Grinberg's Criterion

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Abstract. We generalize Grinberg's hamiltonicity criterion for planar graphs. To this end, we first prove a technical theorem for embedded graphs. As a special case of a corollary of this theorem we obtain Zaks' extension of Grinberg's Criterion (which encompasses earlier work of Gehner and Shimamoto), but the result also implies Grinberg's formula in its original form in a much broader context. Further implications are a short proof for a slightly strengthened criterion of Lewis bounding the length of a shortest closed walk from below as well as a generalization of a theorem due to Bondy and Häggkvist.

Keywords: Hamiltonian cycle, Grinberg's criterion, spanning subgraph

MSC: 05C10, 05C45, 05C70

1 Introduction

This paper extends what has been referred to in the literature as the Kozyrev-Grinberg method, see for instance Tutte [19], Honsberger [10, Chapter 7] or Sachs' original source [15]. However, as Honsberger writes, the mathematical discovery seems to belong to Grinberg, while Kozyrev helped to make the result well known. We therefore will use only Grinberg's name in the following.

All embeddings in this paper are assumed to be 2-cell embeddings on orientable surfaces. We assume the embeddings to be given by an orientation of the edges around each vertex. If we have a subgraph of an embedded graph, the embedding of the subgraph is induced from the orientation around the vertices, so this induced embedding can be in a surface of smaller genus. With Steinitz' Theorem in mind, we call plane 3-connected graphs *polyhedral*. In 1880 Tait conjectured that every cubic polyhedral graph is hamiltonian. Had this conjecture been true, it would have implied the Four Colour Theorem. For historical details and references, we refer to [2]. Tutte [18] was the first to give a counterexample to Tait's conjecture in 1946. In 1968 Grinberg [8] published a necessary condition — which we will refer to as Grinberg's Criterion — for a planar graph to be hamiltonian (given in detail in Section 2). Since its inception, it has turned out to be one of the few powerful tools at our disposal to ascertain that a given planar graph is non-hamiltonian. Following

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Bondy and Murty [4], Kirkman [12] was aware of the identity behind the criterion already in 1881, but Kirkman used it to search for hamiltonian cycles in certain cubic polyhedral graphs, of which he was convinced that they are hamiltonian.

For classic sources on the criterion, consult [1, 10], and for examples of recent related work, see [5, 11, 13, 21]. Grinberg himself used it to construct cubic polyhedral graphs that provide smaller (than Tutte's) counterexamples to Tait's conjecture. The non-hamiltonicity of Tutte's graph can also be proven with the criterion, see Honsberger's account of a proof due to Watts [10, pp. 86–88] or West [20, pp. 303–304]. West provides in his book further applications of the criterion, for instance details on how a similar approach to the one used for Tutte's graph leads to a short proof of the fact that the smallest counterexample to Tait's conjecture, the Lederberg-Bosák-Barnette graph on 38 vertices [9], is indeed non-hamiltonian.

Gehner [6] and Shimamoto [16] extended the criterion in two different (natural) directions, while Zaks [22] unified the two approaches. All of these deal with plane graphs. Here we extend Zaks' version of the criterion in various ways and show that even Grinberg's original equation holds under much weaker assumptions. We also discuss some applications, including a theorem of Lewis [13], as well as a result due to Bondy and Häggkvist [3].

To our knowledge, to date Grinberg's Criterion has been applied only in the context of planar graphs (see e.g. Thomassen's paper [17] or recent work of Jooyandeh et al. [11]), albeit sometimes to prove a result on non-planar graphs. Applications of the latter type include Chia and Thomassen's recent paper [5] in which a unified proof of independent results of Robertson, Bondy, Thomason, and Schwenk about the number of hamiltonian cycles in generalized Petersen graphs is given as well as an article by Wiener [21] in which he constructs the smallest known hypohamiltonian graph with crossing number 1.

2 A technical result

We recall Grinberg's theorem and then present a technical generalization of the theorem. Evaluating the cases when some of the parameters cancel out will result in corollaries that are — though just special cases of the general theorem — results that are interesting in themselves.

Theorem 2.1 Grinberg's Criterion *(Grinberg, 1968 [8]). Given a plane graph with a hamiltonian cycle S and f_k (f'_k) faces of size k inside (outside) of S, we have*

$$\sum_{k\geq 3}(k-2)(f_k - f'_k) = 0.$$

Let $G = (V, E)$ be an embedded graph and $S = (V_S, E_S)$ be a subgraph of G. Let $G_d = (V_d, E_d)$ be the dual graph of G and for $e \in E$ let e_d denote the corresponding edge in E_d. For a face f in G the corresponding vertex in G_d is f_d. Let $C_{d,1}, \ldots, C_{d,k}$ be the components of $(V_d, E_d \setminus \{e_d | e \in E_S\})$. For each such component $C_{d,i}$ we define the (embedded) component $C_i = (V_i, E_i)$ under decomposition by S with V_i, respectively E_i the set of all vertices, respectively edges, contained in a face f so that $f_d \in C_{d,i}$. The rotational order around the vertices is induced by G. Faces that correspond to a vertex $f_d \in C_{d,i}$ are called *internal* faces, the others are called *external*. The genus of an embedded graph G, that is, the genus of the surface it is embedded in, is denoted by $\gamma(G)$, and for a face f its size (that is: the length of the facial walk) is denoted by $s(f)$.

The *decomposition graph* (possibly with loops) $D_{G,S} = (V_{G,S}, E_{G,S})$ of G by S is defined as $V_{G,S} = \{C_1, \ldots, C_k\}$ and $\{C_i, C_j\} \in E_{G,S}$ if and only if there is an edge of E_S with a face of C_i on one side and a face of C_j on the other. To illustrate the construction of $D_{G,S}$, one can consider the setting of Grinberg's original theorem, that is, a plane graph G with a hamiltonian cycle S. Then there are two components C_1, C_2, both outerplanar graphs, and the decomposition graph is K_2. If G is a graph with $\gamma(G) > 0$ and S is a hamiltonian cycle in G, then there are one or two components and the decomposition graph is K_2 or a single vertex with a loop.

If $D_{G,S}$ is bipartite, we can talk about black and white components and we call S a *partitioning* subgraph. In this case each face of G and each vertex in $V \setminus V_S$ can also be called black or white respectively, depending on whether it belongs to a black or a white component. Let F_b denote the set of black faces, V_b the set of black (internal) vertices, and C_b the set of black components. For a black component C_i we define the set of *external faces* B_{C_i} as the set of faces not corresponding to a vertex in $C_{d,i}$ and we define B_b as the disjoint union of all sets of external faces in black components. For a component $C = (V_C, E_C)$ we define the *boundary deficit* $d_C = |E_C \cap E_S| - |V_C \cap V_S|$. The black boundary deficit d_b is the sum over all boundary deficits of black components. Analogously we define F_w, V_w, C_w, B_w and d_w for the white faces, vertices, components, external faces and the boundary deficit. With this notation and calling the inside white and the outside black, the equation in Grinberg's Criterion can be written in a symmetrical way as

$$\sum_{f \in F_w} (s(f) - 2) = \sum_{f \in F_b} (s(f) - 2).$$

We will call this equation *Grinberg's identity*, and deal with it and its generalizations always in this symmetrical way. We will now prove a technical theorem using the many parameters just defined. Studying the cases when the additional parameters vanish or cancel each other out allows the formulation of various interesting consequences, in particular Grinberg's original formula in a much broader context.

Theorem 2.2 *Let G be an embedded graph and $S = (V_S, E_S)$ be a partitioning subgraph of G. Then*

$$|E_S| = \sum_{f \in F_w} (s(f) - 2) - 2|V_w| + 4|C_w| - 4 \sum_{C \in C_w} \gamma(C) - 2|B_w| + 2d_w =$$
$$\sum_{f \in F_b} (s(f) - 2) - 2|V_b| + 4|C_b| - 4 \sum_{C \in C_b} \gamma(C) - 2|B_b| + 2d_b.$$

Proof: Without loss of generality, let $C = (V_C, E_C)$ be an embedded white component and F_C the set of faces of C. The Euler formula gives

$$|V_C| - |E_C| + |F_C| = |V_C| - \frac{\sum_{f \in F_C}(s(f) - 2)}{2} = 2 - 2\gamma(C).$$

F_C is the disjoint union of faces in $F_C \cap B_w$ — we denote the set of these faces as $B_{C,S}$ — and *internal* faces in $F_C \cap F_w$. We write $F_{C,i}$ for the set of all internal faces.

V_C also consists of internal vertices not in S and vertices in S. We denote these sets as $V_{C,i}$, respectively $V_{C,S}$ and analogously $E_{C,i}$ and $E_{C,S}$ for edges. With this notation we have $|V_{C,S}| = |E_{C,S}| - d_C$. We get

$$2|V_{C,i}| + 2|V_{C,S}| - \sum_{f \in F_{C,i}} (s(f) - 2) - \sum_{f \in B_{C,S}} (s(f) - 2) = 4 - 4\gamma(C)$$

$$2(|E_{C,S}| - d_C) - \sum_{f \in B_{C,S}} (s(f) - 2) = \sum_{f \in F_{C,i}} (s(f) - 2) - 2|V_{C,i}| + 4 - 4\gamma(C)$$

$$2|E_{C,S}| - \sum_{f \in B_{C,S}} s(f) = \sum_{f \in F_{C,i}} (s(f) - 2) - 2|V_{C,i}| + 4 - 4\gamma(C) - 2|B_{C,S}| + 2d_C.$$

As each edge in $E_{C,S}$ is counted for exactly one $f \in B_{C,S}$ we have $\sum_{f \in B_{C,S}} s(f) = |E_{C,S}|$ and

$$|E_{C,S}| = \sum_{f \in F_{C,i}} (s(f) - 2) - 2|V_{C,i}| + 4 - 4\gamma(C) - 2|B_{C,S}| + 2d_C.$$

Applying this formula to all white components individually and summing up the left and right sides we get

$$\sum_{C \in C_w} |E_{C,S}| = \sum_{C \in C_w} \left(\sum_{f \in F_{C,i}} (s(f) - 2) - 2|V_{C,i}| \right) + 4|C_w| - 4 \sum_{C \in C_w} \gamma(C) - 2|B_w| + 2d_w.$$

As each edge in S has exactly one white component on one side, we have $|E_S| = \sum_{C \in C_w} |E_{C,S}|$. Thus, we get

$$|E_S| = \sum_{f \in F_w} (s(f) - 2) - 2|V_w| + 4|C_w| - 4 \sum_{C \in C_w} \gamma(C) - 2|B_w| + 2d_w$$

and analogously for the black components

$$|E_S| = \sum_{f \in F_b} (s(f) - 2) - 2|V_b| + 4|C_b| - 4 \sum_{C \in C_b} \gamma(C) - 2|B_b| + 2d_b$$

completing the proof.

■

We have to admit that this theorem looks quite technical and — in contrast to Grinberg's Criterion [8] — lacks beauty. But as we will see, it allows to identify instances when Grinberg's, Zaks' [22] and Shimamoto's [16] theorems are valid in much more general contexts.

Let G be an embedded graph and $S = (V_S, E_S)$ be a partitioning subgraph of G. We define the *black correction term* $T_b(S)$ as

$$T_b(S) = -2|V_b| + 4|C_b| - 4 \sum_{C \in C_b} \gamma(C) - 2|B_b| + 2d_b$$

and analogously for the *white correction term* $T_w(S)$.

We call S a *Grinbergian subgraph* if $T_b(S) = T_w(S)$ or equivalently: if Grinberg's identity holds.

We say that G allows a Grinbergian partition if the faces can be partitioned into sets F_b and F_w so that Grinberg's identity holds — or equivalently: if G has a Grinbergian subgraph.

A hamiltonian cycle in a plane graph is always a Grinbergian subgraph. A hamiltonian cycle in a graph with an embedding in a surface of odd genus is never Grinbergian, while embedded in a surface of even genus it can be Grinbergian as well as non-Grinbergian.

From now on we abbreviate $\sum_{f \in M}(s(f) - 2)$ as Σ_M.

3 Consequences

Corollary 3.1 *Let G be a plane embedded graph and $S = (V_S, E_S)$ a connected spanning partitioning subgraph. Then*

$$\Sigma_{F_w} + 2|C_w| = \Sigma_{F_b} + 2|C_b|.$$

This corollary could be formulated in a more general setting by requiring only the components to be plane and the boundaries of external faces in every component to be a connected graph, but the case of G itself being plane is surely the most interesting and natural case.

Proof:

The prerequisites imply $|V_b| = |V_w| = 0$ and $\sum_{C \in C_b} \gamma(C) = \sum_{C \in C_w} \gamma(C) = 0$.

Let C be a (without loss of generality) black component and $\bar{C} = (\bar{V}_C, \bar{E}_C) = C \cap S$. Note that, as G is plane, also $\bar{C}$ is connected. Then $d_C = |\bar{E}_C| - |\bar{V}_C|$ and the faces of $\bar{C}$ are one internal face containing all the faces of G corresponding to a vertex in the decomposition graph and $|B_C|$ external faces. Applying the Euler formula to $\bar{C}$ we get

$$-(|\bar{E}_C| - |\bar{V}_C|) + (1 + |B_C|) = 2$$

which implies

$$|B_C| - d_C = 1.$$

Summing over all black components we get

$$|B_b| - d_b = \sum_{C \in C_b} (|B_C| - d_C) = |C_b|.$$

Inserting this and its white analogue into the equation of Theorem 2.2 we get the result of the Corollary.

■

Corollary 3.2 *Let G be a plane embedded graph and $S = (V_S, E_S)$ a connected spanning partitioning subgraph so that the two bipartition classes of $D_{G,S}$ have equal size (e.g. because $D_{G,S}$ is regular). Then S is Grinbergian.*

The result follows immediately from Corollary 3.1.

Grinberg's original theorem is the special case when G is plane and S is a spanning cycle, so that $D_{G,S}$ is K_2. Other examples for S that satisfy the requirements of the corollary for plane graphs are e.g. a subdivided spanning $K_{2,2n}$ ($D_{G,S}$ is a cycle of length $2n$), a spanning subdivided octahedron ($D_{G,S}$ is the cube) or the dual of any other 3-regular plane graph where all face sizes are even. Examples for the corollary's generalization to higher genera, for instance for S on the torus, are e.g. subdivisions of duals of bipartite regular tilings of the torus with squares (in which case all vertices have degree 4) or hexagons (in which case all vertices have degree 3). These classes include subdivided $C_k \,\square\, C_\ell$ with k, ℓ even, where "$\square$" denotes the Cartesian product, as well as a subdivided K_7 (compare Figure 3 showing the Heawood graph, the dual graph of K_7 on the torus).

Application: In Figure 1, a non-hamiltonian plane graph found by Thomassen is given. Except for one 10-gon it has only pentagons, so for each decomposition of the set of faces into sets F_b and F_w of black and white faces we have $\Sigma_{F_w} (\bmod 3) \neq \Sigma_{F_b} (\bmod 3)$. This implies that the graph does not allow any spanning Grinbergian subgraph, so it not only contains no hamiltonian cycle, but although it has several vertices of degree 4 or 5, it also contains no spanning subdivided octahedron or spanning subdivided $K_{2,4}$.

3.1 On 2-factors

Let G be an embedded graph. A *partial* 2-factor of G is a 2-factor of a subgraph of G. A 2-factor S of G is called *minimally splitting* if $|V_{G,S}| \geq 2$, but $|V_{G,S'}| = 1$ for every partial 2-factor S' of G that is a proper subgraph of S.

Lemma 3.3 *Let G be an embedded graph and $S = (V_S, E_S)$ a 2-factor in G. Then S is minimally splitting if and only if $D_{G,S} \cong K_2$.*
So minimally splitting 2-factors are also partitioning.

Proof: Assume that S is minimally splitting. If $D_{G,S} \ncong K_2$, it has at least 3 vertices or 2 vertices, but a loop. So there is an edge $e = \{x, y\}$ in $D_{G,S}$ not incident

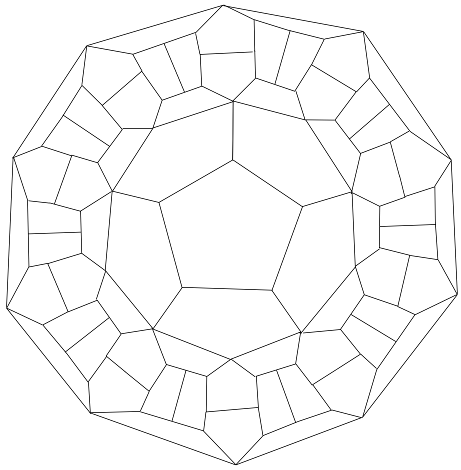

Figure 1: A non-hamiltonian plane graph found by Thomassen [17].

with a vertex v. Let Z be the cycle in S corresponding to this edge. Removing Z from S results in a decomposition graph where x and y are identified, but still different from v, so $|V_{G,S-\{Z\}}| > 1$ in contradiction to S being minimally splitting.

Now assume $D_{G,S} \cong K_2$. If a cycle Z can be removed and the remaining cycles still split G, then Z lies completely inside one component C. This implies that it has faces of C on both sides, in which case $D_{G,S}$ had a loop and would not be isomorphic to K_2.

■

Corollary 3.4 *Let G be an embedded graph of genus g and S a minimally splitting 2-factor with k cycles. Then S is partitioning, $D_{G,S}$ has one black component $C_{(b)}$ and one white component $C_{(w)}$ and*

$$\Sigma_{F_w} - 4\gamma(C_{(w)}) = \Sigma_{F_b} - 4\gamma(C_{(b)}).$$

Furthermore $\gamma(C_{(w)}) + \gamma(C_{(b)}) + k - 1 = g$.

Proof: As S is a 2-factor, all external faces are simple disjoint cycles, so $d_w = d_b = 0$. Furthermore each such cycle is an external face of both, $C_{(b)}$ and $C_{(w)}$, so $|B_w| = |B_b|$. Inserting this together with $|V_w| = |V_b| = 0$ and $|C_w| = |C_b| = 1$ into Theorem 2.2 we get the statement of the corollary.

The second equation follows directly when reconstructing G by identifying components along the cycles and keeping track of the number of vertices, edges and faces. ■

Let G be an embedded graph with genus g and S a minimally splitting 2-factor with $g+1$ cycles. Then S is called *planarizing*: cutting G along S results in two plane graphs G_1, G_2 and G can be reconstructed by identifying cycles of the induced

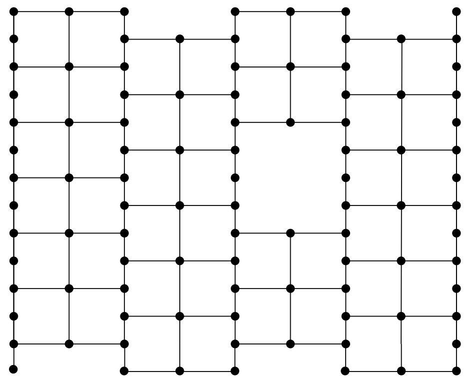

Figure 2: A toroidal graph without a planarizing 2-factor.

2-factors in G_1, G_2. This definition differs from the one used in [14], where the result must be one plane graph.

Corollary 3.5 *Let G be an embedded graph and S a minimally splitting 2-factor, partitioning G into two components with the same genus. Then S is Grinbergian.*

As a special case we have that planarizing 2-factors are Grinbergian.

Examples satisfying the requirements of this corollary are again a hamiltonian cycle in a plane graph, but also two disjoint non-contractible cycles spanning a toroidal graph or analogously for higher genus.

Example applications:

- Let G be the toroidal graph given in Figure 2, where the top and bottom as well as the left and right side have to be identified. In the tradition of Thomassen's example in Figure 1 all faces but one (which is a 12-gon) are pentagons. The drawing can be extended by adding any even number of columns or any number of rows of pentagons. Other graphs can also be obtained by shifting the top and bottom before identifying them. Starting with G sufficiently large (in order not to run out of pentagons), we can construct graphs with larger genus and from genus 2 on also without 2-cuts by identifying pentagons.

  We get $\Sigma_{F_b} (\bmod 3) \neq \Sigma_{F_w} (\bmod 3)$ for all bipartitions into black and white faces and conclude with Corollary 3.5 that none of the graphs can have a planarizing 2-factor.

  Identifying pentagons in Thomassen's graph we also get (immediately 3-connected) embedded graphs of genus 1 or higher without planarizing 2-factors.

- The toroidal embedding of the Petersen graph is given in Figure 3. The sequence of values $(s(f) - 2)$ for all faces f is $3, 3, 3, 4, 7$, which can only be decomposed into two parts with equal sum as $3, 3, 4$ and $3, 7$. This implies that a planarizing 2-factor must split all but one pentagon from the 9-gon. Using this knowledge for the cycle passing through the central vertex, one can uniquely determine a 2-factor which in fact turns out to be planarizing. Note

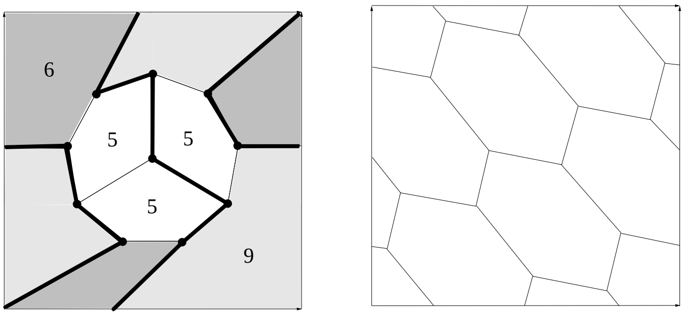

Figure 3: The unique toroidal embeddings of the Petersen graph and the Heawood graph.

that when embedded on the torus, not all 2-factors are equivalent and that other 2-factors are not planarizing for this embedding.

- The toroidal embedding of the Heawood graph is given in Figure 3. The sequence of values $(s(f) - 2)$ for all faces f is $4, 4, 4, 4, 4, 4, 4$, which can not be decomposed into two parts with equal sum, so the toroidal embedding of the Heawood graph has no planarizing 2-factor. As for embeddings of higher genus planarizing 2-factors would need at least three cycles (which is of course impossible in a graph with girth 6 and 14 vertices), there are no embeddings with planarizing 2-factors of the Heawood graph.
- The Heawood graph has a hamiltonian cycle and Corollary 3.4 implies that if there would be a splitting (that is: null-homotopic) hamiltonian cycle in the toroidal embedding, it would have 3 hexagons in the interior. In the dual these hexagons would form an induced path on 3 vertices, but as the dual is K_7, no such path exists and therefore no hamiltonian cycle in the toroidal embedding of the Heawood graph splits the torus.

Also the results of Gehner [6], Shimamoto [16], and Zaks [22] are restricted versions of a special case of Theorem 2.2. Shimamoto and Zaks require that in a 2-regular subgraph of a plane graph no cycle separates two others. The following corollary generalizes this result giving the same formula without this additional restriction.

Corollary 3.6 *Let G be a plane graph and S a partial 2-factor. Then $D_{G,S}$ is a tree and*

$$\Sigma_{F_w} - 2|V_w| + 4|C_w| = \Sigma_{F_b} - 2|V_b| + 4|C_b|.$$

Proof: $D_{G,S}$ is connected and each cycle in S separates the interior from the exterior — so each edge in $D_{G,S}$ is a cut-edge. This implies that $D_{G,S}$ is a tree and therefore bipartite. As each external face in each component is a simple cycle, we can apply Theorem 2.2 with $\gamma(C) = 0$ for each component and $d_b = d_w = 0$. As each cycle occurs as a face in one black and one white component we have $|B_w| = |B_b|$.

Inserting this into the formula from Theorem 2.2 we get the result. ∎

The result of Zaks [22] is the special case where $D_{G,S}$ is not an arbitrary tree, but the star $K_{1,n}$ for some n. It generalizes the result by Shimamoto which can be interpreted as Zaks' theorem with the additional requirement that the interior vertices of the components all lie in the component corresponding to the central vertex of the star and induce a graph with a 2-factor.

3.2 On a theorem of Bondy and Häggkvist

In [3] Bondy and Häggkvist prove a result for two edge-disjoint hamiltonian cycles in a 4-regular plane graph. Already then their proof could have been used in a more general context, but now it is possible to formulate their theorem in a very broad context — still using the same proof.

Let G be an embedded graph and S_1, S_2 two partitioning subgraphs, each with a given assignment of black and white to the two classes of faces. For $i \in \{1, 2\}$ let $F_{i,b}$ (resp. $F_{i,w}$) denote the set of black (white) faces with respect to S_i. We define $F_{b,b} = F_{1,b} \cap F_{2,b}$, $F_{b,w} = F_{1,b} \cap F_{2,w}$, $F_{w,b} = F_{1,w} \cap F_{2,b}$ and finally $F_{w,w} = F_{1,w} \cap F_{2,w}$.

Corollary 3.7 *Let G be an embedded graph and S_1, S_2 two Grinbergian subgraphs. Then*

$$\Sigma_{F_{w,w}} = \Sigma_{F_{b,b}}$$

and

$$\Sigma_{F_{w,b}} = \Sigma_{F_{b,w}}.$$

Proof: $F_{1,w}$ is the disjoint union of $F_{w,w}$ and $F_{w,b}$ and analogously for $F_{1,b}$, $F_{2,w}$ and $F_{2,b}$. Together with the fact that S_1 is Grinbergian, this gives

$$\Sigma_{F_{w,w}} + \Sigma_{F_{w,b}} = \Sigma_{F_{b,b}} + \Sigma_{F_{b,w}}.$$

The fact that S_2 is Grinbergian gives

$$\Sigma_{F_{w,w}} + \Sigma_{F_{b,w}} = \Sigma_{F_{b,b}} + \Sigma_{F_{w,b}}.$$

Subtracting one equation from the other and simplifying the result we get the second identity:

$$\Sigma_{F_{w,b}} = \Sigma_{F_{b,w}}.$$

Inserting this result into one of the equations we get the first identity. ■

Note that for this result it is not necessary that both subgraphs are hamiltonian cycles (let alone edge-disjoint). In fact, it is even possible that they have completely different structures and completely different correction terms. It is for example possible that S_1 is a hamiltonian cycle and S_2 a subdivided octahedron.

If the subgraphs are not Grinbergian, the result does not hold, but at least a weaker version can be proven.

Corollary 3.8 *Let G be an embedded graph and S_1, S_2 partitioning subgraphs with the property that $T_b = T_b(S_1) = T_b(S_2)$ and $T_w = T_w(S_1) = T_w(S_2)$. Then*

$$\Sigma_{F_{w,b}} = \Sigma_{F_{b,w}}.$$

Proof: Theorem 2.2 gives

$$\Sigma_{F_{w,w}} + \Sigma_{F_{w,b}} + T_w = \Sigma_{F_{b,b}} + \Sigma_{F_{b,w}} + T_b$$

and

$$\Sigma_{F_{w,w}} + \Sigma_{F_{b,w}} + T_w = \Sigma_{F_{b,b}} + \Sigma_{F_{w,b}} + T_b.$$

Subtracting one equation from the other leads to

$$\Sigma_{F_{w,b}} = \Sigma_{F_{b,w}}.$$

■

Application: In [3] Bondy and Häggkvist use their result for a necessary criterion for the decomposability of plane medial graphs into hamiltonian cycles. This result is in fact valid for arbitrary Grinbergian subgraphs and also for larger genus. We say that a graph $G = (V, E)$ is decomposed into two subgraphs $S_1 = (V_1, E_1)$ and $S_2 = (V_2, E_2)$ if $V = V_1 \cup V_2$, $E = E_1 \cup E_2$, and $E_1 \cap E_2 = \emptyset$.

The *medial* operation (also called *ambo*) applied to an embedded graph G places a vertex in the center of each edge in an embedded graph and connects it with the vertices in the next and previous edge of each of the two faces. We denote the resulting graph as $M(G)$. If both endpoints had degree at least 3 this does not result in loops or double edges and we get a 4-regular graph embedded in the same surface. Inside each face of G there is a face of $M(G)$ of the same size and around each vertex v of G there is a face f_v with $s(f_v) = \deg(v)$. The bipartition of the set of faces of $M(G)$ reflects these correspondences: one bipartition class consists of the faces coming from faces of G and one consists of the faces coming from vertices of G.

If we have two partitioning subgraphs S_1, S_2 of the medial graph $M(G)$ with a given assignment of the colours black and white, then we have the four classes $F_{b,b}, F_{b,w}, F_{w,b}$, and $F_{w,w}$. As each edge of $M(G)$ belongs to exactly one of S_1, S_2, faces sharing an edge belong to classes that differ by exactly one index and faces opposite to each other at a vertex belong to classes where no index differs or both indices differ. In other words: the bipartition classes of faces are $F_{b,b} \cup F_{w,w}$ and $F_{b,w} \cup F_{w,b}$. As the face sizes in one of $F_{b,b} \cup F_{w,w}$ and $F_{b,w} \cup F_{w,b}$ are the face sizes in G and in the other the face sizes in the dual of G (or equivalently: the vertex degrees in G), G as well as its dual must allow a Grinbergian partition. That implies:

Note 3.9 *The medial graph of an embedded graph without Grinbergian partition or without Grinbergian partition of the dual can not be decomposed into Grinbergian subgraphs.*

Applied to specific graphs this gives e.g.:

- The medial graph of Thomassen's graph in Figure 1 can not be decomposed into two hamiltonian cycles or into a subdivision of $K_{2,4}$ and a cycle of length $|V|-2$ with a vertex in the inside and the outside or a subdivision of the octahedron (saturating 6 vertices completely) and a cycle of length $|V|-6$ and 3 vertices in the inside and the outside, etc. ...
- The medial graph of the toroidal graph in Figure 2 can not be decomposed into any two Grinbergian subgraphs.

3.3 Walks

In non-hamiltonian graphs the length of a shortest spanning closed walk, also called the hamiltonian number [7], is a good measure of how far the graph is from being hamiltonian. We will give a short proof of a theorem due to Lewis [13] bounding such a length from below. It is useful to normalize this value by subtracting the number of vertices in order to have measure 0 for hamiltonian cycles. To this end we define the *repetition number* $r(s)$ of a spanning closed walk s with length $l(s)$ in a graph with n vertices as $r(s) = l(s) - n$.

For a spanning closed walk s in a plane graph G, we define the graph G_s as the graph G where each edge e that is used $k > 1$ times by s is replaced by k parallel paths of length 2 following each other in the rotational order around the endpoints of e. In G_s the closed walk s is modified on the parallel paths to a walk $\bar{s}$ so that each new edge is used once. We have $r(s) = r(\bar{s})$. As each vertex in G_s has an even degree in the subgraph S_s formed by $\bar{s}$, the dual is bipartite and S_s is partitioning.

Theorem 3.10 (Lewis [13]) *Let G be a plane graph, $F_{(w)}, F_{(b)}$ a partition of the faces of G into black and white faces, so that $|\Sigma_{F_{(w)}} - \Sigma_{F_{(b)}}|$ is minimal and s a spanning closed walk in G.*

Then $r(w) \geq |\Sigma_{F_{(w)}} - \Sigma_{F_{(b)}}|/2$.

More specifically we can say that with $F_{s,w}, F_{s,b}$ given by the partitioning subgraph S_s in G_S
$r(s) = (\Sigma_{F_{s,w}} - \Sigma_{F_{s,b}})/2 + 2|C_w| - 2 = (\Sigma_{F_{s,b}} - \Sigma_{F_{s,w}})/2 + 2|C_b| - 2.$ $(*)$

Note that, as we will show, $\Sigma_{F_{s,b}} - \Sigma_{F_{s,w}}$ can be considered as referring only to face sizes in G, while C_b and C_w may contain components with faces not present in G.

As in the case s is a cycle (so $G_s = G$, $r(w) = 0$, $|C_w| = |C_b| = 1$) we get Grinberg's original theorem back, this theorem can also be considered as a generalization of Grinberg's theorem.

Proof: Each face f in G has a corresponding face f_s in G_s.

Let $F_w = \{f | f \text{ is a face of } G, f_s \in F_{s,w}\}$ and analogously for F_b. This is (up to interchanging black and white) the same partition one would get by assigning two faces sharing an edge the same colour if the edge is traversed an even number of times and a different colour if it is traversed an odd number of times. If we modify G_s to form a multigraph by replacing the paths of length 2 by edges, two face sizes change, but $\Sigma_{F_{s,b}} - \Sigma_{F_{s,w}}$ does not change, as each such path separates a white from a black face, so in every step $\Sigma_{F_{s,b}}$ as well as $\Sigma_{F_{s,w}}$ are decreased by one. In the resulting multigraph the faces not present in G contribute 0 to the sum, so $\Sigma_{F_b} - \Sigma_{F_w} = \Sigma_{F_{s,b}} - \Sigma_{F_{s,w}}$. Together with the fact that $|C_w| \geq 1$ and $|C_b| \geq 1$, equation $(*)$ implies the first inequality.

In order to prove $(*)$ we can apply Corollary 3.1 and get

$$\Sigma_{F_{s,w}} + 2|C_w| = \Sigma_{F_{s,b}} + 2|C_b|. \qquad (**)$$

In the graph $S_s = (V_s, E_s)$ the components in C_b and C_w correspond to faces, so we can apply the Euler formula and get $|V_s| - |E_S| + |C_b| + |C_w| = 2$ or equivalently $|C_b| = r(s) - |C_w| + 2$. Inserting this into $(**)$ we get

$$\Sigma_{F_{s,w}} + 2|C_w| = \Sigma_{F_{s,b}} + 2(r(w) - |C_w| + 2),$$

that is

$$r(w) = (\Sigma_{F_{s,w}} - \Sigma_{F_{s,b}})/2 + 2|C_w| - 2$$

and analogously for white and black interchanged.

■

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