Regular Non-Hamiltonian Polyhedral Graphs

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Abstract. Invoking Steinitz' Theorem, in the following a *polyhedron* shall be a 3-connected planar graph. From around 1880 till 1946 Tait's conjecture that cubic polyhedra are hamiltonian was thought to hold—its truth would have implied the Four Colour Theorem. However, Tutte gave a counterexample. We briefly survey the ensuing hunt for the smallest non-hamiltonian cubic polyhedron, the Lederberg-Bosák-Barnette graph, and prove that there exists a non-hamiltonian essentially 4-connected cubic polyhedron of order n if and only if $n \geq 42$. This extends work of Aldred, Bau, Holton, and McKay. We then present our main results which revolve around the quartic case: combining a novel theoretical approach for determining non-hamiltonicity in (not necessarily planar) graphs of connectivity 3 with computational methods, we dramatically improve two bounds due to Zaks. In particular, we show that the smallest non-hamiltonian quartic polyhedron has at least 35 and at most 39 vertices, thereby almost reaching a quartic analogue of a famous result of Holton and McKay. As an application of our results, we obtain that the shortness coefficient of the family of all quartic polyhedra does not exceed 5/6. The paper ends with a discussion of the quintic case in which we tighten a result of Owens.

Keywords. Non-hamiltonian; non-traceable; polyhedron; planar; 3-connected; regular graph

MSC 2010. 05C45, 05C10, 05C38

1 Introduction

Due to Steinitz' classic theorem that the 1-skeleta of 3-polytopes are exactly the 3-connected planar graphs [34], we shall call such a graph a *polyhedron*. While the rigorous study of hamiltonian cycles goes back to at least 1766, when Euler treated the knight's tour problem, Hamilton and Kirkman were the first to study spanning cycles in polyhedra. We refer to [4] for further historical details. By Euler's formula, there are k-regular polyhedra for three values of k: 3, 4, or 5. We will call these *cubic*, *quartic*, and *quintic*, respectively. We use the word "regular" exclusively in the graph-theoretical sense of having all vertices of the same degree. The paper is split naturally into three sections—cubic, quartic, and quintic polyhedra—, and we give in each of these separately the motivation for treating the respective problem. Our results mainly revolve around the following

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six numbers: let c_k (p_k) denote the order of the smallest k-regular non-hamiltonian (non-traceable) polyhedron for $k \in \{3, 4, 5\}$.

For a possibly disconnected graph G, let $\omega(G)$ denote the number of connected components of G. In this article, a "cut" shall always be a vertex-cut, i.e. a vertex-set X in a graph G such that $\omega(G - X) \geq 2$. If we are referring to edge-cuts (defined analogously to vertex-cuts), we will explicitly mention this. We tacitly use the fact that for a 3-cut X in a polyhedron G, we have $\omega(G - X) = 2$, and that by Tutte's theorem [39] stating that 4-connected polyhedra are hamiltonian, a non-hamiltonian polyhedron contains at least one 3-cut. For more on the interplay between polyhedra, cuts, and hamiltonicity, we refer to the survey [32]. Let G be a polyhedron of connectivity 3 and $X = \{u, v, w\}$ a 3-cut in G. X is *trivial* if one of the components of $G - X$ is K_1. If G' is a component of $G - X$, then $G[V(G') \cup X]$ is called a *fragment* with *attachments* u, v, w (where $G[V(G') \cup X]$ denotes the subgraph of G induced by $V(G') \cup X$). A fragment F with attachments u, v, w is called an *ijk-fragment* if the degrees of u, v, w in F are i, j, k, respectively.

2 The Cubic Case

Tait conjectured [35] in 1884 that every cubic polyhedron is hamiltonian. This conjecture became famous because it implied the Four Colour Theorem (which at that time was itself open): by Jordan's Curve Theorem, any hamiltonian cycle $\mathfrak{h}$ in a cubic polyhedron G naturally divides the plane into an unbounded region A and a bounded region B with $A \cap B = \mathfrak{h}$. The duals of the planar graphs $G \cap A$ and $G \cap B$ are trees, so we may colour their vertices alternatingly. Thus, we can colour the faces of G with four colours such that no two adjacent faces receive the same colour. One can reduce the case of general polyhedra to cubic polyhedra, and thus, Tait's conjecture would have implied the Four Colour Theorem. However, Tait's conjecture turned out to be false and the first to construct a counterexample was Tutte in 1946, see [38], using in his approach three copies of a graph that has become to be known as "Tutte-fragment". The smallest counterexample is due to Lederberg (and independently, Bosák and Barnette), has order 38, and is also based on Tutte-fragments. (To be precise, there are six structurally very similar such graphs [19].) That this is indeed the smallest possible counterexample was shown by Holton and McKay [19] after a long series of papers by various authors, see for instance work of Butler [10], Barnette and Wegner [2], and Okamura [28]. The second part of the next theorem follows directly from the Holton-McKay result by successively substituting a vertex with a triangle.

Theorem 1 (Holton and McKay [19])**.** *We have that*

$$c_3 = 38.$$

Furthermore, for each even $n \geq 38$ there is a non-hamiltonian cubic polyhedron on n vertices.

Balinski asked whether non-traceable cubic polyhedra exist. Brown and independently Grünbaum and Motzkin proved the existence of such graphs. Klee asks for determining p_3. (We refer to Klee's excellent [13, Chapter 17] for references and further details.) The best bounds that are known are as follows.

Theorem 2 (Knorr [23] and T. Zamfirescu [44])**.** *We have that*

$$54 \leq p_3 \leq 88.$$

Furthermore, for each even $n \geq 88$ there exists a non-traceable cubic polyhedron of order n.

The lower bound was proven by Knorr in 2010, see [23]. The upper bound due to T. Zamfirescu [44], although published in 1980, has resisted all improvement attempts hitherto. Knorr's lower bound is based on work of Okamura [28] and improves a result of Hoffmann [18], while the upper bound is based on Tutte-fragments and improves a result of Brown [13, p. 362]. That indeed there exists a non-traceable cubic polyhedron of order n for every even $n \geq 88$ can be shown again by simply replacing vertices with triangles.

Recently, McKay raised the question [25] which plane graphs have a spanning tree such that at least one edge of each face is in the tree, specifically asking whether triangulations always have such a tree. Following an argument of Kynčl [25], a triangulation has a spanning tree with the required property if and only if its dual is traceable. Thus, every graph providing a positive answer to Balinski's question yields an example for a negative answer to McKay's question, and vice-versa.

A 3-connected graph is *essentially* 4*-connected* if all of its 3-cuts are trivial. Essentially 4-connected k-regular polyhedra that are not 4-connected only exist for $k = 3$. Let c_3^* (p_3^*) be the order of the smallest non-hamiltonian (non-traceable) essentially 4-connected cubic polyhedron. We shall see that c_3^* is known. However, we first give a definition and a well-known lemma (see for instance [12]), the proof of which we include for completeness' sake.

In a graph G, a set S of k edges is called a *k-edge-cut* if $G - S$ is disconnected and if no proper subset of S satisfies this property. It is easy to see that $\omega(G - S) = 2$. These two components are called *k-pieces*. A k-edge-cut is called *non-trivial* if each of its k-pieces contains a cycle. We say that a cubic graph is *cyclically k-edge-connected* if it has no non-trivial t-edge-cuts for $0 \leq t \leq k-1$. For $X, Y \subset V(G)$ we denote with $E(X, Y)$ the set of all edges with one endpoint in X and the other endpoint in Y. Abusing notation, when $X = \{x\}$, we shall write $E(x, Y)$ instead of $E(\{x\}, Y)$. For a vertex $v \in V(G)$ and a subgraph H of G, we denote with $N_H(v)$ the set of neighbours of v in H, and put $N_G(v) = N(v)$.

Lemma 1. *A* 3*-connected cubic graph of order* $\neq 6$ *is essentially* 4*-connected if and only if it is cyclically* 4*-edge-connected.*

Proof. Assume G is essentially 4-connected, but not cyclically 4-edge-connected. Then there is a 3-edge-cut M such that each 3-piece contains a cycle. Since $G \not\cong K_4$ and $|V(G)| \neq 6$, we have that $|V(G)| \geq 8$, so one of these pieces H (say) contains at least four vertices. If $|V(H)| = 4$, then its sum of vertex-degrees would be odd (namely 9), which is impossible. So $|V(H)| \geq 5$. Taking the endpoints of the edges in M lying in H, we obtain a non-trivial k-cut with $k \leq 3$, a contradiction.

Suppose that G is cyclically 4-edge-connected. We first claim that G has no 3-edge-cut M such that $G - M$ has two components H_1, H_2 none of which is isomorphic to K_1. Otherwise, let $\{u_i, v_i, w_i\} = V(M) \cap V(H_i)$ for $i \in \{1, 2\}$. Since G is 3-connected, by Menger's Theorem there are three vertex-disjoint paths in G connecting any two vertices of $\{u_1, v_1, w_1\}$. Note that two of these lie in H_1 and one lies in H_2. This implies that H_1 must contain a cycle. Furthermore, there are three paths in H_2 connecting any two vertices of $\{u_2, v_2, w_2\}$, which means that H_2 contains a cycle, a contradiction.

Let $X = \{u, v, w\}$ be a 3-cut of G and H_1, H_2 be the two components of $G - X$. Then we claim that X is an independent set of G, i.e. no two vertices of X are adjacent. Otherwise, assume $uv \in E(G)$. Then $|E(u, H_1)| = |E(u, H_2)| = |E(v, H_1)| = |E(v, H_2)| = 1$ and $wv, wu \notin E(G)$. Either $E(\{u, v\}, H_2) \cup E(w, H_1)$ (if $|E(w, H_1)| = 1$) or $E(\{u, v\}, H_1) \cup E(w, H_2)$ (if $|E(w, H_2)| = 1$) is a non-trivial 3-edge-cut of G, a contradiction.

Thus $|E(X, H_1)| + |E(X, H_2)| = 9$. If $|E(X, H_1)| \geq 4$ and $|E(X, H_2)| \geq 4$, then, without loss of generality, assume $|E(X, H_1)| = 5$ and $N_{H_1}(w) = \{w'\}$. Then $X' = \{u, v, w'\}$ would be a 3-cut such that exactly one component H' (say) of $G - X'$ with $|V(H')| \geq 2$ satisfies $|E(X', H')| = 3$, a contradiction. Then, by symmetry, $|E(X, H_1)| = 3$, which means $|V(H_1)| = 1$. Thus G is essentially 4-connected. □

The "$\neq 6$" in above statement stems from the fact that the triangular prism is essentially 4-connected, but not cyclically 4-edge-connected.

Motivated by his counterexample to Tait's conjecture [38], in 1960 Tutte [40] proved that there exists a non-hamiltonian cyclically 4-edge-connected cubic planar graph. It took another forty years to determine the order of the smallest such graph:

Theorem 3 (Aldred, Bau, Holton, and McKay [1])**.** *Every cyclically* 4*-edge-connected cubic planar graph of order at most* 40 *is hamiltonian. Furthermore, there exist non-hamiltonian examples of order* 42*. In particular*

$$c_3^* = 42.$$

This result raises the natural question for which n non-hamiltonian essentially 4-connected cubic polyhedra of order n exist. Here, the naïve approach of substituting vertices by triangles fails. We now present the strongest form of an answer to this question, for which we require an operation of Thomassen: let G be a graph containing a 4-cycle $v_1v_2v_3v_4v_1 = C$ and consider vertices $v_1', v_2', v_3', v_4' \notin V(G)$. We denote by $T(G_C)$ the graph obtained from G by deleting the

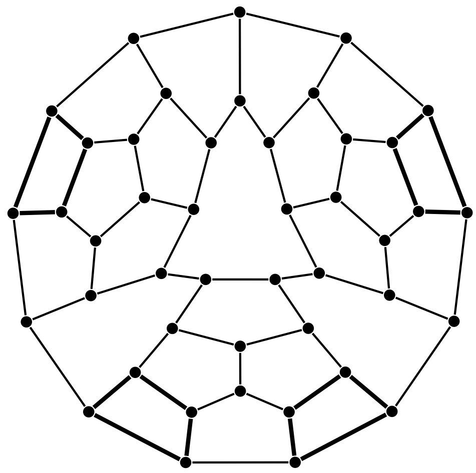


Figure 1: The unique non-hamiltonian essentially 4-connected cubic polyhedron on 44 vertices containing a 4-face. The three cubic essentially 4-connected cubic polyhedra on 44 vertices that are non-hamiltonian are available from [26], and only one of them contains a 4-face.

edges v_1v_2, v_3v_4 and adding a new 4-cycle $v_1'v_2'v_3'v_4'v_1'$ and the edges v_iv_i', $1 \leq i \leq 4$. Abusing notation, when we speak of "the graph $T(G_C)$" and C is a not further specified 4-cycle, we refer to (an arbitrary but fixed) one of the two (possibly isomorphic) graphs obtained when applying T. When any cubic 4-cycle will do, we simply write $T(G)$.

In 1981, Thomassen [37] showed that if G is a non-hamiltonian graph containing a cubic 4-cycle, then $T(G)$ is non-hamiltonian, as well. The operation T preserves planarity and 3-regularity, and it is easy to verify that if G is essentially 4-connected, then so is $T(G)$. We are now in the position to extend the second part of Theorem 3.

Theorem 4. *There exists a non-hamiltonian essentially* 4*-connected cubic polyhedron of order* n *if and only if* n *is even and* $n \geq 42$.

Proof. Combining Lemma 1 and Theorem 3, we obtain that $c_3^* = 42$. Let G_1 and G_2 be essentially 4-connected cubic polyhedra of order 42 and 44, respectively, each containing a quadrilateral—take for instance [1, Figure NH42.a] and the graph depicted in Figure 1 as G_1 and G_2, respectively. Iteratively applying the operation T to G_1 and G_2 completes the proof. □

Let us briefly address the non-traceable case. We are not aware of any bounds concerning p_3^*. Knorr's result [23] immediately implies $p_3^* \geq 54$. To establish an upper bound, we will use a technique devised by Horton [20] and generalised by Thomassen [36]. We apply to five pairwise disjoint polyhedra $G_1, ..., G_5$ the following operation, which we shall denote with L. For each $i \in \{1, ..., 5\}$, let x_i, y_i be adjacent cubic vertices in G_i and put $N(x_i) = \{a_i, b_i, y_i\}$ and $N(y_i) = \{c_i, d_i, x_i\}$ such that a_i, b_i, c_i, d_i are pairwise different. Put $H_i = G_i - \{x_i, y_i\}$. Now consider $\bigcup_i H_i$ and add the edges c_ia_{i+1} and d_ib_{i+1} for all $i \in \{1, ..., 5\}$, indices mod 5. This yields the graph $L(G_1, ..., G_5)$.

Theorem 5. *There exists a non-traceable essentially* 4*-connected cubic polyhedron of order* n *for every even* $n \geq 200$*. In particular*

$$p_3^* \leq 200.$$

Proof. Let $G_1, ..., G_5$ be pairwise disjoint non-hamiltonian essentially 4-connected cubic polyhedra and consider all definitions given above the theorem's statement. Put $G = L(G_1, ..., G_5)$. Thomassen already established (see [36, Lemma 2.1]) that G is non-traceable. By construction, if each G_i is cubic and planar, then G is cubic and planar as well. We now prove that every 3-cut in G is trivial. As shown in the proof of Lemma 1, in a cubic essentially 4-connected graph of order at least 7, no two vertices lying in a 3-cut are adjacent. Thus, if two adjacent vertices are deleted from a cubic essentially 4-connected graph, then the resulting graph is 2-connected, so G is 3-connected.

It remains to show that G is essentially 4-connected. Assume there exists a non-trivial 3-cut S in G. Note that since G is a polyhedron, $\omega(G - S) = 2$.

Let G'_1 be the subgraph of G isomorphic to G_1 minus two adjacent vertices x_1, y_1. First, we treat the case that all vertices of S lie in G'_1. If $G'_1 - S$ has a component which contains neither x_1 nor y_1, then S is a non-trivial 3-cut in G_1 as well, a contradiction to the fact that G_1 is essentially 4-connected. Thus, each component of $G'_1 - S$ must contain either x_1 or y_1. This however is impossible, since x_1 and y_1 were adjacent.

We are left with the case that not all vertices of S lie in the same G'_i, for all i. As described in the first paragraph, if each vertex of S lies in a different G'_i, then the removal of S cannot disconnect G. With similar arguments we deal with the case when two of the vertices of S lie in G'_1 and one lies in G'_i for $i \neq 1$. Together with Theorem 4, the proof is complete. □

We end this first section by stating a conjecture on a structural property of the smallest non-traceable cubic polyhedron.

Conjecture 1. *The smallest non-traceable cubic polyhedron is not cyclically 4-edge-connected.*

3 The Quartic Case

Quartic planar graphs arise in various settings: they are for instance line-graphs of cubic planar graphs, as well as the duals of plane quadrangulations. We recall that the hamiltonian cycle problem is NP-complete in quartic planar graphs [21]. In 1970, Lovász conjectured that every quartic planar graph G can be drawn in the plane using a set of circles, such that $V(G)$ corresponds to the intersection and touching points of the circles and $E(G)$ to the arc segments among pairs of intersection and touching points of the circles. In 2015, Bekos and Raftopoulou [3] settled this conjecture by proving that precisely for the polyhedral case, it is possible, and that there are non-3-connected counterexamples. Broersma, Duijvestijn, and Göbel generated in [9] all quartic polyhedra from the octahedron graph. (See also an earlier article of Lehel for more context [24].) Their work is motivated by results from chemistry—for the details, see King [22].

Bondy and Häggkvist [5] have studied edge-disjoint hamiltonian cycles in planar quartic graphs via Grinberg's hamiltonicity criterion, the 3-connected case playing an important role. David High [17] investigates knots using planar hamiltonian quartic graphs.

As mentioned in [6], non-hamiltonian quartic polyhedra have been known for a long time. Following work of Walther [41] and Sachs [33] in the late sixties, Zaks [43] proved that there exists a non-hamiltonian quartic polyhedron of order 209. Note that this number (and several other numbers answering similar problems) is not as stated in [43], since the number given in [43] is false, as pointed out in Owens' work [29]—therein the correct numbers can be found. Owens [30] also investigated regular non-hamiltonian polyhedra that contain only two types of faces.

Lemma 2. *For each $n \geq 6$ there exists a 233-fragment on n vertices of a quartic polyhedron.*

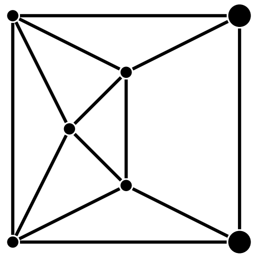
Proof. Let G be a polyhedron containing two adjacent cubic vertices and all other vertices quartic. Let x be one of the cubic vertices. If we consider the cut formed by the neighbourhood of x, then one fragment (namely the one containing x) is trivial, while the other fragment is a 233-fragment on $|V(G)| - 1$ vertices of a quartic polyhedron.

The left-hand side of Figure 2 shows a 7-vertex polyhedron containing two adjacent cubic vertices (emphasised in the figure) and five quartic vertices. Removing one of the cubic vertices results—as described above—in a 233-fragment (on six vertices) of a quartic polyhedron.

The right-hand side of Figure 2 depicts an operation to transform a polyhedron containing two adjacent cubic vertices and k quartic vertices into a polyhedron containing two adjacent cubic vertices and $k+1$ quartic vertices. Using this transformation and the graph on the left-hand side of Figure 2, we can construct a 233-fragment on n vertices of a quartic polyhedron for any $n \geq 7$. □

The following shall be an essential tool in our efforts; we here focus on the planar case, since this is what we are interested in, but its applicability has nothing to do with planarity.

In a hamiltonian polyhedron containing a 3-cut, every hamiltonian cycle visits the graph as shown in Figure 3, i.e. one vertex of the 3-cut only has neighbours on the hamiltonian cycle in one of the fragments, while the other two vertices of the 3-cut have a neighbour on the hamiltonian cycle



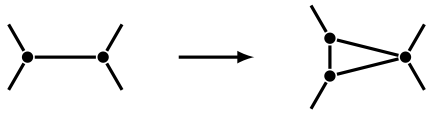


Figure 2: On the left-hand side a polyhedron on seven vertices is depicted. It contains two adjacent cubic vertices (shown in bold), and all other vertices quartic. On the right-hand side we present an operation to transform a polyhedron containing two adjacent cubic vertices and k quartic vertices into a polyhedron containing two adjacent cubic vertices and $k + 1$ quartic vertices—that this transformation preserves planarity is evident, and that the resulting graph is 3-connected can be shown easily with Menger's Theorem.

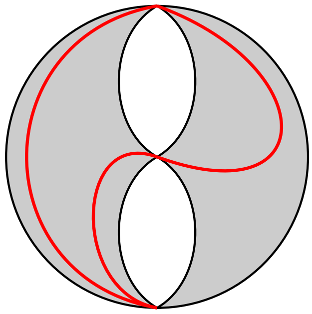


Figure 3: A hamiltonian polyhedron containing a 3-cut.

in both fragments. This leads us to the following definition. Let G be a hamiltonian polyhedron of connectivity 3, consider X to be a 3-cut of G, put $G - X = G_X$, and let F be a fragment of G with attachments X. If for a hamiltonian cycle $\mathfrak{h}$ in G we have that $\mathfrak{h} \cap F$ is connected, i.e. a path (as depicted in the left fragment of the graph from Figure 3), then F is called a *strong half of* G_X *w.r.t.* $\mathfrak{h}$. If not, then $\mathfrak{h} \cap F$ is a path and an isolated vertex and F is called a *weak half of* G_X *w.r.t.* $\mathfrak{h}$, see the fragment on the right-hand side of the graph from Figure 3. When G and X are clear from context and $\mathfrak{h}$ may be chosen arbitrarily with the same outcome, we will simply speak of a strong or weak half. Note that every trivial fragment is weak.

Lemma 3. *For each $n \geq 21$ there exists a 222-fragment on n vertices of a quartic polyhedron that cannot be the weak half of a hamiltonian quartic polyhedron.*

Proof. We make use of the notation from Figure 6, which shows a construction that combines three 233-fragments of a quartic polyhedron to form a 222-fragment of a quartic polyhedron. We now show that the resulting fragment cannot be the weak half of a hamiltonian quartic polyhedron.

Owing to the symmetry of the construction we only need to prove that there is no path from x to y that visits each vertex except for z. Let us assume that such a path exists. First suppose that this path would first visit F_2 from x. After leaving F_2 the path can either go to F_1 or F_3, but cannot visit the other one before reaching y. All other cases are completely analogous, so such a path does not exist.

The order of the constructed 222-fragment is the sum of the orders of the three 233-fragments plus 3. Together with Lemma 2, the proof is complete. □

Theorem 6. *We have that*

$$c_4 \leq 39.$$

Furthermore, for each $n \geq 39$ there is a non-hamiltonian quartic polyhedron on n vertices.

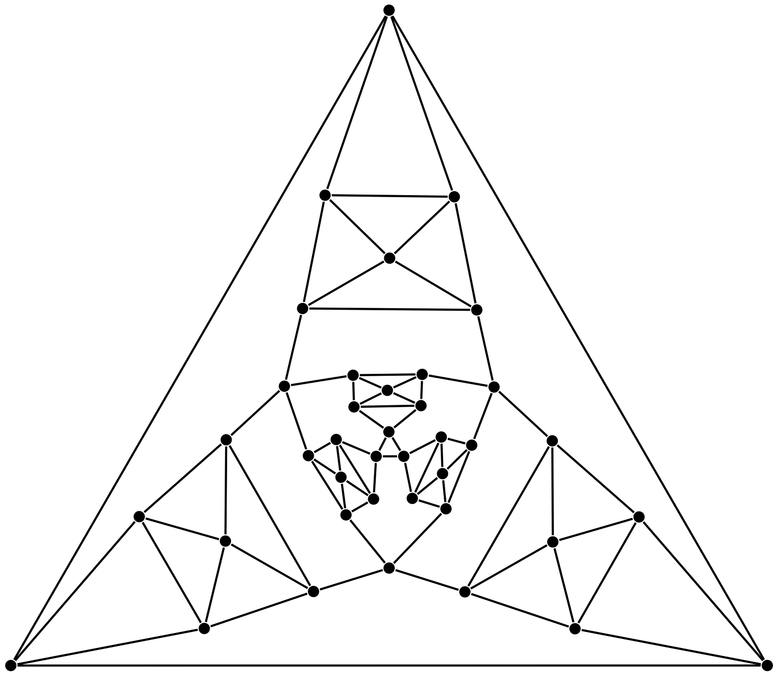


Figure 4: A non-hamiltonian quartic polyhedron of order 39. It illustrates that although polyhedra having no 3-vertex-cuts must be hamiltonian, non-hamiltonian polyhedra with no 3-edge-cuts do exist. We leave to the reader the verification that this graph is 1-tough (so its non-hamiltonicity cannot be obtained through a toughness argument) and its automorphism group is $\mathbb{Z}/3\mathbb{Z}$. In stark contrast to the Lederberg-Bosák-Barnette graph, this graph has no cycles of length one less than its order. We prove in Theorem 11 that this graph is not homogeneously traceable.

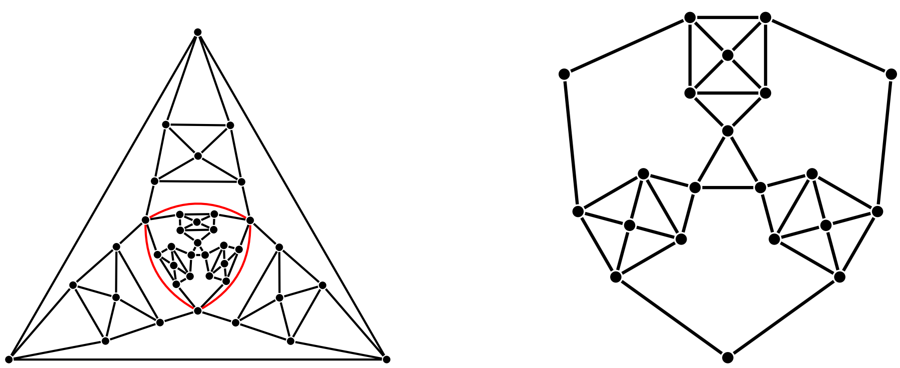


Figure 5: The graph from Figure 4 contains a 3-cut (left-hand side) such that we can split the graph into two identical closed components (right-hand side).

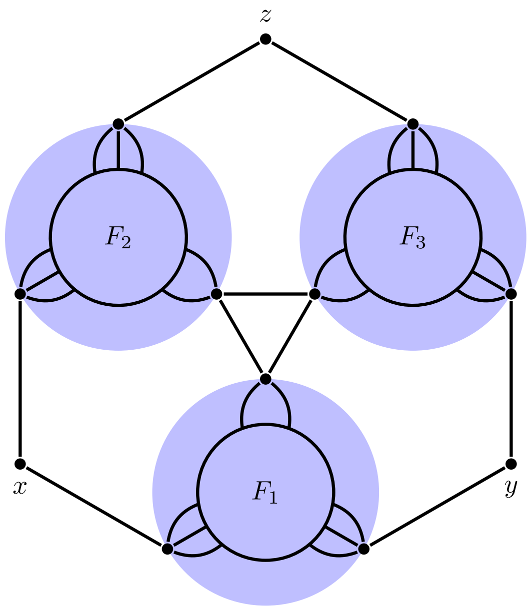


Figure 6: Combining three 233-fragments of a quartic polyhedron to obtain a 222-fragment of a quartic polyhedron that cannot be the weak half of a hamiltonian quartic polyhedron.

Proof. By Lemma 3, for every $n \geq 21$ there exists a non-weak n-vertex 222-fragment of a quartic polyhedron. Combining such fragments of order n_1 and n_2, we get a non-hamiltonian quartic polyhedron of order $n_1 + n_2 - 3$. In particular, if $n_1 = n_2 = 21$, we obtain the existence of a non-hamiltonian quartic polyhedron of order 39 (see Figure 5). □

If it turns out that the graph from Figure 4 is indeed the smallest non-hamiltonian quartic polyhedron, then there is a clear discrepancy between the cubic and the quartic case. In the smallest non-hamiltonian cubic polyhedra (there are six such graphs, one of which is the Lederberg-Bosák-Barnette graph), there are many cycles of length $n - 1$, where n is the order of the graph. For the Lederberg-Bosák-Barnette graph G, Neyt [27] determined all vertices v such that $G - v$ is hamiltonian. In the quartic case, no vertex-deleted subgraph of the graph depicted in Figure 2 is hamiltonian—in fact, its longest cycles have length 34, while its order is 39. (We shall come back to this observation at the end of this section.) It would be interesting to further pursue these gaps in the cycle spectrum, but we shall here restrict ourselves to the most obvious direction for further research:

Conjecture 2. *The graph depicted in Figure 2 is the smallest non-hamiltonian quartic polyhedron. We emphasise that we here also conjecture its uniqueness.*

In order to improve the lower bound of c_4, we implemented a backtracking algorithm to look for a hamiltonian cycle in a quartic polyhedron. Since most graphs for which this program will run are hamiltonian, it turned out that applying too many optimisations to bound early did not yield a significant decrease in the running time—experimentally we established that in many cases they even increased the running time.

When executing the program to check all quartic polyhedra for being hamiltonian, we see that the percentage of the time used to generate the graphs versus the time needed to check them for being hamiltonian rapidly decreases. For 32 vertices, less than 2% of the time is needed to generate the graphs. This was the motivation to spend slightly more time generating the graphs, but trying to eliminate early as many graphs as possible that cannot occur as candidates for the smallest non-hamiltonian quartic polyhedron. To this end, we first identified certain subgraphs which cannot be present in a smallest non-hamiltonian quartic polyhedron.

Lemma 4. *A smallest non-hamiltonian quartic polyhedron does not contain a subgraph of type 1 or type 2 (see Figure 7) such that the pendent edges are connected to distinct vertices.*

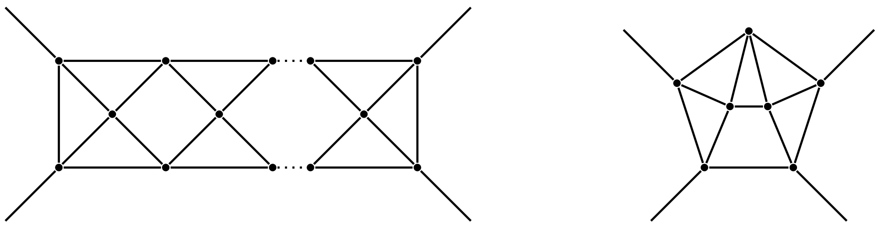


Figure 7: Two (families of) subgraphs that cannot be present in a smallest non-hamiltonian quartic polyhedron. Type 1 is shown on the left-hand side, and type 2 is shown on the right-hand side.

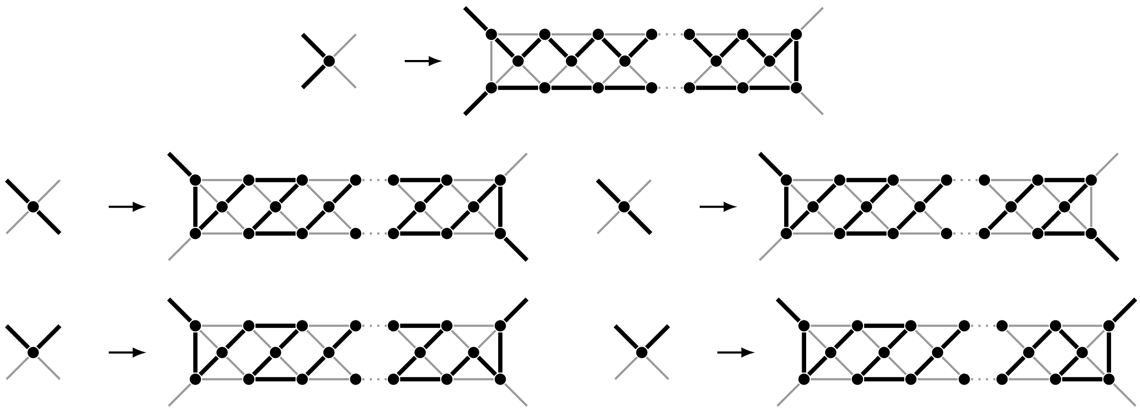


Figure 8: Transforming a hamiltonian cycle in a contracted quartic polyhedron into a hamiltonian cycle in the original quartic polyhedron for a subgraph of type 1. Note that two of the three cases exhibit a slightly different behaviour depending on the number of vertices in the subgraph: $3k+5$ with k either even or odd.

Proof. Assume we have a quartic polyhedron G containing either a subgraph of type 1 or a subgraph of type 2. Both types of subgraph have four pendent edges, so we can contract one of the subgraphs to a single vertex in order to obtain a new quartic polyhedron G'. We will show that G' is non-hamiltonian if G is non-hamiltonian. Assume G is hamiltonian. In Figure 8 and Figure 9 we give all (up to equivalence) possibilities for a hamiltonian cycle to go through the new vertex in G, and a transformation to a hamiltonian cycle in G' for each of these cases. □

We generated all quartic polyhedra up to 34 vertices using `plantri` [7, 8]. The graphs containing any of the subgraphs described in the lemma above were removed, and the remaining graphs were checked for being hamiltonian using the program described above. We found that up to 34 vertices all quartic polyhedra are hamiltonian. On a cluster of Intel E5-2670 processors running at 2.6 GHz these programs needed roughly 38 CPU years to check the quartic polyhedra on 34 vertices. The result of these computations is the theorem below.

Theorem 7. *We have that*

$$c_4 \geq 35.$$

On [6, p. 132] a method is given to construct a 4-regular graph from a cubic graph while preserving non-hamiltonicity and planarity. It is claimed that converting the Lederberg-Bosák-Barnette graph, which is one of the six smallest non-hamiltonian cubic polyhedra and has 38 vertices, with the aforementioned method gives a non-hamiltonian quartic polyhedron of order 161. However, the correct number should be $\frac{9}{2} \cdot 38 = 171$. This method goes back to work of Sachs—we present his result and a straightforward but tedious extension to the non-traceable case, the proof of which is therefore omitted.

Theorem 8 (Sachs [33])**.** *If there exists a non-traceable (non-hamiltonian) cubic polyhedron of order n, then there exists a non-traceable (non-hamiltonian) quartic polyhedron on $9n/2$ vertices.*

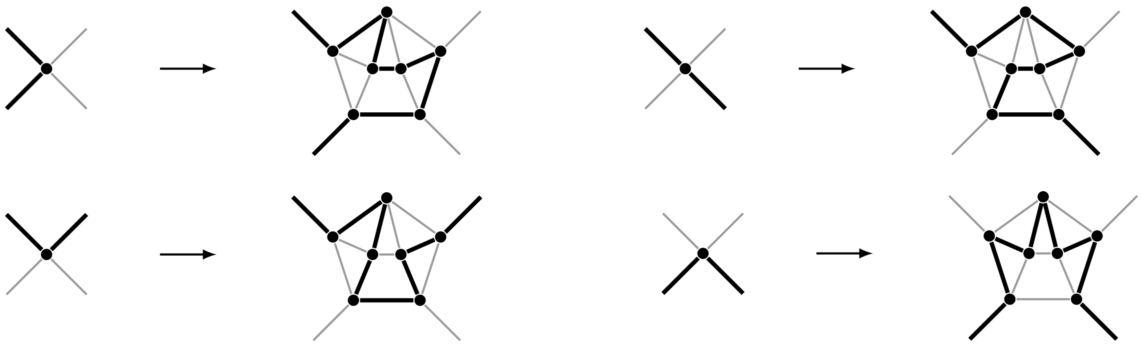


Figure 9: Transforming a hamiltonian cycle in a contracted quartic polyhedron to a hamiltonian cycle in the original quartic polyhedron for a subgraph of type 2.

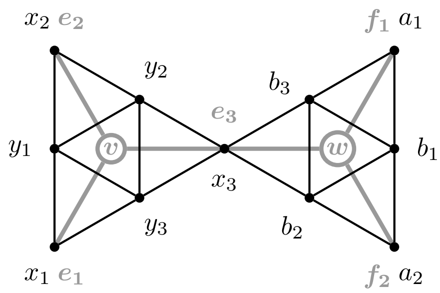


Figure 10: The transformation used by Sachs [33] to construct a non-hamiltonian quartic polyhedron from a non-hamiltonian cubic polyhedron.

In particular

$$p_4 \leq \frac{9}{2} p_3 \quad \text{and} \quad c_4 \leq \frac{9}{2} c_3.$$

The transformation used to obtain the quartic polyhedra in the theorem above is shown in Figure 10.

Zaks had shown that $p_4 \leq 484$. With Sachs' theorem and T. Zamfirescu's 88-vertex graph [44], we immediately obtain:

Corollary 1. *There exists a non-traceable quartic polyhedron of order* 396*.*

However, we can do much better—but we first extend the definition of weak and strong half. Let G be a traceable polyhedron of connectivity 3 and X a 3-cut of G, put $G - X = G_X$, and consider F to be a fragment of G with attachments X. If for a hamiltonian path $\mathfrak{p}$ in G we have that $\mathfrak{p} \cap F$ is a path, then F is called a *strong half of* G_X *w.r.t.* $\mathfrak{p}$. If $\mathfrak{p} \cap F$ is a path and an isolated vertex, then F is called a *weak half of* G_X *w.r.t.* $\mathfrak{p}$, and whenever $\mathfrak{p} \cap F$ is a path and two isolated vertices, F is called a *very weak half of* G_X *w.r.t.* $\mathfrak{p}$. If G and X are clear from context and $\mathfrak{p}$ can be chosen arbitrarily with the same outcome, we will simply speak of a strong or weak half. In the case of hamiltonian cycles, for every 3-cut either half was either strong or weak. For hamiltonian paths this distinction is not as easily made.

Lemma 5. *For each* $n \geq 21$ *there exists a 222-fragment on* n *vertices of a quartic polyhedron that cannot be a very weak half of a traceable quartic polyhedron.*

Proof. The fragments we will construct coincide with the ones described in Lemma 3. We use the notation introduced in Figure 6.

Owing to the symmetry of the construction we only need to prove that there is no path from x that visits each vertex except for y and z. Let us assume that such a path exists. Without loss of generality we may suppose that this path would first visit F_1 from x. After leaving F_1 the path visits F_i, but thereafter cannot reach F_{5-i}, where $i \in \{2, 3\}$.

The order of the constructed 222-fragment is the sum of the orders of the three 233-fragments plus 3. Together with Lemma 2 this completes the proof. □

Theorem 9. *There exists a quartic non-traceable polyhedron of order n for every $n \geq 78$. In particular*

$$p_4 \leq 78.$$

Proof. The family of polyhedra which we will construct can be formed by taking four 222-fragments as described in Lemma 3 and Lemma 5. These fragments are connected as shown on the left-hand side in Figure 12 to form the graph G, and we use the notation as established in that figure.

Assume first that there is a hamiltonian path $\mathfrak{p}$ in G which has one of the connection points as a starting vertex. Without loss of generality, we can assume that u is one of the endpoints of $\mathfrak{p}$, and that the neighbour of u on $\mathfrak{p}$ lies in F_1. This means that the other endpoint of $\mathfrak{p}$ has to lie in F_2 and should be different from x and z, since otherwise F_2 would be a very weak half, which contradicts Lemma 5. Since the other endpoint of $\mathfrak{p}$ is not contained in F_1, we have that $\mathfrak{p} \cap F_1$ is not the union of two paths, so necessarily $\mathfrak{p} \cap F_1$ is a single path, i.e. F_1 is a strong half. Ignoring analogous cases, we can assume that $\mathfrak{p} \cap F_1$ is a uv-path which contains w. This implies that F_4 is a weak half, which contradicts Lemma 3. So we find that G does not contain a hamiltonian path which has one of the connection points as a starting vertex. This also implies that G is not hamiltonian.

Next suppose that there is a hamiltonian path $\mathfrak{p}$ in G which has both endpoints in the same fragment. We can assume that both endpoints are contained in F_1, and that $\mathfrak{p} \cap F_1$ is the union of a ua-path, a vb-path and possibly the isolated vertex w with a and b two distinct vertices in F_1. If $\mathfrak{p} \cap F_1$ does not contain the isolated vertex w, then w might be contained in either the ua-path or the vb-path, but this is not necessarily the case. We then have that F_2 is a strong half and that $\mathfrak{p} \cap F_2$ is a uz-path containing x. This however implies that F_3 is a weak half, which contradicts Lemma 3.

Finally, assume there is a hamiltonian path $\mathfrak{p}$ in G which has endpoints in different fragments. Without loss of generality, we can assume that one endpoint is contained in F_1, and the other endpoint is contained in F_2. We denote these endpoints by a and b, and without loss of generality we can assume that $\mathfrak{p} \cap F_1$ contains a ua-path and that $\mathfrak{p} \cap F_2$ contains a xb-path. This implies that $\mathfrak{p} \cap F_4$ contains a path which has z as one of the endpoints. Since F_4 contains no endpoint of $\mathfrak{p}$, it is necessarily a strong half, and so $\mathfrak{p} \cap F_4$ is either a zw-path containing y, or a zy-path containing w. The former case implies that F_3 is a weak half, which contradicts Lemma 3. The latter case implies that F_1 is a very weak half, which contradicts Lemma 5. □

The smallest member of the family constructed in the theorem above has 78 vertices, and is shown in Figure 11.

If a graph is hamiltonian, then it is also traceable. This implies that any lower bound for c_4 is also a lower bound for p_4. However, we can slightly improve on this automatic implication.

Theorem 10. *We have that*

$$p_4 \geq c_4 + 1.$$

Proof. In [7] it was proven that every quartic polyhedron, except the antiprisms, can be reduced to a smaller quartic polyhedron using either operation R_1 or R_2 (see Figure 13). When we use reduction R_i to reduce a quartic polyhedron G, we will denote the resulting smaller quartic polyhedron by $R_i(G)$. Note that there are usually multiple ways to apply any of the operations, so $R_i(G)$ is normally not uniquely determined, but that is not an issue for what follows. We will show that if $R_i(G)$ is hamiltonian, then G is traceable. Since every antiprism is hamiltonian, and thus certainly traceable, this immediately implies the theorem.

If $R_1(G)$ is hamiltonian, then there are up to symmetry only two distinct ways in which the hamiltonian cycle can pass through the 'new' vertex in $R_1(G)$. Both ways can be translated to a hamiltonian cycle in G, so for this reduction, we even have a stronger result.

If $R_2(G)$ is hamiltonian, then there are multiple ways in which the hamiltonian cycle can pass through the involved vertices in $R_2(G)$. Figure 14 shows all possible ways up to equivalence, and a translation to a hamiltonian path or cycle in G for every case. □

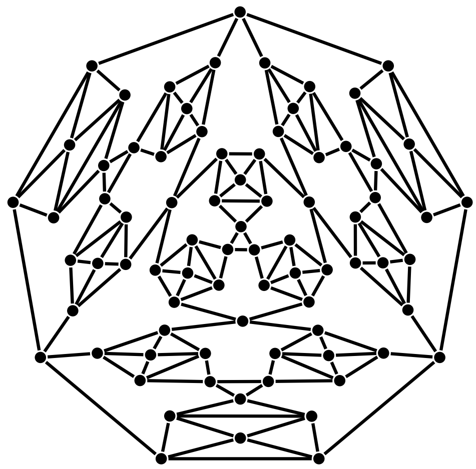


Figure 11: A non-traceable quartic polyhedron on 78 vertices.

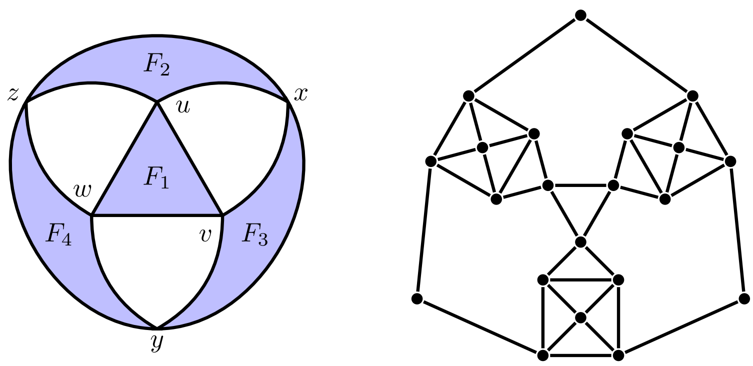


Figure 12: The structure of the graph shown in Figure 11. Each of the four triangles on the left-hand side is replaced with the fragment shown on the right-hand side. The vertices of degree 2 are identified at the intersections of the fragments to form vertices of degree 4.

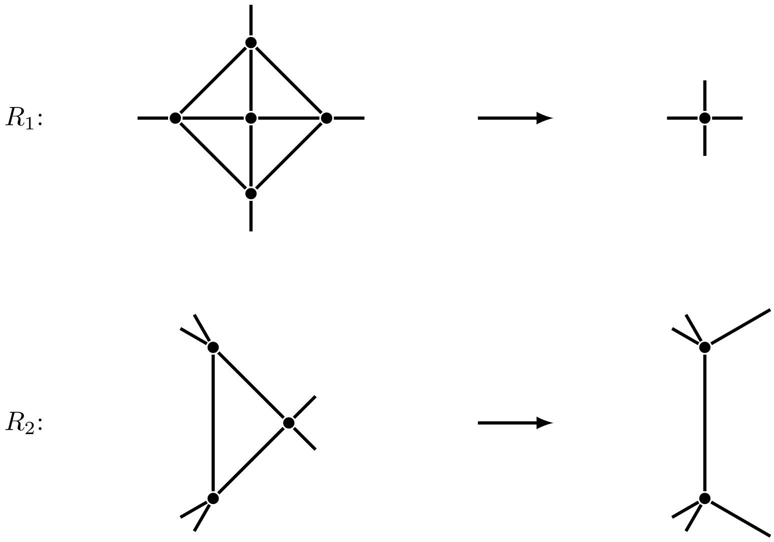


Figure 13: The two reductions that can be used to reduce a quartic polyhedron to a smaller quartic polyhedron.

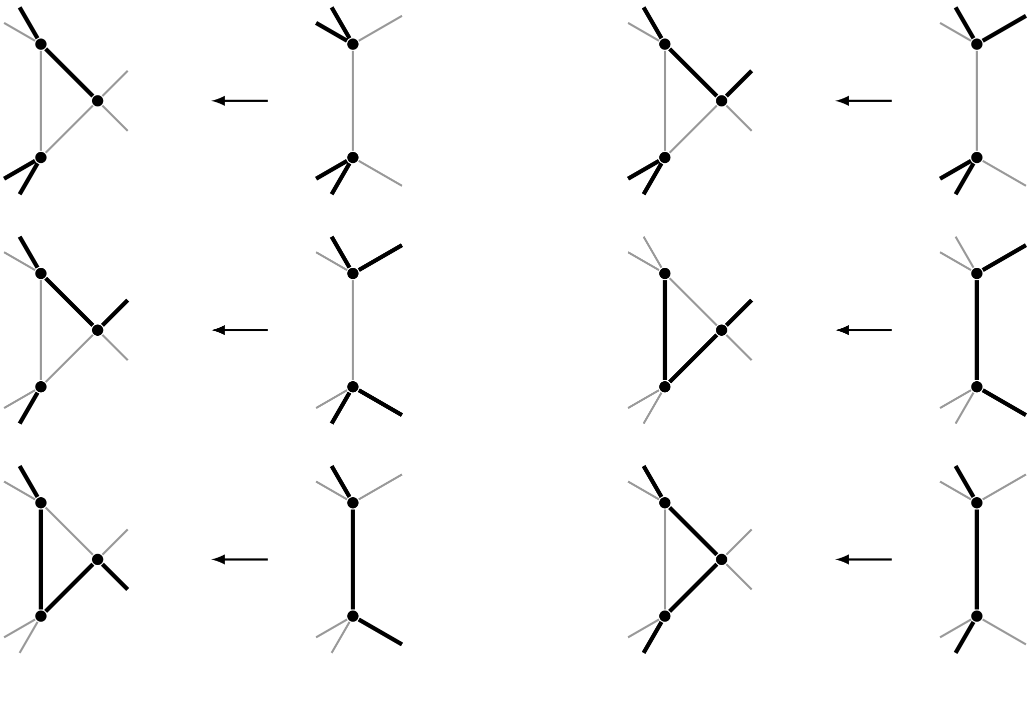


Figure 14: Translations of hamiltonian cycles in $R_2(G)$ to hamiltonian paths/cycles in G.

Corollary 2. *We have that*

$$p_4 \geq 36.$$

Proof. This immediately follows from Theorem 10 and Theorem 7. □

A graph is *homogeneously traceable* if for every vertex there exists a hamiltonian path beginning at that vertex [11]. In [44] T. Zamfirescu presented a cubic polyhedron on 44 vertices (two fewer than Tutte's example) that is not homogeneously traceable. Knorr [23] showed that all cubic polyhedra on at most 42 vertices are homogeneously traceable. We now prove a quartic analogue of T. Zamfirescu's result.

Theorem 11. *There is a quartic polyhedron on* 39 *vertices that is not homogeneously traceable.*

Proof. The quartic polyhedron shown in Figure 4 is not homogeneously traceable. More specifically, it has no hamiltonian path with endpoints in one of the vertices of the 3-cut that splits the graph into two equal halves. Denote the vertices in this 3-cut by u, v, w. Assume there is a hamiltonian path $\mathfrak{p}$ which starts at u, and denote the half that contains the neighbour of u on $\mathfrak{p}$ by H_1, and the other half by H_2. If H_1 is a strong half, then H_2 is a very weak half, which is not possible. Therefore the only option is that, without loss of generality, $\mathfrak{p} \cap H_1$ is a uv-path and a wz-path with z some vertex in H_1. This would however imply that H_2 is a weak half, which is again not possible. □

The above proof immediately gives us that there is a non-homogeneously traceable quartic polyhedron on n vertices for every $n \geq 39$.

We end this section with an upper bound for the so-called shortness coefficient of quartic polyhedra, directly inferrable from our work presented above. Given an infinite family of graphs $\mathcal{G}$, Grünbaum and Walther [14] introduced its *shortness coefficient* as

$$\rho(\mathcal{G}) = \liminf_{G \in \mathcal{G}} \frac{\text{circ}(G)}{|V(G)|},$$

where the *circumference* circ(G) denotes the length of a longest cycle in a given graph G. The purpose of ρ is to measure how far an infinite family of graphs is from being hamiltonian. For a survey, we refer the reader to [31].

Lemma 6. *Let G_k be a graph constructed from six (identical) 233-fragments, each on k vertices, by combining two copies of the 222-fragment shown in Figure 7. Let C be a cycle in G_k. Then the length of C is bounded above by $5k + 4$.*

Proof. The 222-fragment F from Figure 7 has $3k + 3$ vertices (assuming that F_1, F_2, F_3 each have k vertices), so we only need to concern ourselves with cycles C which contain vertices of both copies of F. The 3-cut $\{x, y, z\}$ (using the labels in Figure 7) splits C into two paths P_1 and P_2 each lying completely in one of the copies of F. At most one of these paths can contain all three vertices from the 3-cut $\{x, y, z\}$. Without loss of generality, we can assume that P_1 misses one of the vertices from the 3-cut. It follows from the proof of Lemma 3 that P_1 has length (i.e. number of edges) at most $2k + 2$, since it completely misses one of the copies of the 233-fragment except for at most one vertex. Since F has $3k + 3$ vertices, the length of P_2 is at most $3k + 2$. □

Theorem 12. *For the family of all quartic polyhedra $\mathcal{G}$, we have $\rho(\mathcal{G}) \leq 5/6$.*

Proof. Due to the above lemma, the circumference of the graph G_k is at most $5k + 4$. The graph G_k contains $6k + 3$ vertices. Therefore

$$\rho(\mathcal{G}) \leq \lim_{k \to \infty} \frac{5k + 4}{6k + 3} = \frac{5}{6}.$$

□

4 The Quintic Case

For results on hamiltonicity in 5-regular polyhedra, we refer the reader to work of Walther [42], as well as Harant, Owens, Tkáč, and Walther [15]. The best upper bounds to be found in the literature are due to Owens [29], who showed that $c_5 \leq 76$ and $p_5 \leq 128$. Although not explicitly mentioned in [29], the construction that shows that $c_5 \leq 76$ can also be used to show that there exists a non-hamiltonian quintic polyhedron on n vertices for each even $n \geq 76$. One step in the construction is to replace a vertex of degree 5 in a given multigraph by a vertex-deleted icosahedron. This step can actually be performed using any vertex-deleted quintic polyhedron as there are no further restrictions on the polyhedron. It follows from [16] that a quintic polyhedron on n vertices exists if and only if n is even and $n \geq 12$.

Theorem 13 (Owens [29])**.** *We have that*

$$c_5 \leq 76.$$

Furthermore, for each even $n \geq 76$ there is a non-hamiltonian quintic polyhedron on n vertices.

We shall now provide non-trivial lower bounds for both c_5 and p_5, and slightly improve Owens' upper bound for p_5.

Indubitably, certain subsets of the family of all quintic polyhedra must have been checked for being hamiltonian, but these verifications appear to not have been reported. A plugin for `plantri` which was also used to generate the table of counts in [16] yielded all quintic polyhedra up to 36 vertices. These were then checked for being hamiltonian with the program described in the quartic case. The result of these computations is the following theorem.

Theorem 14. *We have that*

$$c_5 \geq 38.$$

We now improve a bound of Owens [29].

Theorem 15. *There exists a quintic non-traceable polyhedron of order n for every even $n \geq 108$. In particular*

$$p_5 \leq 108.$$

Proof. We use the Fruchard graph G depicted in Figure 15. This graph is bipartite as shown in that figure. The eight white vertices are cubic, and the six black vertices are quartic. Due to the difference in the sizes of the parts, G has no path visiting all the white vertices. We will apply a transformation to construct a quintic non-traceable polyhedron from the Fruchard graph. Consider a maximum matching M of G, e.g. the matching shown with arrows in Figure 15, and mark the white vertices incident to this matching with α and the remaining two white vertices with β.

Replace each edge of M by a subgraph as shown in the top half of Figure 16, and replace each vertex labelled β by a subgraph as shown in the bottom half of Figure 16. It is easily verified that each hamiltonian path of the resulting quintic polyhedron can be translated back to a path that visits all white vertices (it might miss some black vertices), however, as noted before, such a path does not exist.

The subgraphs have respectively thirteen and fifteen vertices, so the resulting quintic polyhedron has $6 \cdot 13 + 2 \cdot 15 = 108$ vertices. By using the operation shown in Figure 17 for one (respectively two, three, or four) edges in M, we can construct a non-traceable quintic polyhedron with 110 (respectively 112, 114, or 116) vertices, since the replacing subgraph in that operation has fifteen instead of thirteen vertices. We can further increase the number of vertices by an arbitrary multiple of 10 by successively replacing any vertex of degree 5 except the vertices corresponding to the black vertices in the Fruchard graph (i.e. the vertices of degree 2 in the subgraphs replacing the edges of M) by a vertex-deleted icosahedron. □

The results we obtain in the theorem above are best possible with this approach: On one hand, if we take a subgraph on n vertices that is a candidate to replace a vertex labelled β, and connect the three vertices of degree 4 to a new vertex, then we obtain a polyhedron with one vertex of degree 3, and all other vertices of degree 5. Next we take a icosahedron, and subdivide a single edge. Finally we identify the vertex of degree 3 and the vertex of degree 2, so we obtain a quintic plane graph with vertex-connectivity 1 and $n+13$ vertices. The smallest quintic plane graph with vertex-connectivity 1 has 28 vertices [16], so the subgraph with 15 vertices in the bottom half of Figure 16 is smallest possible.

On the other hand, if we consider a subgraph on n vertices that is a candidate to replace an edge of M, and connect the two vertices of degree 4 and the vertex of degree 2 to a new vertex, then we obtain a polyhedron with two adjacent vertices of degree 3, and all other vertices of degree 5. We wrote a straight-forward plugin for `plantri` to generate these polyhedra. The program took less than a second on a standard laptop to determine that the smallest such polyhedron has fourteen vertices, so the subgraph with thirteen vertices in the top half of Figure 16 is smallest possible.

As an immediate corollary of Theorem 14, we obtain the following.

Corollary 3. *We have that*

$$p_5 \geq 38.$$

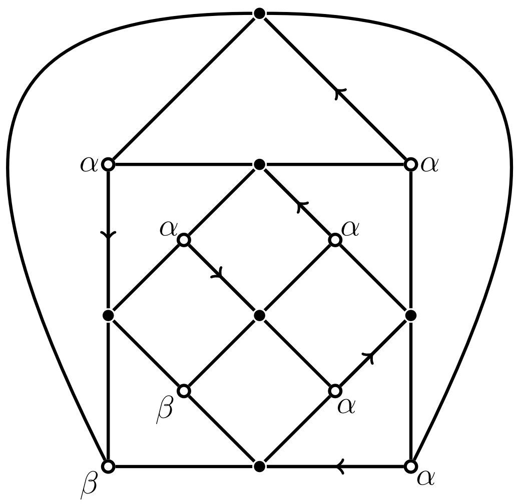


Figure 15: The Fruchard graph, a non-traceable polyhedron.

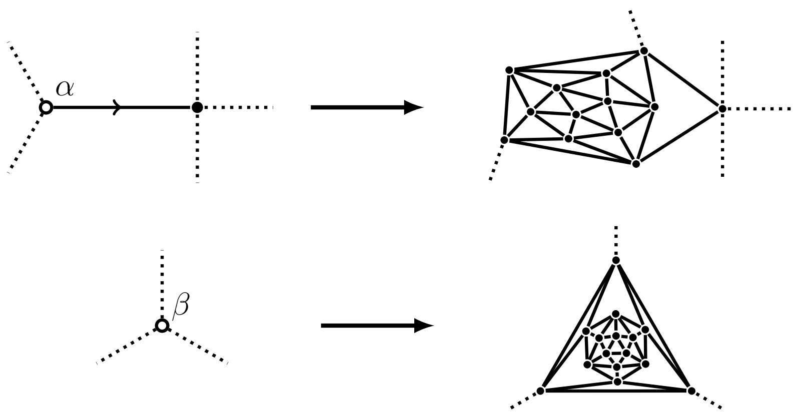


Figure 16: The two operations used to construct a quintic non-traceable polyhedron from the Fruchard graph.

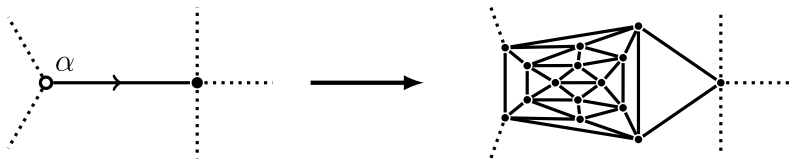


Figure 17: An alternative form of the operation shown in the top half of Figure 16.

Acknowledgements

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