, no characterisation of those planar graphs which occur as induced subgraphs of planar hypohamiltonian graphs exists.

Theorem 5 did not use the imposed planarity in its proof, but since we know that the Petersen graph (which is the smallest hypohamiltonian graph) does not contain the graph from Fig. 3, this is a moot point. A strengthening of Theorem 5 to "a smallest planar hypohamiltonian does not contain a quadrilateral containing only cubic vertices" might be true. McKay [19] showed that a smallest planar cubic hypohamiltonian graph of girth 5 has order 76, and Araya and Wiener [4] proved that there exist planar cubic hypohamiltonian graphs of order 70. Since a cubic hypohamiltonian graph cannot have girth 3, a smallest planar cubic hypohamiltonian graph has girth 4. Thus, the statement "a smallest planar cubic hypohamiltonian does not contain a quadrilateral (containing only cubic vertices)" is certainly false.

Let us characterise 3-fragments of hypohamiltonian graphs. In a polyhedron G embedded in the plane, we call vertices *cofacial* if they lie on the same face. Following Chvátal [8], in a graph G a pair of vertices (x, y) is called *good in* G if there exists a hamiltonian xy-path in G. H is a non-trivial 3-fragment of a hypohamiltonian graph iff there exist three pairwise non-adjacent vertices $x_1, x_2, x_3 \in V(H)$ such that

(1) for every $v \in V(H)$ there exist i, j with $i \neq j$ s.t. (x_i, x_j) is good in $H - v$ and
(2) (x_i, x_j) is not good in H for every i, j with $i \neq j$.

We now focus on non-trivial (i.e. $\neq K_{1,3}$) 3-fragments of planar hypohamiltonian graphs, which we will call *hypo-fragments* in the following. For a characterisation of such fragments, we need to add to (1) and (2) the condition that x_1, x_2, x_3 are cofacial. Denote an arbitrary but fixed hypo-fragment by F, and its set of vertices of attachment by $X = \{x, y, z\}$. By Lemma 3, $E(F[X]) = \emptyset$. Adding to F the edges xy, yz, zx, we obtain a graph B which we will call a *closed hypo-fragment*. By [7, Lemma 4], B is a polyhedron, and in the embedding of B—we write "the", since it is unique due to a result of Whitney [30]—, xyz is a facial triangle. B contains at least one triangle, namely $B[X] = xyz$, so it has girth 3. Every vertex of a triangle in B has degree at least 4: for the vertices in X, this follows from Lemma 4, and for all other triangles this follows from Lemma 3.

F may be hamiltonian or non-hamiltonian, but due to (1), every closed hypo-fragment is hamiltonian. Identifying, using a bijection, the leaves of $K_{1,3}$ with the vertices of attachment of a hypo-fragment F yields a planar hypohamiltonian graph iff F is hamiltonian, and a planar almost hypohamiltonian graph with a cubic exceptional vertex iff F is non-hamiltonian. Goedgebeur and the author showed [11, 12] that the smallest planar hypohamiltonian (planar almost hypohamiltonian) graph has order at least 23 (22). Subsequently, Goedgebeur verified (personal communication) that there exist no planar almost hypohamiltonian graphs with a cubic exceptional vertex and of order 22. In consequence, the smallest hypo-fragment has order at least 22. The smallest known planar hypohamiltonian graphs have order 40, see [16]. Thus, gluing two hypo-fragments (using Lemma 2) cannot help improve this bound, or in other words:

Theorem 6 (Goedgebeur and Zamfirescu)**.** *Every planar hypohamiltonian graph on at most* 40 *vertices is essentially* 4*-connected.*

The smallest planar hypohamiltonian graph containing a non-trivial 3-cut has order at least 41 and at most 67. A 67-vertex graph is constructed by gluing two copies of the hypo-fragment obtained when deleting the (cubic) exceptional vertex from the 36-vertex planar almost hypohamiltonian graph G presented in [12]. The graph G was independently found by Wiener [32]. The smallest hypohamiltonian graph containing a non-trivial 3-cut has order 15 (see [15, Fig. 3.2]) and is obtained by gluing two copies of the vertex-deleted Petersen graph.

A smallest planar hypohamiltonian graph has maximum degree at least 4, as Aldred et al. [3] showed that there is no planar cubic hypohamiltonian graph on 42 or fewer vertices. The girth of a smallest planar hypohamiltonian graph is 3 or 4, since the smallest planar hypohamiltonian graph of girth 5 has order 45, as proven by Jooyandeh et al. [16].

With the same idea as in the proof of Corollary 2, we obtain the following.

Proposition 1. *Every hypo-fragment contains, among its vertices which are not vertices of attachment, at least two cubic vertices. In consequence, a hypohamiltonian graph containing a hypo-fragment has at least two vertices which are cubic.*

A brief remark on the old question of Thomassen whether hypohamiltonian graphs with minimum degree at least 4 exist [26]: From Theorem 1 we can conclude that such a graph cannot be planar. From Proposition 1 it follows that every (necessarily non-trivial) 3-fragment of such a graph must be non-planar.

Let $\mathcal{F}_h$ ($\mathcal{F}_{nh}$) be the set of all hamiltonian (non-hamiltonian) hypo-fragments, put $\mathcal{F} = \mathcal{F}_h \dot\cup \mathcal{F}_{nh}$, and denote the gluing of $F, F' \in \mathcal{F}$ (as in Lemma 2) with $F \vdots F'$, which is a planar hypohamiltonian graph. In fact, by Theorem 1, *every* planar hypohamiltonian graph can be obtained as $F \vdots K_{1,3}$ for the appropriate $F \in \mathcal{F}_h$, i.e.:

Proposition 2 (Thomassen)**.** *Every planar hypohamiltonian graph contains a hamiltonian hypo-fragment.*

If $F \in \mathcal{F}_{nh}$, then $F \vdots K_{1,3}$ is an almost hypohamiltonian graph—these will be discussed in the next section. We also point out the following direct corollary of Theorem 6.

Corollary 3. *In every planar hypohamiltonian graph on at most* 40 *vertices, every hypo-fragment is hamiltonian.*

Despite the fact that removing a 3-cut from a hypohamiltonian graph will always yield exactly two components, more than two 3-fragments can be combined simultaneously in order to obtain a hypohamiltonian graph. This is a generalisation of

Thomassen's [26, Corollary 1] in a different direction than Lemma 2, but its proof relies on the same idea—we therefore leave it to the reader.

Proposition 3. *Let $F_0, ..., F_{k-1}$, $k \geq 2$, be pairwise disjoint* 3*-fragments of hypohamiltonian graphs. Denote the vertices of attachment of F_i with x_{i1}, x_{i2}, x_{i3}. In $\bigcup_{i=0}^{k-1} F_i$, identify all x_{i2}, and identify each x_{i3} with $x_{i+1,1}$, indices taken* mod. k. *The resulting graph Γ is hypohamiltonian and of order $\sum_{i=0}^{k-1} |V(F_i)| - 2k + 1$. If every F_i is planar, the identification can be performed such that Γ is planar, as well.*

This shows that the maximum degree of a planar hypohamiltonian graph can become arbitrarily large. (Thomassen [27] proved that there exist hypohamiltonian graphs of order n and maximum degree $\frac{1}{2}n - 44$. Herz, Duby, and Vigué [14] showed that the maximum degree is less than $\frac{1}{2}(n-3)$.) It also shows that in planar hypohamiltonian graphs, the ratio of the minimum number of cubic vertices (which increases linearly) to the maximum number of 3-cuts (which increases quadratically) vanishes asymptotically.

In the following, in the spirit of Corollary 2, our aim is to show that the minimum number of cubic vertices in an essentially 4-connected planar hypohamiltonian can be increased beyond what Theorem 3 gives us under certain circumstances—recall that by Theorem 6, a smallest planar hypohamiltonian graph is essentially 4-connected. Apart from this, we believe that Theorem 8 is in itself an interesting observation concerning the hamiltonicity of essentially 4-connected planar graphs.

Consider a polyhedron G embedded in the plane, let $v \in V(G)$, and denote with $F_1, ..., F_k$ the faces incident with v. We call $\bigcup F_i$ a *v-patch* if, except for two cofacial neighbours u, w of v, every vertex in $N(v)$ has degree at least 4. We emphasise that u, v, w may have any degree which is at least 3.

The set of cubic vertices $V_3(G)$ of G is a *non-obstructive k-set* if $V_3(G)$ consists of $k-1$ (cubic) vertices S' and (i) $V_3(G) \setminus S'$ is a cubic ℓ-path, $\ell \leq 5$, or (ii) $V_3(G) \setminus S'$ lies in a v-patch for some $v \in V(G)$. (We have chosen this notation since, when counting cubic vertices, the set $V_3(G) \setminus S'$ behaves like one additional cubic vertex when dealing with hamiltonicity—this will become evident in the following paragraphs.) We will appeal frequently to [7, Theorem 12] by Brinkmann and the author, of which we now state only the part being used in the present article. For its full form, see [7].

Theorem 7 (Brinkmann and Zamfirescu [7])**.** *Let G be an essentially* 4*-connected planar graph and F a face of G containing a vertex v of degree at least* 4 *with neighbouring edges uv, vw in the boundary of F. Furthermore, assume that except for u, w—which can have any degree of at least* 3*—there are at most two cubic vertices in G. Then there is a hamiltonian cycle of G containing uvw.*

Theorem 8. *Essentially* 4*-connected planar graphs whose set of cubic vertices is a non-obstructive* 3*-set are hamiltonian.*

Proof. Let G be a graph satisfying the theorem's conditions. We give the proof depending on whether we are dealing with a cubic path or a patch, i.e. case (i) or (ii) as defined above.

(i) Let $P = v_1...v_5$ be a cubic 5-path in G. We now treat the three essentially different situations.

Case 1: There exists a face F in G such that $V(P) \subset V(F)$. We construct the graph G' by adding to G a vertex v in the interior of F and joining it to all vertices in $V(F)$. Theorem 7 yields that there exists a hamiltonian cycle in G' containing v_1vv_2. Replacing v_1vv_2 with v_1v_2, we obtain a hamiltonian cycle in G.

Case 2: No four vertices among $v_1, ..., v_5$ are cofacial. Construct G' by adding to G the edges v_1v_3, v_2v_4, and v_3v_5. Note that G' is planar. Theorem 7 implies that there exists a hamiltonian cycle in G' containing $v_2v_3v_4$. Since the edges v_1v_3, v_2v_4, and v_3v_5 are not contained in this cycle, we have shown that G is hamiltonian, as well.

Case 3: v_1, v_2, v_3, v_4 are cofacial, but $v_1, ..., v_5$ are not. We separate between two cases: In Subcase 3.1, v_1 and v_4 are non-adjacent. Denote the neighbour of v_4 not in P with w. v_1, v_2, v_3, v_4, and w lie on the same face F. We construct the graph G_v by adding to G a vertex v in F and joining it to all vertices in $V(F)$. Furthermore, we denote with G' the graph obtained from G_v by removing the edge v_4w and adding the edge vv_5. G_v and G' are planar. It is well-known that G_v is 3-connected, since $|V(F)| \geq 3$. It remains to prove that G' is 3-connected. For every $x, y \in V(G_v) \setminus \{v\}$, $x \neq y$, by Menger's Theorem there exist three pairwise internally disjoint xy-paths in G—two paths are *internally disjoint* if the intersection of their vertex sets coincides with their end-vertices—, the union of which we will call P. If $v_4w \notin E(P)$, we are done, so assume $v_4w \in E(P)$. We replace in P the edge v_4w with v_4vw and obtain the desired three paths in G'. Now we show that there exist three pairwise internally disjoint vz-paths in G' for every $z \in V(G') \setminus \{v\}$. Since G_v is 3-connected, there exist three pairwise internally disjoint vz-paths in G_v. Assume one of these paths, which we call Q, uses v_4w. If, on Q, v_4 (w) is closer to z than w (v_4), then we replace the subpath of Q from v to v_4 (w) with vv_4 (vw), and are finished. It is now not difficult to show that G' is in fact essentially 4-connected. We apply Theorem 7 and obtain a hamiltonian cycle in G' using v_4vv_5. Replacing v_4vv_5 with v_4v_5 we have shown that G is hamiltonian.

In Subcase 3.2, v_1 and v_4 are adjacent. In this situation, $v_1v_2v_3v_4$ is a quadrilateral face F of G. Furthermore, v_1, v_4, v_5 lie on the same face F'. We construct the planar graph G' by adding to G a vertex v in F and joining it to all vertices in $V(F)$, adding a vertex v' in F' and joining it to v_1, v, v_4, v_5, and removing the edge v_1v_4. As above we show that G' is essentially 4-connected. We apply Theorem 7 and obtain a hamiltonian cycle $\mathfrak{h}$ of G' containing $v'vv_4$. Clearly $v'v_4 \notin E(\mathfrak{h})$. If $v_1v'vv_4 \subset \mathfrak{h}$ $(v_5v'vv_4 \subset \mathfrak{h})$, then we replace $v_1v'vv_4$ $(v_5v'vv_4)$ with v_1v_4 (v_5v_4) and obtain a hamiltonian cycle in G.

If P is a cubic k-path with $k \leq 3$, all vertices of P are cofacial and we are in the situation of Case 1. If P is a cubic 4-path and not all vertices of P are cofacial (if they are, we are in Case 1), we use the argument from Case 2.

(ii) Denote with F the face containing u, v, and w. We construct a graph G' by adding to G as many edges emanating from v as possible, but excluding vertices in $V(F)$, such that no multiple edges occur and G' is planar. We call the set of added edges $\mathcal{E}$. If v is not cubic in G we use Theorem 7 and obtain that there exists a hamiltonian cycle in G' containing uvw. The same hamiltonian cycle exists in $G' - \mathcal{E} = G$, and we are done. Now assume v is cubic in G. We denote its neighbour which is neither u nor w by x. If $G[\{x, v, w\}]$ and $G[\{x, u, v\}]$ are not both triangles, we are finished as well, so suppose both are triangles. Neither u nor w are cubic in G, as otherwise we would have a non-trivial 3-cut. But in this case we may apply directly (i.e. without modifying G) Theorem 7 and obtain that there exists a hamiltonian cycle in G. $\square$

From Theorem 8 we can directly deduce that an essentially 4-connected planar graph whose cubic vertices are a non-obstructive 3-set cannot be hypohamiltonian. For example, let G be an essentially 4-connected planar graph. If $V_3(G)$ consists of seven vertices, five of which induce a path, then G cannot be hypohamiltonian.

We end this section with a brief discussion of a natural question raised by Fabrici and Madaras during the Cycles and Colourings Workshop in 2011: does every planar hypohamiltonian graph contain *adjacent* cubic vertices? Note that the graph from Fig. 2 contains a cubic vertex v (the bottom right vertex) with no cubic neighbours—vertices such as v have the property that removing them from a hypohamiltonian polyhedron yields a (hamiltonian) polyhedron, whence, they are not contained in any 3-cut.

Let the *weight* of an edge be the sum of the degrees of its end-vertices. In his talk entitled "Local structure of planar hypohamiltonian graphs" held at the "Kolloquium über Kombinatorik" in Ilmenau (Germany) in 2013, Fabrici presented the result that every planar hypohamiltonian graph contains an edge of weight at most 9. From a theorem of Lebesgue [17] it follows that every planar hypohamiltonian graph of girth at least 4 contains an edge of weight at most 8, and that planar hypohamiltonian graphs of girth 5 contain a cubic 4-path.

Let μ be the minimum number of pairs of adjacent cubic vertices in a planar hypohamiltonian graph. Assume $\mu = 1$, and let G be a planar hypohamiltonian graph with exactly one pair of adjacent cubic vertices v, w. Let H be the non-trivial 3-fragment of G whose set of vertices of attachment is $N(v)$. Gluing two copies of H via Lemma 2, we obtain a planar hypohamiltonian graph with no adjacent cubic vertices, so $\mu = 0$, a contradiction. Thus $\mu \neq 1$.

3 Planar almost hypohamiltonian graphs

It is not difficult to prove that there exist planar 1-hamiltonian graphs with minimum degree 4—think of triangulations—, so we cannot replace in Theorem 1 the condition of non-hamiltonicity with hamiltonicity. Thomassen's question whether hypohamiltonian graphs with minimum degree at least 4 exist remains open [26], so we do not know whether the condition of planarity can be dropped in Theorem 1. We *do* know that almost hypohamiltonian graphs with minimum degree 4 (and even 4-connected such graphs) exist [34]. Problem 5 in [34] asks whether every planar almost hypohamiltonian graph contains a cubic vertex. If the exceptional vertex of the graph is cubic, the following stronger statement containing the affirmative answer is a direct corollary of Theorem 3. Its proof, which we leave to the reader, is based on Lemma 2 and Proposition 1, and uses the same idea of gluing 3-fragments as we have already seen.

Proposition 4. *Let G be a planar almost hypohamiltonian graph. If the exceptional vertex w of G is cubic, and $N(w)$ contains k cubic vertices, then G contains at least $k + 3$ cubic vertices.*

We now treat the case when the exceptional vertex is not cubic. We will use tacitly the fact that in a graph of connectivity 3, every vertex of any 3-cut has at least one neighbour which is not a vertex of attachment, in every 3-fragment. Despite what was said concerning Lemma 4, we do have the following.

Lemma 6. *Let G be an almost hypohamiltonian graph with exceptional vertex w and H a 3-fragment in G with set of vertices of attachment A_H. If $w \notin A_H$, H is non-trivial, and either $w \notin V(H)$ or $w \in V(H)$ has no neighbour in A_H, then every vertex in A_H has at least two neighbours in $H \setminus A_H$.*

If $w \in A_H$, we put $A_H = \{u, v, w\}$, and the following hold.

(i) *$uv \notin E(G)$.*

(ii) *If H is non-trivial, then u and v each have at least two neighbours in $H \setminus A_H$.*

(iii) *If w has exactly one neighbour in H (which must lie in $H \setminus A_H$), then there is no hamiltonian uv-path in $H - w$.*

(iv) *If $vw \in E(G)$ and $uw \notin E(G)$, then $G + uw$ is almost hypohamiltonian, as well.*

Proof. Denote by $H' \neq H$ the other 3-fragment in G whose set of vertices of attachment is A_H. For the first statement, we argue as in the proof of Lemma 4.

(i) Assume $uv \in E(G)$. Since $G-u$ and $G-v$ are hamiltonian, there is a hamiltonian vw-path (hamiltonian uw-path) in $H - u$ ($H' - v$). These paths together with the edge uv give a hamiltonian cycle in G, a contradiction.

(ii) Assume v has exactly one neighbour in $H \setminus A_H$ which we call z. $G - z$ is hamiltonian, so there exists a hamiltonian uw-path in H'. Furthermore, there is a hamiltonian uw-path in $H - v$. The union of these paths is a hamiltonian cycle in G, a contradiction.

(iii) Denote by z' the neighbour of w in $H \setminus A_H$. Since $G - z'$ is hamiltonian, there exists a hamiltonian uv-path in H'. Since $G - w$ is non-hamiltonian, there cannot be a hamiltonian uv-path in $H - w$.

(iv) Assume $G + uw$ contains a hamiltonian cycle $\mathfrak{h}$. Since G is non-hamiltonian, $uw \in E(\mathfrak{h})$. W.l.o.g. $H' - w$ contains a hamiltonian uv-path. Since $G - v$ is hamiltonian, there exists a hamiltonian uw-path in $H - v$. These paths together with the edge vw provide a hamiltonian cycle in G, a contradiction. We also need to verify that $G + uw - w$ is non-hamiltonian, but this is certainly the case, since $G + uw - w = G - w$ and $G - w$ is non-hamiltonian. □

Proposition 5. *Let G_1 and G_2 be disjoint almost hypohamiltonian graphs. For $i \in \{1, 2\}$, let G_i have exceptional vertex w_i, and let there be a non-trivial 3-fragment H_i in G_i with set of vertices of attachment X_i such that $w_i \in X_i$ and $E(G_i[X_i]) \neq \emptyset$. In $G_1 \cup G_2$, identifying X_1 with X_2 using a bijection such that w_1 is identified with w_2, and deleting all but one edge between adjacent vertices, we obtain an almost hypohamiltonian graph Γ whose exceptional vertex is the vertex obtained when identifying w_1 with w_2. If G_1 and G_2 are planar, then so is Γ.*

Proof. Let $X_i = \{x_{i1}, w_i, x_{i2}\}$, and denote by J_i the 3-fragment in G_i which is not H_i and whose set of vertices of attachment is X_i. We call $x_j \in V(\Gamma)$ the vertex obtained when identifying x_{1j} with x_{2j}, and w the vertex obtained when identifying w_1 and w_2.

Claim. *There is no hamiltonian $x_{i1}x_{i2}$-path in $H_i - w_i$.*

Proof of the Claim. Assume there is such a path $\mathfrak{p}$. As $E(G_i[X_i]) \neq \emptyset$, due to Lemma 6 (i), x_1w or x_2w lie in $E(\Gamma)$, say the former. Since G_i is almost hypohamiltonian, there exists a hamiltonian path $\mathfrak{p}'$ in $J_i - x_{i1}$ with end-vertices w_i, x_{i2}. $\mathfrak{p} \cup \mathfrak{p}' + x_{i1}w_i$ is a hamiltonian cycle in G_i, a contradiction.

Assume Γ is hamiltonian. Treating H_i as a subgraph of Γ, for appropriate i there exists a hamiltonian x_1w-path in H_i, which we deal with as in the proof of Lemma 2, or a hamiltonian x_1x_2-path; in this case $H_j - w_j$, $i \neq j$, contains a hamiltonian $x_{j1}x_{j2}$-path. But this is impossible by the above claim.

Since $G_i - x_{ij}$ is hamiltonian, there exists a hamiltonian $x_{ik}w_i$-path $\mathfrak{p}_{ij}$ in $H_i - x_{ij}$, where $k \neq j$, for all i, j. Then $\mathfrak{p}_{1i} \cup \mathfrak{p}_{2i}$ gives a hamiltonian cycle in $\Gamma - x_i$. Consider $v \in V(H_1) \setminus X_1$. Using the claim, we have a hamiltonian w_1x_{1i}-path $\mathfrak{p}$ in $H_1 - v$ for appropriate i. (Note that x_{1j}, $i \neq j$, necessarily lies in $\mathfrak{p}$. Assume it does not, and consider the hamiltonian cycle $\mathfrak{h}$ in $G_1 - v$ which contains $\mathfrak{p}$. $\mathfrak{h} \cap J_1$ is a hamiltonian path in J_1 with the same end-vertices as $\mathfrak{p}$, i.e. w_1, x_{1i}, but containing the vertex x_{1j}, $i \neq j$. We then have that $(\mathfrak{h} \cap J_1) \cup \mathfrak{p}_{1j}$, $i \neq j$, is a hamiltonian cycle in G_1, a contradiction.) Then $\mathfrak{p} \cup \mathfrak{p}_{2j}$, $i \neq j$, gives the desired hamiltonian cycle in $\Gamma - v$. The same argument yields a hamiltonian cycle in $\Gamma - v'$ for $v' \in V(H_2) \setminus X_2$.

Due to the claim, $\Gamma - w$ cannot be hamiltonian. □

In the above proposition, the condition "$E(G_i[X_i]) \neq \emptyset$" is annoying, but we were not able to prove the statement without it. This is due to the fact that the statement becomes false without the requirement: in [12, Fig. 3] an almost hypohamiltonian graph is shown; it has a non-trivial 3-cut X containing the exceptional vertex. Gluing two copies of the smaller of the two 3-fragments whose set of vertices of attachment is X such that the exceptional vertices are identified with each other, we obtain a graph with minimum degree 2, which cannot be almost hypohamiltonian.

In the following, we need [7, Lemma 14]—we reproduce here only the part of it relevant to our current pursuit.

Lemma 7 (Brinkmann and Zamfirescu [7])**.** *Consider a polyhedron G with at most one 3-cut, and let G contain a triangular face xyz with* $\deg(y) \geq 4$. *Then there is a hamiltonian xz-path in G containing no edges of xyz.*

Theorem 9. *Every planar almost hypohamiltonian graph contains a cubic vertex which is not the exceptional vertex.*

Proof. In the light of Proposition 4, let G be a planar almost hypohamiltonian graph with exceptional vertex w, and assume that G has minimum degree at least 4.

If there exists a 3-cut X in G such that $w \notin X$, denote by H the 3-fragment (which is necessarily non-trivial, since there are no trivial 3-cuts) whose set of vertices of attachment is X and which does not contain w. Applying Lemma 2 to two copies of H yields a planar hypohamiltonian graph containing no cubic vertices, contradicting Theorem 1.

So we may assume that every 3-cut in G contains w. [7, Lemma 5] states that G has a 3-cut $Y = \{u, v, w\}$ such that at least one closed hypo-fragment B of G whose set of vertices of attachment is Y has no 3-cuts, i.e. that it is 4-connected or isomorphic to K_4. The latter case is impossible here, since G contains no cubic vertices. As $w \in Y$, Lemma 6 (ii) implies that u has degree at least 4 in B. By Lemma 7, there is a hamiltonian vw-path $\mathfrak{p}$ in B containing no edge in $E(B[Y])$. Let H' be the 3-fragment $\neq H$ in G whose set of vertices of attachment is Y. Since G is almost hypohamiltonian and u is non-exceptional, there exists a hamiltonian vw-path $\mathfrak{p}'$ in $H' - u$. We are led to a contradiction, since $\mathfrak{p} \cup \mathfrak{p}'$ is a hamiltonian cycle in G. □

We have obtained the second strengthening of Thomassen's Theorem 1:

Corollary 4. *Let G be a planar non-hamiltonian graph containing a vertex w such that $G - v$ is hamiltonian for every $v \in V(G) \setminus \{w\}$. Then G contains a cubic vertex which is not w.*

The smallest known planar almost hypohamiltonian graph has order 31 and was discovered by Wiener [32]. Its exceptional vertex has degree 4. Goedgebeur and the author [12] showed that the smallest planar almost hypohamiltonian graph has order at least 22.

Corollary 5. *Every planar almost hypohamiltonian graph contains a hamiltonian 3-fragment.*

A graph G is *hypotraceable* if G is non-traceable (i.e. contains no hamiltonian path) and $G - v$ is traceable for every $v \in V(G)$. The following result of Wiener together with Corollary 4 provides a first step in establishing an analogue of Theorem 1 for hypotraceable graphs. Note that every 2-edge-cut in a hypotraceable graph consists of two independent edges.

Theorem 10 (Wiener [32])**.** *If G is a hypotraceable 2-fragment containing an edge-cut $\{e_1, e_2\}$, and G_1 and G_2 are the components of $G - \{e_1, e_2\}$, then G_1 and G_2 are vertex-deleted hypohamiltonian or almost hypohamiltonian graphs. In both cases the deleted vertex must be cubic, and in the latter case the exceptional vertex is being deleted.*

Corollary 6. *Each 2-fragment of edge-connectivity 2 in a planar hypotraceable graph contains a cubic vertex.*

Let us briefly discuss 2-hypohamiltonian graphs, since the behaviour with respect to cubic vertices changes from the 1-hypohamiltonian to the 2-hypohamiltonian case. As described in [34], it is easy to see that there exists a planar 2-hypohamiltonian graph with no cubic vertex: take a 4-cycle $v_1v_2v_3v_4$, add the vertex v_5, and the edges v_1v_3, v_1v_5, and v_3v_5. Thus, a 2-hypohamiltonian analogue of Theorem 9 would certainly be false if we would not demand 3-connectedness. Whether every 2-hypohamiltonian polyhedron contains a cubic vertex is unknown. In a similar vein, it would be interesting to determine a small integer k such that there exists a k-hypohamiltonian polyhedron of minimum degree at least 4.

4 Hypohamiltonian graphs with crossing number 1

An informal definition of the crossing number of a graph G follows. Consider all drawings of G in the plane, and choose one with the fewest edge crossings. The number of crossings in this drawing is the *crossing number* $\mathrm{cr}(G)$ of G. For a rigorous definition and a survey, see Székely's work [23].

Lemma 8 (Brinkmann [5]). *Let G be a 4-connected graph with crossing number 1. Denote by e and e' the crossing edges. Then $G - \{e, e'\}$ is hamiltonian.*

Proof. Let G be a 4-connected graph with crossing number 1. We denote its crossing edges by aa' and cc'. We apply the transformation shown in Fig. 4 and obtain a planar graph G'.

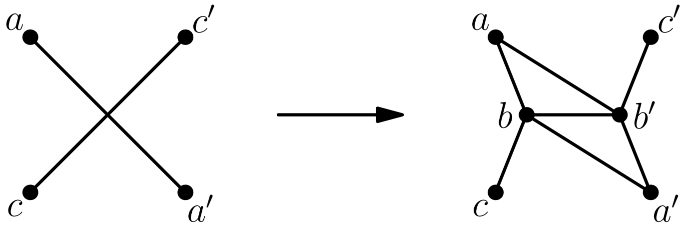

Fig. 4: Transforming a crossing.

Let us show that G' is 4-connected by using Menger's Theorem and proving that for every pair of vertices of G' there exist four pairwise internally disjoint paths between them. Let $v, w \in V(G') \setminus \{b, b'\}$. We see $G - \{aa', cc'\}$ as subgraph of G'. Denote the union of four pairwise internally disjoint vw-paths in G with P. If $aa', cc' \notin P$, we are done. If $aa' \in P$ and $cc' \notin P$, replace in P the edge aa' with aba', and we have four pairwise disjoint vw-paths in G'. (We proceed in the same manner if $aa' \notin P$ and $cc' \in P$.) If $aa', cc' \in P$, replace aa' and cc' with abc and $a'b'c'$ or $ab'c'$ and $a'bc$ (depending on P), respectively, and we are finished as well.

We now show that there are four pairwise internally disjoint bb'-paths in G'. Three such paths are given by bb', bab', and $ba'b'$. Since G is 4-connected, there exists a cc'-path Q in G which is not cc' and which uses neither a nor a'. Treating Q as lying in G', we obtain our fourth bb'-path by considering $bc + Q + c'b'$.

Finally, let $v \in V(G') \setminus \{b, b'\}$. We prove that there are four pairwise internally disjoint vb-paths in G'. It is well-known that for a vertex $\beta \notin V(G)$, the graph $G'' = (V(G) \cup \{\beta\}, E(G) \cup \{a\beta, c\beta, a'\beta, c'\beta\})$ is 4-connected, as well. There exist four pairwise internally disjoint $v\beta$-paths P_a, P_c, $P_{a'}$, $P_{c'}$ in G'', where the index denotes the vertex through which β is reached. Then

$$(P_a - a\beta) + ab,\ (P_c - c\beta) + cb,\ (P_{a'} - a'\beta) + a'b,\ (P_{c'} - c'\beta) \cup c'b'b$$

are the four desired paths in G'. (The same argument can be used to prove that there are four pairwise internally disjoint vb'-paths in G'.)

Thomas and Yu [24] showed that 4-connected polyhedra are 2-hamiltonian. Thus $G' - \{b, b'\} = G - \{aa', cc'\}$ is hamiltonian. □

Using the same approach, and applying [7, Theorem 12], we can obtain the following result which is stronger than Lemma 8. However, we skip here the proof, as Lemma 8 is sufficient for our purposes.

Theorem 11 (Brinkmann [5]). *Let G be a 4-connected graph with crossing number 1. Denote by e and e' the crossing edges. Then G contains a hamiltonian cycle through* *neither e nor e', a hamiltonian cycle through e but not e', as well as a hamiltonian cycle through e' but not e. Consequently, G contains at least three hamiltonian cycles.*

Lemma 9. *Consider a hypohamiltonian graph G with crossing number 1 containing a 3-cut X. Let H and H' be the 3-fragments whose set of vertices of attachment is X. Then H or H' is planar.*

Proof. Due to Lemma 3, H and H' are edge-disjoint induced subgraphs of G. Assume H and H' are non-planar. By Kuratowski's Theorem, each contains a Kuratowski subgraph, i.e. a subgraph homeomorphic to $K_{3,3}$ or K_5. Thus G contains at least two edge-disjoint Kuratowski subgraphs. G has crossing number 1, so we can delete one edge from G such that the graph G' we obtain is planar. In at least one of the Kuratowski subgraphs no edge was deleted—but this contradicts the planarity of G'. Therefore H or H' must be planar. □

We now present the third extension of Theorem 1.

Theorem 12. *Every hypohamiltonian graph with crossing number at most 1 contains a cubic vertex.*

Proof. For graphs with crossing number 0, this is Thomassen's Theorem 1. Let G be a hypohamiltonian graph with crossing number 1 and minimum degree at least 4. By Lemma 8, G has connectivity 3. By Lemma 9, one of the (necessarily non-trivial) 3-fragments of G, which we call H, is planar. Consider H and glue it with a copy of itself as described in Lemma 2. We obtain a planar hypohamiltonian graph with no cubic vertices, in contradiction with Theorem 1. □

In the light of above arguments, extending Tutte's Theorem to 4-connected graphs with crossing number 2 seems within reach. However, Grünbaum and Nash-Williams independently conjectured that every 4-connected toroidal graph is hamiltonian—this has proven to be a very challenging problem—, and on the double torus there even exist 4-connected triangulations that are non-hamiltonian.

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