Hamiltonian properties of polyhedra with few 3-cuts— A survey

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Abstract

We give an overview of the most important techniques and results concerning the hamiltonian properties of planar 3-connected graphs with few 3-vertex-cuts. In this context, we also discuss planar triangulations and their decomposition trees. We observe an astonishing similarity between the hamiltonian behavior of planar triangulations and planar 3-connected graphs. In addition to surveying, (i) we give a unified approach to constructing non-traceable, non-hamiltonian, and non-hamiltonian-connected triangulations, and show that planar 3-connected graphs (ii) with at most one 3-vertex-cut are hamiltonian-connected, and (iii) with at most two 3-vertex-cuts are 1-hamiltonian, filling two gaps in the literature. Finally, we discuss open problems and conjectures.

Keywords: planar, 3-connected, polyhedron, triangulation, decomposition tree, hamiltonian, traceable, hamiltonian-connected
2010 MSC: 05C10, 05C45, 05C38, 05C40, 05C42

1. Introduction

Hamiltonian cycles constitute a major branch of research in modern graph theory. Historically, such spanning cycles were already studied by Euler in 1759 in the closed variant of the "knight's tour" problem [22], which in non-mathematized form was treated by scholars in 9th century Baghdad and Kashmir. Applications of hamiltonicity range from combinatorial optimization and operations research [46] over coding theory [5], molecular chemistry [48] and

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fault-tolerance in networks [38] to music [1]. As far as we know, the hamiltonicity of polyhedra was first investigated in the 1850's, when Kirkman and Hamilton re-invented the concept independently [44, 26]—for an excellent historical account, we refer the reader to [6]. Due to Steinitz' Theorem [61], we will here view polyhedra as 3-connected planar graphs, and *triangulations* shall be polyhedra in which every face is a triangle.

Investigating the hamiltonian properties of polyhedra gained popularity due to its connection with one of the most famous problems in all of mathematics: in 1884, Tait conjectured [62] that every cubic (i.e. 3-regular) polyhedron is hamiltonian, and had this conjecture been true, it would have implied the Four Color Theorem (which, then, was itself open). However, the conjecture turned out to be false and the first to construct a counterexample was Tutte in 1946, see [69]. The smallest counterexample is due to Noble Prize recipient Lederberg (and independently, Bosák and Barnette [37]) and has order 38. Deciding whether a given graph is hamiltonian is one of Karp's 21 NP-complete problems [42]. It remains NP-complete restricted to cubic polyhedra [27].

This survey will focus on the hamiltonian properties of polyhedra with few 3-cuts. (All cuts in this survey are vertex-cuts.) The first result in this line of research was obtained by Whitney in 1931: he showed that all 4-connected planar triangulations are hamiltonian [73]. In 1956, this was generalized by Tutte to all 4-connected polyhedra in his seminal work [70]. The works of Whitney and Tutte have seen many extensions; for instance to graphs of non-zero genus, both in the orientable and non-orientable case, graphs with small crossing number, as well as treating hamiltonian properties other than hamiltonicity itself, for instance traceability, hamiltonian-connectedness, or 1-hamiltonicity. Concerning these extensions, we refer the reader to the recent articles by Kawarabayashi and Ozeki [43], and Ozeki and Zamfirescu [57], and restrict ourselves to mentioning here a major conjecture of Grünbaum [29] and Nash-Williams [51]: *Every 4-connected toroidal graph is hamiltonian.* Since the result of Tutte, all of these theorems—be it in the planar or non-planar case—were proved using technical lemmas revolving around what are now known as Tutte-subgraphs and bridges. In this survey we shall concentrate on the planar case: in Table 1 we provide an overview of the current state of hamiltonicity and related concepts in polyhedra with few 3-cuts.

In Section 2 we introduce the basic terminology and concepts used throughout this paper. We will give an overview of all negative results from Table 1 in Section 3, while in Section 4 we introduce Tutte subgraphs and present the positive results from Table 1, as well as the tools used to prove them. In Section 5 we direct our attention to the subclass of triangulations and ascertain the changes with respect to the polyhedral case—for many readers, this will reveal a surprising insight: triangulations and polyhedra exhibit near-identical behavior concerning their hamiltonian properties, despite the fact that their sizes (i.e. the number of edges) may vary drastically for a fixed order. We finish in Section 6 by surveying open problems and partial results.

	Traceable	Hamiltonian	Hamiltonian-connected	2-edge-hamiltonian-connected	1-hamiltonian	2-hamiltonian
0	✓	✓(A,B)	✓(E)	✓(H)	✓(C)	✓(F)
1	✓	✓(D)	✓(K,L)	✗	✓	✗
2	✓	✓		✗	✓(L)	✗
3	✓	✓(G,J)		✗		✗
4	✓(J)		✗(K)	✗	✗	✗
5			✗	✗	✗	✗
6		✗(I)	✗	✗	✗	✗
7		✗	✗	✗	✗	✗
8	✗(J)	✗	✗	✗	✗	✗

A. *Whitney (1931) [73] (for triangulations)*
B. Tutte (1956) [70]
C. Nelson (1973) [52]
D. Thomassen (1978) [66]
E. Thomassen (1983) [68]
F. Thomas and Yu (1994) [64]
G. *Jackson and Yu (2002) [40] (for triangulations)*
H. Ozeki and Vrána (2014) [55]
I. Brinkmann, Souffriau, and Van Cleemput (2016) [10]
J. Brinkmann and Zamfirescu (2016) [11]
K. *Van Cleemput (2016) [72] (for triangulations)*
L. Ozeki, Van Cleemput, and Zamfirescu (this paper)

Table 1: Hamiltonian properties of polyhedra with few 3-cuts. Green cells (marked ✓) indicate that every polyhedron with the specified number of 3-cuts satisfies the property, red cells (marked ✗) signify that there exist polyhedra which do not satisfy the property, and blank cells designate unknown behavior.

2. Basic terminology and concepts

All graphs considered are finite, simple, and undirected. If $G = (V(G), E(G))$ is a graph and $X \subseteq \binom{V(G)}{2}$, then we denote the graph $(V(G), E(G) \cup X)$ by $G \cup X$, and the graph $(V(G), E(G) \setminus X)$ by $G \setminus X$. If $Y \subset V(G)$, then $G - Y$ denotes the graph induced by G on the vertices $V(G) \setminus Y$, i.e. $G[V(G) \setminus Y]$. Abusing notation, we put $G - y = G - \{y\}$ for $y \in V(G)$. For an edge between vertices v and w we write vw. We say that a planar graph G is *plane* if we consider an embedding of G in the plane. Recall that by a theorem of Whitney polyhedra have unique planar embeddings [74].

A *hamiltonian cycle* (*hamiltonian path*) in a graph G is a cycle (path) which spans the vertex set $V(G)$. A graph is called *hamiltonian* if it contains a hamiltonian cycle, *traceable* if it contains a hamiltonian path, and *hamiltonian-connected* if there is a hamiltonian path connecting any two vertices in the graph. For a non-negative integer k, we say that a graph of order at least $k + 3$ is k-*hamiltonian* if the removal of any set of k vertices from the graph leaves a graph which is hamiltonian. We can consider hamiltonian as being 0-hamiltonian.

A *linear forest* is a forest in which each component is a path. For a non-negative integer k, we say that a graph G of order at least $k + 1$ is k-*edge-hamiltonian-connected* if for any $X \subset \{x_1x_2 : x_1, x_2 \in V(G)\}$ with $|X| \leq k$ and the graph $(V(X), X)$ is a linear forest, $G \cup X$ has a hamiltonian cycle containing all edges of X. We can consider hamiltonian as being 0-edge-hamiltonian-connected, and hamiltonian-connected as being 1-edge-hamiltonian-connected.

The concept was first introduced for $k = 2$ in [45], and mentioned in its general form in [55].

For the rest of this section we treat certain necessary conditions for hamiltonicity. Note that the remainder of this section is also valid for non-planar graphs. Most negative entries in Table 1 rely on the same type of property to show that the graph is not hamiltonian: deleting some set of vertices splits the graph into too many components. In what follows we will denote the number of components of a graph G by $c(G)$.

Chvátal introduced the *toughness* of a graph in [17]. Intuitively, it is a measure for how tightly the graph holds together. Rigorously, the toughness $t(G)$ of a non-complete graph G is defined as

$$t(G) = \min \left\{ \frac{|X|}{c(G - X)} : X \subseteq V(G), c(G - X) > 1 \right\}.$$

The toughness of a complete graph is convened to be ∞. A graph G is *t-tough* whenever $t \leq t(G)$. This gives us a necessary condition for hamiltonicity.

Theorem 2.1 (Chvátal [17], 1973). *Every hamiltonian graph is* 1*-tough.*

This theorem follows easily from observing that by removing t vertices in a cycle, the cycle can be split into at most t components. This reveals an a priori surprising connection between the cardinality of a cut, the number of components obtained when removing the cut, and the hamiltonicity of the graph. We shall see that in order to also investigate other hamiltonian properties, the following notion is useful.

In [41] the *scattering number* $s(G)$ of a non-complete graph G is defined as

$$s(G) = \max\{c(G - X) - |X| : X \subseteq V(G), c(G - X) > 1\}.$$

In [36] it is noted that $s(G) \leq 1$, respectively 0 or -1, is a necessary condition for G to be traceable, respectively hamiltonian or hamiltonian-connected. In [55] this is further extended to show that $s(G) \leq -k$ is a necessary condition for G to be k-hamiltonian or k-edge-hamiltonian-connected.

Observation 2.2. *Let G be a graph.*

(i) If G is k-edge-hamiltonian-connected, then G is $(k+2)$-connected.

(ii) If G is k-hamiltonian, then G is $(k+2)$-connected.

Proof. Let $S = \{v_1, \dots, v_{k+1}\}$ be a $(k+1)$-cut in G.

(i) If the graph $G \cup \{v_1v_2, \dots, v_kv_{k+1}\}$ has a hamiltonian cycle containing the edges $v_1v_2, \dots, v_kv_{k+1}$, then by removing the cut S from the graph the remainder of the hamiltonian cycle forms a hamiltonian path of the graph $G \setminus S$. This contradicts the fact that S is a $(k+1)$-cut.

(ii) The graph $G \setminus \{v_1, \dots, v_k\}$ is not 2-connected and thus not hamiltonian.

Lemma 2.3. *Let G be a non-complete graph.*

(i) If G is traceable, then $s(G) \leq 1$.

(ii) If G is k-edge-hamiltonian-connected, then $s(G) \leq -k$.

(iii) If G is k-hamiltonian, then $s(G) \leq -k$.

Proof. (i) Consider a hamiltonian path in G. Removing any set of vertices of size k will split the graph into at most $k+1$ components, and this number will be even smaller if any of the removed vertices are adjacent on the hamiltonian path, or any of the removed vertices are end-vertices of the path, or any of the components are connected by edges that are not on the hamiltonian path.

(ii) Let $X \subseteq V(G)$ with $c(G-X) > 1$. By Observation 2.2, we have $|X| \geq k+2$. Since there is a hamiltonian path after removing any $k+1$ vertices, we obtain $c(G-X) \leq |X| - (k+1) + 1 = |X| - k$. This gives us $\max\{c(G - X) - |X| : X \subseteq V(G), c(G-X) > 1\} \leq -k$.

(iii) This proof is very similar to the one above and therefore omitted.

3. Non-hamiltonian polyhedra

In this section we discuss all negative entries in Table 1. Note that all graphs given in this section are not only polyhedra, but in fact triangulations (which are also known as maximal planar graphs, since the addition of any edge renders the graph non-planar).

Theorem 3.1 (Brinkmann, Souffriau, and Van Cleemput [10], 2016). *For each $k \geq 6$ there exists a non-hamiltonian triangulation with exactly k 3-cuts.*

These non-hamiltonian triangulations are constructed by taking the basis graph shown in the center in Figure 1 and subdividing each face, including the outer face, with a vertex. Since the number of faces in the basis graph is greater by one than the number of vertices, if we remove all vertices of the basis graph, then the number of components is precisely one greater than the number of vertices that were removed. Hence, the scattering number is at least 1. We shall come back to this idea in Section 5.

Theorem 3.2 (Van Cleemput [72], 2018+). *For each $k \geq 4$ there exists a triangulation with exactly k 3-cuts which is not hamiltonian-connected.*

Again the result is based on describing a family of triangulations which have a scattering number that is too high. These graphs are constructed by taking the basis graph shown on the left in Figure 1 and subdividing each face with a vertex. Since the number of faces in the basis graph is equal to the number

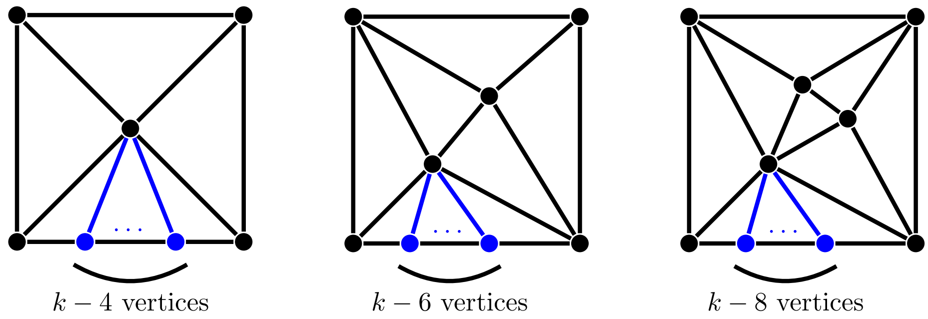

Figure 1: The bases for a non-hamiltonian-connected (left), non-hamiltonian (center), or non-traceable (right) polyhedron with k 3-cuts for $k \geq 4$, respectively $k \geq 6$, or $k \geq 8$. The non-hamiltonian-connected—respectively non-hamiltonian or non-traceable—polyhedron is obtained by subdividing each face of the basis with a vertex, and the 3-cuts correspond to the vertices of the facial triangles of this basis.

of vertices, if we remove all vertices of the basis graph, then the number of components is equal to the number of vertices that were removed. Hence the scattering number is at least 0.

A scattering number of at most -1 is a necessary condition for a graph to be 1-hamiltonian, so this last family gives us the negative entries in the column corresponding to the 1-hamiltonian case, as well.

Theorem 3.3. *For each $k \geq 4$ there exists a triangulation with exactly k 3-cuts which is not 1-hamiltonian.*

The previous constructions can be extended in order to give a family of non-traceable graphs with k 3-cuts for $k \geq 8$.

Theorem 3.4 (Brinkmann and Zamfirescu [11], 2018+). *For each $k \geq 8$ there exists a triangulation with exactly k 3-cuts which is not traceable.*

The triangulations we describe here are not exactly the graphs mentioned by Brinkmann and Zamfirescu, but using our construction provides a systematic approach regarding examples for traceability, hamiltonicity, and hamiltonian-connectedness. The non-traceable graphs are constructed by taking the basis graph shown on the right in Figure 1 and subdividing each face with a vertex. Since the number of faces in the basis graph is two higher than the number of vertices, if we remove all vertices of the basis graph, then the number of components is two higher than the number of vertices that were removed. Hence the scattering number is at least 2.

The negative entries in Table 1 for 2-edge-hamiltonian and 2-hamiltonian can be explained in a more general setting, since we do not need to rely on the planarity of the graph: they follow immediately from Observation 2.2.

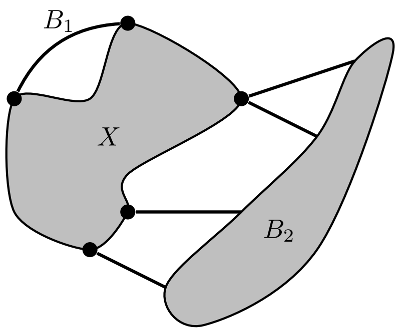

Figure 2: An example of a Tutte subgraph X together with two X-bridges: B_1 is a trivial X-bridge, and B_2 is a non-trivial X-bridge.

4. Tutte subgraphs

The field of research discussed in this survey was initiated by a result of Whitney for triangulations and its generalization by Tutte. For the former we refer to the next section which discusses the results for triangulations in more detail. The latter is given below.

Theorem 4.1 (Tutte [70], 1956). *4-connected polyhedra are hamiltonian.*

The class of subgraphs used by Tutte to establish this result have later been called Tutte subgraphs. They have proven to be one of the few powerful tools for proofs concerning hamiltonian properties of graphs. We therefore present the basis of this technique here. These ideas were first used by Tutte in [70]—later he gave a more detailed description of his approach in [71].

Let X be a subgraph of a graph G. An *X-bridge* of G is either a single edge of $G \setminus E(X)$ with both ends on X, or a component B of $G - V(X)$ together with the edges with one end vertex in X and one in B and the end vertices of these edges in X. The bridges of the first type are called *trivial*. The bridges of the second type are called *non-trivial*. If B is an X-bridge of G, then the *attachments* of B are the vertices in $V(B) \cap V(X)$. The subgraph X is a *Tutte subgraph* if every X-bridge of G has at most three attachments. Note that this additional condition is automatically met for trivial bridges. See Figure 2 for an example of a Tutte subgraph.

Let Y be another subgraph of G. The subgraph X is a *Y-Tutte subgraph* if X is a Tutte subgraph and every X-bridge that contains an edge of Y has at most two attachments. See Fig. 3 for an example of a Y-Tutte subgraph. Note that this additional condition is automatically met for trivial bridges. If X is a Tutte subgraph and X is a cycle (path), then we refer to X as a *Tutte cycle* (*Tutte path*).

One may think that the definition of Tutte subgraph (and also Tutte cycle, path) is strange and complicated. This comes from highly technical reasons,

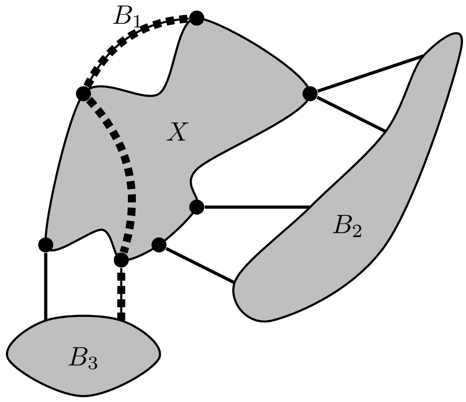

Figure 3: An example of a Y-Tutte subgraph X together with three X-bridges: B_1 is a trivial X-bridge, B_2 is a non-trivial X-bridge that does not contain an edge of Y and B_3 is a non-trivial X-bridge that contains an edge of Y. The subgraph Y is shown as a dotted path of length 3.

where the properties are chosen to fit for the inductive argument to show the existence of such subgraphs. We briefly explain the strategy to show the following theorem. See [21] for details.

Theorem 4.2 (Thomassen [68], 1983). *Let G be a plane graph with outer facial cycle D, $x \in V(D)$, $y \in V(G) \setminus \{x\}$, and let $e \in E(D)$. If G contains a path from x to y through e, then G contains a D-Tutte path from x to y through e.*

We use induction on $|V(G)|$ to prove Theorem 4.2. By considering block decomposition, it is easy to reduce the proof to the case where G is 2-connected. Furthermore, suppose that G contains a 2-cut S, where G_1 and G_2 are the components of $G - S$, such that neither of them consists of only the vertex x or y. Then we use the induction hypothesis on $G[V(G_1) \cup S]$ and/or $G[V(G_2) \cup S]$ with an appropriate choice for x, y, e, and find a desired path in G by combining the obtained paths if necessary. Thus, we may further assume that G has no 2-cut except for the neighborhood of x or y. Let $e = uv$, and let Q be a subpath of D that connects x and u, but does not contain v. If Q contains y, then change the role of x and y. This choice and the limited number of 2-cuts allow us to see that $G' = G - V(Q)$ is connected. Let D' be the outer facial cycle of G' and let e' be the edge both in D and D' such that an end vertex of e' is adjacent with x in D. Then we use the induction hypothesis on G' to find a D'-Tutte path P' in G' from v to y through e'. Consider each $(Q \cup P')$-bridge B of G. If B has at most three attachments, then we leave it without any change. Suppose that B contains at least four attachments. Since P' is a D'-Tutte path in G', $B - V(Q)$ has at most three attachments on P', and hence B has an attachment on Q. This means that $B - V(Q)$ contains an edge in D'. Thus, again since P' is a D'-Tutte path in G', B has at least two attachments on Q. Let x_B (y_B, respectively) be the attachment of B on Q that is closest to x (to u, respectively) on Q. By induction hypothesis, B contains a Tutte path P_B from x_B to y_B through an appropriate edge. Then we detour Q from x_B to y_B with P_B. Since G is a plane graph, we can perform this detouring independently for each $(Q \cup P')$-bridge B of G with at least four attachments. This finally gives us a desired D-Tutte path.

Historically, the first explicit strengthening of Tutte's famous result towards polyhedra with few 3-cuts was the following. We shall see that, implicitly, this had been done earlier by Nelson.

Theorem 4.3 (Thomassen [66], 1978). *Polyhedra with at most one* 3*-cut are hamiltonian.*

A surprising application of this result and Thomassen's motivation for establishing it, is the fact that every planar *hypohamiltonian* graph—these are non-hamiltonian graphs in which every vertex-deleted subgraph is hamiltonian—contains a cubic vertex [66]. Whether the statement remains true if we drop the word 'planar' is a 40-year old open problem of Thomassen [66]. With the theorem which now follows one can even show that planar hypohamiltonian graphs must contain at least four cubic vertices [75].

Theorem 4.4 (Brinkmann and Zamfirescu [11], 2018+). *Polyhedra with at most three* 3*-cuts are hamiltonian.*

For the proof of the above theorem we refer the reader to [11]. One of the proof's critical components is Lemma 4.5—we need one more definition to state it. A *circuit graph* is a pair (G, D) such that G is a 2-connected plane graph and D is a facial cycle of G such that for any 2-cut S of G, each component of $G - S$ contains a vertex of D. In a polyhedron, each facial cycle has this property.

Lemma 4.5 (Jackson and Yu [40], 2002). *Let (G, D) be a circuit graph, x, y be vertices of G, and $e \in E(D)$. Then G contains a D-Tutte cycle through e, x, and y.*

Tutte's result has been strengthened in other directions, as well. The following theorem follows directly from Theorem 4.2.

Theorem 4.6 (Thomassen [68], 1983). 4*-connected polyhedra are hamiltonian-connected.*

Thomassen's proof of this theorem contained a small omission which was later corrected by Chiba and Nishizeki [15].

Sanders proved a generalization of this theorem.

Theorem 4.7 (Sanders [59], 1997). *Let G be a 4-connected polyhedron. Let x, y be two distinct vertices of G. Let e be an edge of G different from xy. Then G contains a hamiltonian path from x to y through e.*

Corollary 4.8 (Sanders [59], 1997). *Let G be a 4-connected polyhedron. Let e_1, e_2 be two distinct edges of G. Then G contains a hamiltonian cycle through e_1 and e_2.*

By counting the different ways we can select two edges at a vertex of maximum degree, we immediately get the following corollary.

Corollary 4.9. *Let G be a 4-connected polyhedron. Let Δ be the maximum degree of G. Then G contains at least $\binom{\Delta}{2} \geq 6$ hamiltonian cycles.*

With additional work, the following corollary can be derived from a lemma of Sanders, which we now state.

Lemma 4.10 (Sanders [59], 1997). *Let G be a 2-connected plane graph with outer facial cycle D, let $x, y \in V(G)$ with $x \neq y$ and let $e \in E(D)$. Then G contains a D-Tutte path from x to y through e.*

Corollary 4.11. *Polyhedra containing at most one 3-cut are hamiltonian-connected.*

Proof. Let G be a polyhedron containing at most one 3-cut, and let $x, y \in V(G)$ with $x \neq y$. It suffices to find a hamiltonian path in G from x to y. By Theorem 4.6 we can assume that G is not 4-connected.

Let $S = \{u, v, w\}$ be the unique 3-cut of G. By symmetry, we may assume that $u \neq x$ and $u \neq y$. Since S is a 3-cut of G, G contains a facial cycle D_{uv} with $u, v \in V(D_{uv})$. Let $D'_{uv} = D_{uv}$ if $uv \notin E(D_{uv})$; otherwise, let D'_{uv} be the facial cycle of $G \setminus \{uv\}$ that is not facial in G. (So $E(D_{uv}) \setminus \{uv\} \subset E(D'_{uv})$.) This implies that the two non-trivial $(S, \emptyset)$-bridges in G both contain an edge in D'_{uv}, where $(S, \emptyset)$ is the graph with the three vertices u, v, w and no edges. Similarly to the above, we consider a facial cycle D_{uw} of G with $u, w \in V(D_{uw})$, and D'_{uw} from D_{uw}. Since G is 3-connected and S is the unique 3-cut of G, we have $V(D'_{uv}) \cap V(D'_{uw}) = \{u\}$. Therefore, by symmetry, we may assume that $|V(D'_{uv}) \cap \{x, y\}| \leq 1$.

Let $G' = G$ if $uv \notin E(D_{uv})$; otherwise let $G' = G \setminus \{uv\}$. Note that in either case, G' is 2-connected and D'_{uv} is a facial cycle of G'. We re-embed G' into the plane so that D'_{uv} is the outer facial cycle. Since $|\{x, y\} \cap V(D'_{uv})| \leq 1$, there exists an edge e in D'_{uv} such that e is incident with u and the other end-vertex of e is neither x nor y.

By Lemma 4.10, G' contains a D'_{uv}-Tutte path P from x to y through e. By the choice of e, the two end-vertices of e, x and y are pairwise distinct, and hence $|P| \geq 4$. We show below that P is a hamiltonian path in G', and hence also in G.

Suppose that G' contains a non-trivial P-bridge B. Note that B contains at most three attachments on P. Since $|P| \geq 4$, we have $V(P) \setminus V(B) \neq \emptyset$.

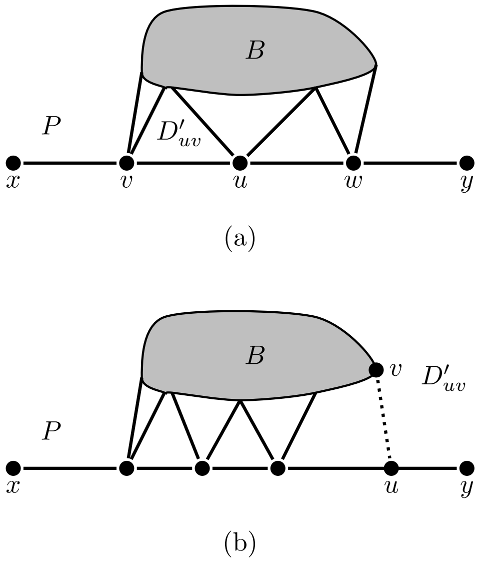

Figure 4: The two possibilities for a non-trivial P-bridge B in the proof of Corollary 4.11

Therefore, the attachments of B form a cut of order at most 3 of G' separating non-attachments of B from $V(P) \setminus V(B)$. Since S is the unique 3-cut of G, one of the following two must occur:

(a) The attachments of B are exactly u, v, and w (see Figure 4(a)), or

(b) the vertex v is contained in $B \setminus V(P)$, and the edge uv connects $B \setminus V(P)$ to $P \setminus V(B)$. (See Figure 4(b). Note that $u \in V(P)$ since $e \in E(P)$.)

In either case, we see that B contains an edge in D'_{uv}. Since P is a D'_{uv}-Tutte path in G', B contains at most two attachments, and hence the attachments of B form a cut of order at most 2 of G'. Since G is 3-connected, (b) must occur, but this implies that G contains a 3-cut different from S, a contradiction.

Therefore, G' contains no non-trivial P-bridge, and hence P is a hamiltonian path in G' and also in G.

Using a recent result of Ozeki and Vrána, we shall later give another proof of Corollary 4.11.

Let G be a polyhedron and $S = \{u, v, w\}$ be a 3-cut in G. If G' is a component of $G - S$, then $G[V(G') \cup S]$ is called a *closed component* of $G - S$. (Thomassen calls $G[V(G') \cup S]$ a 3-*fragment* with *vertices of attachment* S [65].) If G'' is a closed component of $G - S$, then $(V(G''), E(G'') \cup \{uv, vw, wu\})$ is called an *edge closed component* of $G - S$.

While in triangulations, which we deal with in the other main section of this survey, 3-cuts are separating triangles that lie properly inside each other, the relative position in polyhedra can be more complicated: in a polyhedron, vertices of a 3-cut S may end up in different edge closed components of a 3-cut S'. This makes it worthwhile to explicitly state Lemma 4.12 (vii), which is trivial for triangulations.

Lemma 4.12 (Brinkmann and Zamfirescu [11], 2018+). *Let G be a polyhedron with k 3-cuts and S be a 3-cut in G. Let G' be an edge closed component of $G - S$.*

(i) $G - S$ has exactly two components.

(ii) G' is planar and the vertices of S form a facial triangle in the (unique) embedding of G'.

(iii) Edge closed components of $G - S$ are polyhedra.

(iv) For any two vertices in G' that are not both in S, there are at least as many vertex disjoint paths joining them in G' as there are in G.

(v) Each 3-cut in G' is also a 3-cut in G.

(vi) The number of 3-cuts in all edge closed components of $G - S$ together is at most $k - 1$.

(vii) G contains a 3-cut S' such that at least one edge closed component of $G-S'$ has no 3-cuts, i.e. the edge closed component is 4-connected or isomorphic to K_4.

Lemma 4.13 (Brinkmann and Zamfirescu [11], 2018+). *If all polyhedra with at most k 3-cuts are hamiltonian, then all polyhedra with at most $k + 1$ 3-cuts are traceable.*

Nelson showed in his master's thesis that the techniques of Tutte could be strengthened to also handle the case of 1-hamiltonicity. This was then further improved by Thomas and Yu.

Theorem 4.14 (Nelson [52], 1973). *4-connected polyhedra are 1-hamiltonian.*

Theorem 4.15 (Thomas and Yu [64], 1994). *4-connected polyhedra are 2-hamiltonian.*

This theorem settled a conjecture of Plummer [58]. (In the same paper, Thomas and Yu solved a conjecture of Grünbaum [29] by proving that 4-connected projective planar graphs are hamiltonian.) In fact, Thomas and Yu proved the following lemma, which is slightly stronger than Theorem 4.15.

Lemma 4.16 (Thomas and Yu [64], 1994). *Let G be a 4-connected polyhedron, let x, y be two vertices of G, and let $H = G - \{x, y\}$. Let F be the face of H containing x, and let e be an edge of F. Then there exists a hamiltonian cycle in H that contains e.*

Evidently, the result of Thomas and Yu cannot be strengthened to "3-hamiltonian". In order to emphasize the connection of the preceding two theorems to this survey's topic, the following lemma is useful.

Lemma 4.17 (Brinkmann and Zamfirescu [11], 2018+). *A polyhedron with k 3-cuts contains a spanning subgraph that can be obtained from a 4-connected polyhedron by deleting at most k vertices.*

With Lemma 4.17, a direct corollary of Theorem 4.15 is that every polyhedron with at most one 3-cut is 1-hamiltonian, and that every polyhedron with at most two 3-cuts is hamiltonian. With this lemma, Thomassen's Theorem 4.3 in fact follows from Nelson's earlier result, Theorem 4.14.

Combining Theorem 4.4 and Lemma 4.13, we immediately obtain the following.

Corollary 4.18 (Brinkmann and Zamfirescu [11], 2018+). *Polyhedra with at most four 3-cuts are traceable.*

Combining Theorems 3.1, 3.4, 4.4, and 4.18, the obvious open questions here are:

- Are polyhedra with four or five 3-cuts hamiltonian?
- Are polyhedra with five, six or seven 3-cuts traceable?

Unfortunately, the hope for an easy answer to either question is extinguished by the following two results.

Theorem 4.19 (Brinkmann and Zamfirescu [11], 2018+). *Polyhedra with at most five 3-cuts are 1-tough, and polyhedra with at most seven 3-cuts have scattering number at most 1.*

Theorem 4.20 (Brinkmann and Zamfirescu [11], 2018+). *There are no non-hamiltonian polyhedra with at most five 3-cuts on up to 19 vertices, and there are no non-traceable polyhedra with at most seven 3-cuts on up to 18 vertices.*

The following result by Ozeki and Vrána is a strengthening of Corollary 4.8, which states that for any pair of edges in a 4-connected polyhedron there is a hamiltonian cycle through these edges. The following theorem shows that this remains valid even if these edges were not part of the polyhedron.

Theorem 4.21 (Ozeki and Vrána [55], 2014). *Every 4-connected polyhedron is 2-edge-hamiltonian-connected.*

We establish the following application of the above result—it yields a shorter proof of Corollary 4.11.

Proof of Corollary 4.11 using Theorem 4.21. Let G be a polyhedron with exactly one 3-cut. Lemma 4.17 states that there exists a 4-connected polyhedron G' containing a vertex v such that $G'-v$ is a spanning subgraph of G. Let $x,y \in V(G') \setminus \{v\}$. By Theorem 4.21 there exists a hamiltonian cycle C in $G' \cup \{xv, vy\}$ which contains the edges xv and vy. Now $C-v$ is a hamiltonian path in a spanning subgraph of G which has end-vertices x and y.

Theorem 4.22. *Polyhedra with at most two* 3*-cuts are* 1*-hamiltonian.*

Proof. Let G be a polyhedron with at most two 3-cuts, and consider $x \in V(G)$. It suffices to show that $G' = G - x$ is hamiltonian.

If G contains at most one 3-cut, then the theorem follows immediately from Theorem 4.15 and Lemma 4.17, so assume that G has exactly two 3-cuts: S_1 and S_2. Note that it is not necessarily so that $S_1 \cap S_2 = \emptyset$.

We first consider the case where a component of $G - S_i$ consists of only x for some i. By symmetry we can assume that x is a component of $G - S_1$, and we denote the vertices in S_1 by u, v and w. If one of the edges uv, uw, and vw is not present in G, we can add an edge from x to $G - (S \cup \{x\})$. The resulting graph G'' is then a polyhedron with at most one 3-cut. As above it follows immediately from Theorem 4.15 and Lemma 4.17 that G'' is 1-hamiltonian, so $G'' - x = G'$ is hamiltonian. On the other hand, if G contains uv, uw, and vw, then G' is a polyhedron with exactly one 3-cut, and it follows from Theorem 4.3 that G' is hamiltonian.

Next we consider the case where x is not a component of $G - S_i$ for any i. Let D' be the facial cycle in G' that is not facial in G. So, in G, D' separates x from $G - (\{x\} \cup V(D'))$. We may assume that G' is embedded into the plane so that D' is the outer facial cycle. Let $e \in E(D')$. Note that (G', D') is a circuit graph.

For each $i \in \{1, 2\}$, we pick a vertex z_i in $G' - S_i$, depending on whether $x \in S_i$ or not, as follows;

- If $x \notin S_i$, then S_i is a 3-cut of G' separating into the inside and the outside (i.e. the side containing D'). In this case, let z_i be a vertex in the inside of S_i with $z_i \notin S_i$.
- If $x \in S_i$, then $S_i - x$ is a 2-cut of G' that separates into two parts both containing an edge in D'. In this case, choose the one not containing e, and let z_i be a vertex in it with $z_i \notin S_i$.

Furthermore, since $S_1 \neq S_2$, we may assume that $z_1 \neq z_2$. Since for each i we have that z_i and e belong to different components of $G' - S_i$, the vertex z_i is not contained in e.

It follows from Lemma 4.5 that G' contains a D'-Tutte cycle C through e, z_1 and z_2. Since neither z_1 nor z_2 is an end vertex of e and $z_1 \neq z_2$, we have $|C| \geq 4$. We show that C is a hamiltonian cycle in G'.

Suppose that G' contains a non-trivial C-bridge B. Let A_B be the set of attachments of B on C. Since C is a Tutte cycle in G', we have $|A_B| \leq 3$. Since $|C| \geq 4$, we have $V(C) \setminus A_B \neq \emptyset$. Therefore, A_B is a cut of order at most 3 of G' separating $B \setminus A_B$ from $V(C) \setminus A_B$.

If B contains no edge in D', then A_B is also a $\leq$ 3-cut in G. This means that $A_B = S_i$ for some $i \in \{1, 2\}$, and B must be either the inside of S_i or the outside of S_i. However, in either case, this contradicts that C contains both e and z_i.

Therefore, B must contain an edge in D'. Since C is a D'-Tutte cycle, we have $|A_B| \leq 2$. Since G is 3-connected, we have $A_B \cup \{x\} = S_i$ for some $i \in \{1, 2\}$. In this case, by the choice of z_i, B has to contain either e or z_i, which contradicts that C contains both of them.

Therefore, G' contains no non-trivial C-bridge, and hence C is a hamiltonian cycle in G'.

It is also worth mentioning that Tutte subgraphs have been used for other purposes than pure hamiltonicity-related results. In [24, 25] Gao, Richter, and Yu used Tutte subgraphs to prove that any polyhedron has a closed 2-walk, i.e. a closed walk visiting each vertex at least once and at most twice. They also show that such a walk can be constructed such that the only vertices that are visited twice must belong to at least one 3-cut. In [14, 18] G. Chen, Fan, and Yu, respectively Cui, Hu, and Wang, use Tutte subgraphs to construct long cycles in 4-connected polyhedra. In [50] Nakamoto, Nozawa, and Ozeki show that any projective-planar graph has a 6-page book embedding, and that 3 pages, respectively 5 pages, suffice if the graph is 5-connected, respectively 4-connected. The proof idea is based on Tutte paths.

Proofs using Tutte subgraphs are inherently non-constructive. Recently Schmid and Schmidt [60] extended the decomposition of Gao, Richter, and Yu in such a way that all pieces into which the graph is decomposed are edge-disjoint. This allows them to give a polynomial-time algorithm to compute closed 2-walks in polyhedra.

We end this section by observing that in the case of cubic (i.e. 3-regular) polyhedra, no insight can be gained from the number of non-trivial 3-cuts (i.e. not neighborhoods of cubic vertices), since there exist non-hamiltonian cubic polyhedra in which every 3-cut is trivial, e.g. the infinite family of cubic hypohamiltonian polyhedra constructed by Thomassen in [67]. That indeed these graphs have only trivial 3-cuts follows from a result by Thomassen [66, Corollary 1]. However, the smallest non-hamiltonian cubic polyhedron, the Lederberg-Bosák-Barnette graph [37], does contain non-trivial 3-cuts. It has 38 vertices, and there are in total six non-hamiltonian cubic polyhedra on 38 vertices. The smallest non-hamiltonian cyclically 4-connected cubic polyhedra have 42 vertices and there are three such graphs [2]. The smallest non-hamiltonian cyclically 5-connected cubic polyhedron, the Grinberg graph, has 44 vertices [23, 3].

5. Triangulations and decomposition trees

Triangulations form a subclass of the polyhedra and are also called maximal planar graphs, since the addition of any edge creates a graph which is not planar. Thus triangulations are among the embedded graphs those with the greatest number of edges. A triangulation has the property that a minimal cut corresponds to a cycle. More specifically, a 3-cut corresponds to a non-facial triangle. (In stark contrast, we know nothing about the adjacencies of vertices in a 3-cut of a polyhedron.)

Before we turn our attention to triangulations with few separating triangles, let us first have a look at the other end of the spectrum, namely *Apollonian networks*. An Apollonian network is formed by recursively subdividing a triangle by adding a vertex inside the triangle and connecting it to the three vertices of the triangle. By starting with the tetrahedron this results in triangulations with the maximum number of separating triangles for a given number of vertices. It was erroneously claimed [63] that all Apollonian networks are hamiltonian. Moon and Moser showed [49] the existence of Apollonian networks whose longest path is of length $O(n^{\log 2/ \log 3})$, where $\log 2/ \log 3 = 0.63...$.

As was laid out earlier, in a natural development many of the results for general polyhedra were first stated and proved for triangulations. However, due to the many edges, one might expect that triangulations have much stronger hamiltonicity properties than general polyhedra. It however turns out that were we to reconstruct Table 1 for triangulations that we would obtain exactly the same table. Up to this survey, the only exception consisted of one additional positive entry for 1-hamiltonicity and two 3-cuts—due to the proof given above, this has now been settled to also be identical. Note that certain results were first proven for triangulations before being generalized to polyhedra. We will include these for completeness.

All examples given in Section 3 for the negative entries in Table 1, which concerns polyhedra, are in fact triangulations, so they also serve as examples for the negative entries in Table 1 restricted to triangulations.

5.1. Positive results for triangulations

We begin by focusing on hamiltonicity and later discuss related properties. The first result on hamiltonian triangulations is in fact the starting point of the topic discussed in this survey.

Theorem 5.1 (Whitney [73], 1931). *4-connected triangulations are hamiltonian.*

In [4], Asano, Kikuchi, and Saito give a simpler proof of above result and present a linear-time algorithm for finding a hamiltonian cycle in a 4-connected triangulation. It turns out that the same holds for 4-connected polyhedra, as shown by Chiba and Nishizeki [16]. Their approach is based on Thomassen's proof for hamiltonian-connectedness.

A strengthening of Theorem 5.1 was proven by C. Chen.

Theorem 5.2 (Chen [13], 2003). *Triangulations with one separating triangle are hamiltonian.*

This result is in fact implied by an older result of Thomassen, see Theorem 4.3. Furthermore, seven years before Chen's article, Böhme, Harant, and Tkáč published a stronger result. However both this paper and Thomassen's result appear to have escaped the attention of people working in this field.

Theorem 5.3 (Böhme, Harant, and Tkáč [8], 1996). *Triangulations with at most two separating triangles are hamiltonian.*

They also show that in a non-hamiltonian triangulation with three separating triangles these separating triangles are at distance at least two, and that there exists a non-hamiltonian triangulation with six separating triangles that are pairwise edge-disjoint. They conclude their paper by asking whether a non-hamiltonian triangulation with disjoint separating triangles exists. Applying Remark 1.4 given in the paper [7] by Böhme and Harant, we immediately obtain that for any non-negative integer d there exists a non-hamiltonian triangulation with seven separating triangles every two of which lie at distance at least d. This constitutes an affirmative answer to the aforementioned question. Recently, Ozeki and Zamfirescu [56] proved that this result holds even if we replace seven with six. It is unknown whether we can further lower this number to five or four, since every known triangulation with fewer than six separating triangles is hamiltonian—certainly, three or less separating triangles are impossible by Theorem 5.4.

Still one year before Chen's article, Jackson and Yu published a result that is stronger in several senses. We first give their result restated purely in terms of the number of separating triangles.

Theorem 5.4 (Jackson and Yu [40], 2002). *Triangulations with at most three separating triangles are hamiltonian.*

The theorem they actually prove is much more powerful, but in order to give this result we first need to introduce additional concepts. Note that Theorem 5.4 is generalized by Theorem 4.4.

The *decomposition tree* of a triangulation is defined as follows. A triangulation containing a separating triangle can be split into two triangulations: the subgraphs inside and outside of the separating triangle (here we assume that the triangulation is embedded in the plane—recall that this embedding is unique since triangulations greater than a triangle are planar and 3-connected [74]), with a copy of the separating triangle contained in both. (This is a special case of edge closed components, see Section 4.) By iteratively applying this procedure to a triangulation with k separating triangles, we obtain a collection of $k + 1$ triangulations without separating triangles. These pieces form the vertices of the decomposition tree. Two vertices share an edge if the corresponding pieces share a separating triangle in the original triangulation. It follows from the decomposition theory of Cunningham and Edmonds [19] that the decomposition tree is indeed a tree and is uniquely defined. As observed by Jackson and Yu [40], Herschel's graph provides a dramatic example establishing that a direct translation of the decomposition tree approach to the setting of polyhedra fails.

Theorem 5.5 (Jackson and Yu [40], 2002). *Let G be a triangulation with decomposition tree T. If T has maximum degree at most three, then G is hamiltonian.*

This result by Jackson and Yu is clearly much stronger, because even triangulations with an arbitrary number of separating triangles can be hamiltonian depending on the relative positions of these separating triangles. To prove this theorem they again first established a technical lemma on Tutte cycles.

Theorem 5.6 (Jackson and Yu [40], 2002). *Consider a 4-connected triangulation G and let T, T_1, T_2 be distinct triangles in G with $V(T) = \{u, v, w\}$. Then there exists a hamiltonian cycle C of G and edges $e_1 \in E(T_1)$, $e_2 \in E(T_2)$, such that uv, vw, e_1, e_2 are distinct and contained in C.*

Lemma 5.7 (Jackson and Yu [40], 2002). *Let G be a triangulation with decomposition tree T. Let F be a facial triangle of G and let e_1, e_2 be two edges of F. If T has maximum degree at most two, then G contains a hamiltonian cycle through e_1 and e_2.*

This lemma by Jackson and Yu implies the following corollary, which is not stated by them explicitly.

Corollary 5.8. *Let G be a triangulation with decomposition tree T. If T has maximum degree at most two, then G is 1-hamiltonian.*

Proof. If $\{u, v, w\}$ is the set of vertices of a facial triangle of G, then owing to Lemma 5.7 G contains a hamiltonian cycle C through uv and uw. The cycle $(C \setminus \{uv, uw\}) \cup \{vw\}$ is a hamiltonian cycle of $G - u$.

Corollary 5.9. *Triangulations with at most two separating triangles are 1-hamiltonian.*

Proof. If the triangulation has at most two separating triangles, then its decomposition tree has maximum degree at most two.

These results about the existence of hamiltonian cycles through specific edges in plane triangulations are also used for finding better lower bounds for the number of hamiltonian cycles in plane triangulations with few separating triangles [30, 12]. (The analogous problem for polyhedra is widely open, but seems extremely challenging.)

The approach above by Jackson and Yu warrants a closer study of the related hamiltonian properties in terms of the degree sequence of the decomposition tree of a triangulation. An overview of what we know is given in Table 2, where the degree sequence is the list of degrees of the decomposition tree sorted in descending order.

	Traceable	Hamiltonian	Hamiltonian-connected	2-edge-hamiltonian-connected	1-hamiltonian	2-hamiltonian
0	✓	✓	✓	✓	✓	✓
1,1	✓	✓	✓	✗	✓	✗
2,...	✓	✓		✗	✓	✗
3,...	✓	✓		✗		✗
4,{1,2,3},...	✓		✗	✗	✗	✗
4,4,...		✗	✗	✗	✗	✗
5,{1,2,3},...			✗	✗	✗	✗
5,{4,5},...		✗	✗	✗	✗	✗
6,{1,2,3},...		✗	✗	✗	✗	✗
6,{4,5,6},...	✗	✗	✗	✗	✗	✗
7,{1,2,3},...		✗	✗	✗	✗	✗
7,{4,5,6,7},...	✗	✗	✗	✗	✗	✗
8,...	✗	✗	✗	✗	✗	✗

Table 2: Hamiltonian properties of triangulations with decomposition trees with specified degree sequences. Green cells (marked ✓) indicate that every triangulation with a decomposition tree with the specified degree sequence satisfies the property, red cells (marked ✗) signify that there exist triangulations which do not satisfy the property, and blank cells designate unknown behavior.

Theorem 5.10 (Van Cleemput [72], 2018). *Let G be a triangulation. Let uv be an edge which is contained in all separating triangles of G. Then G is hamiltonian-connected.*

An immediate corollary of this result gives us Corollary 4.11 restricted to triangulations.

Corollary 5.11 (Van Cleemput [72], 2018). *Let G be a triangulation with exactly one separating triangle. Then G is hamiltonian-connected.*

Hamiltonicity implies traceability, so all triangulations with decomposition trees with maximum degree at most 3 are traceable owing to Theorem 5.5. A slight improvement in the case of traceability is easily shown.

Theorem 5.12. *Let G be a triangulation with decomposition tree T. If T has one vertex of degree 4 and all others of degree at most 3, then G is traceable.*

We have seen a series of positive results on hamiltonian properties of triangulations. A natural question is how to construct triangulations which are certainly non-hamiltonian. One prominent idea is due to Klee (whose approach generalizes the Apollonian networks we had seen earlier): starting from a graph with more faces than vertices, in each face add a vertex and join it to the face's vertices—the non-hamiltonicity of the triangulation one obtains follows from a simple toughness argument. This idea can also be used to construct higher-dimensional non-hamiltonian polytopes, which are sometimes called *Kleetopes*. For more on this subject, see [28]. We now discuss a different approach.

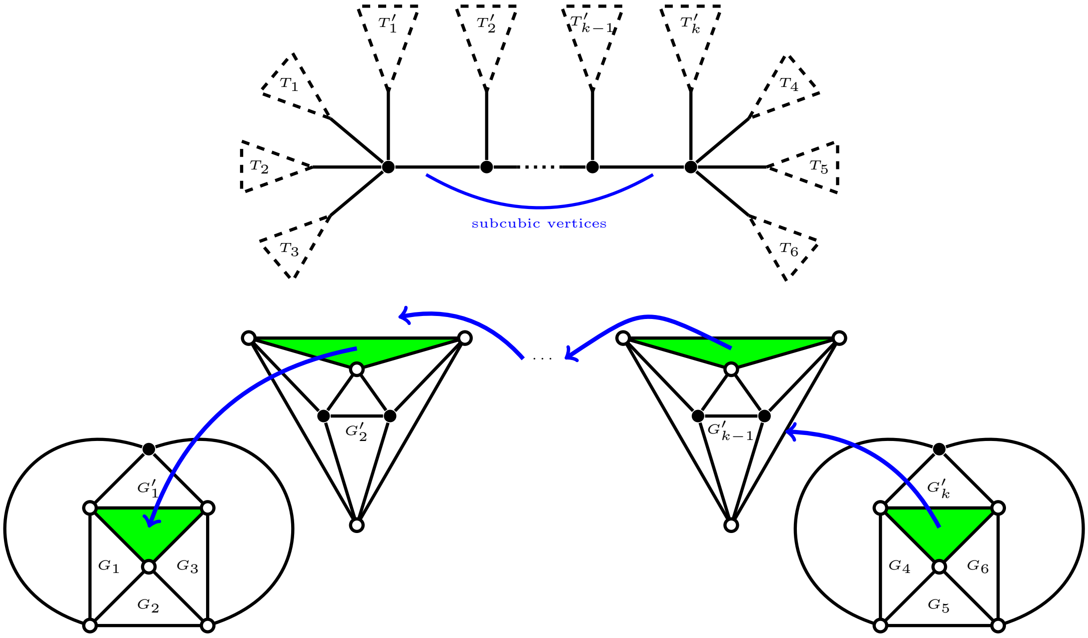

Figure 5: Constructing a non-hamiltonian triangulation for a tree T with at least two vertices of degree 4 or 5. A subcubic vertex is a vertex with degree at most 3.

5.2. Possible decomposition trees for non-hamiltonian triangulations

Theorem 5.13 (Brinkmann, Souffriau, and Van Cleemput [10], 2016). *For each tree T with $\Delta(T) \geq 6$, there exists a triangulation G with decomposition tree T that is non-hamiltonian.*

The basis for these triangulations is yet again the family of graphs shown in Figure 1 (center). A sufficiently large graph from this family is chosen to represent a vertex in the decomposition tree with degree at least 6, i.e. the number of triangular faces has to be equal to the degree of the vertex. Instead of subdividing the triangular faces with single vertices, they are subdivided with arbitrary triangulations which have the subtrees that remain after removing the chosen vertex from the given tree as their decomposition tree.

Theorem 5.14 (Brinkmann, Souffriau, and Van Cleemput [10], 2016). *For each tree T with at least two vertices of degree 4 or 5, there exists a triangulation G with decomposition tree T that is non-hamiltonian.*

We illustrate Theorem 5.14 in Figure 5. We can assume that the maximum degree of the tree is at most 5. In order to select a non-hamiltonian triangulation we start by selecting two vertices u and v in the tree with degree at least 4 such that all vertices on the path between u and v have degree at most 3. We subdivide the faces G_i with an arbitrary triangulation having the corresponding subtree as its decomposition tree. If u or v has degree 5, or any of the vertices on the connecting path has degree 3, then we can also subdivide the corresponding face G'_i. If the connecting path contains $k-2$ vertices, then the resulting triangulation has $5+k-2+(5-3)=5+k$ hollow vertices, and by removing them the graph is split into $4+k-2+4=6+k$ components. This proves that the triangulation is not hamiltonian.

A similar construction for non-hamiltonian triangulations with decomposition trees with one vertex of degree 4 or 5 and all other vertices of degree at most 3 is unfortunately not possible, since a result similar to Theorem 4.19 can be shown.

Theorem 5.15 (Brinkmann, Souffriau, and Van Cleemput [10], 2016). *Let G be a triangulation with decomposition tree T. If T has only one vertex of degree $k \in \{4,5\}$, and all other vertices have degree at most 3, then G is 1-tough.*

This implies that non-hamiltonian triangulations with a decomposition tree with one vertex of degree 4 or 5 and all other vertices of degree at most 3, have to be structurally very different from the non-hamiltonian triangulations constructed above. Non-hamiltonian triangulations that are 1-tough have also been studied [53, 31, 54], but all of the triangulations constructed have decomposition trees with multiple vertices of degree 4 or 5, or a decomposition tree with a maximum degree that is too high.

The construction above for non-hamiltonian triangulations with specific decomposition trees can also be used for related properties.

Theorem 5.16 (Van Cleemput [72], 2018+). *For each tree T with $\Delta(T) \geq 4$, there exists a triangulation with decomposition tree T that is not hamiltonian-connected.*

The technique used to construct these triangulations is similar to that used for the non-hamiltonian triangulations: a basis graph is chosen such that by subdividing the faces one obtains a graph with a high enough scattering number. The basis graphs in this case are the wheel graphs with at least four spokes.

This approach cannot be used to exclude any subcubic tree as the decomposition tree of a hamiltonian-connected triangulation. In [10] it is shown that in a triangulation G with a subcubic decomposition tree, we have for each cut set S: $c(G-S) < |S|$. So the scattering number of G is at most -1.

Theorem 3.4 can also be extended for triangulations and decomposition trees.

Theorem 5.17. *For each tree T with $\Delta(T) \geq 8$, there exists a triangulation with decomposition tree T that is non-traceable.*

These triangulations are constructed completely analogously to the previous cases, but this time the basis is the family of graphs shown on the right in Figure 1.

6. Closing remarks and open problems

From the tables in this survey a rich collection of natural problems emerges: each blank entry represents an open problem. However, we still want to give an overview and point out some partial results. We want to explicitly repeat the most important one, whose answer would give us, in a sense, a strongest form of Tutte's famous theorem [70].

Problem 6.1. *Is every polyhedron with four or five* 3*-cuts hamiltonian?*

Similarly, we are interested in the following.

Problem 6.2. *Is every polyhedron with five, six, or seven* 3*-cuts traceable?*

Concerning pancyclicity, Malkevitch [47] conjectured that for every 4-connected polyhedron G and any $3 \leq \ell \leq |V(G)|$ with $\ell \neq 4$, the graph G contains a cycle of length exactly ℓ. This is trivial for triangulations, but for polyhedra the general case is still unknown. Partial results can be found in [14, 18]. We here pose the following meta-question.

Problem 6.3. *Treat Malkevitch's question for polyhedra with few* 3*-cuts.*

We turn our attention to triangulations and begin with a question concerning both triangulations and polyhedra.

Problem 6.4. *Is there a cell in Table 1 which holds for triangulations but is untrue for polyhedra?*

Conjecture 6.5. *For $k \in \{4, 5\}$, every triangulation with a decomposition tree with one vertex of degree k and all other vertices of degree at most* 3 *is hamiltonian.*

This conjecture would complete the column concerning hamiltonicity in Table 1 restricted to triangulations, and in Table 2. Clearly it is not possible to generalize Theorem 5.6 to also include a third triangle or even a fourth triangle, since this would imply that all triangulations with a decomposition trees with maximum degree 4, respectively 5, would be hamiltonian, which is known to not be true. In order to prove the conjecture above it would be sufficient to prove the following two conjectures.

Conjecture 6.6. *Let G be a* 4*-connected triangulation. Let T_1, T_2, T_3, T_4 be distinct, pairwise vertex-disjoint triangles in G. Then there exists a hamiltonian cycle C of G and edges $e_1 \in E(T_1)$, $e_2 \in E(T_2)$, $e_3 \in E(T_3)$, $e_4 \in E(T_4)$, such that e_1, e_2, e_3, e_4 are contained in C.*

Conjecture 6.7. *Let G be a* 4*-connected triangulation. Let T_1, T_2, T_3, T_4, T_5 be distinct triangles in G. Then there exists a hamiltonian cycle C of G and edges $e_1 \in E(T_1)$, $e_2 \in E(T_2)$, $e_3 \in E(T_3)$, $e_4 \in E(T_4)$, $e_5 \in E(T_5)$, such that e_1, e_2, e_3, e_4, e_5 are distinct and contained in C.*

The following lemma is the reason that in the case of four triangles, it is sufficient to look at pairwise vertex-disjoint triangles.

Lemma 6.8 (Brinkmann, Souffriau, and Van Cleemput [10], 2016). *Let G be a 4-connected triangulation. Let T_1, T_2, T_3, T_4 be distinct triangles in G such that (at least) two of them share a vertex. Then there exists a hamiltonian cycle C of G and edges $e_1 \in E(T_1)$, $e_2 \in E(T_2)$, $e_3 \in E(T_3)$, $e_4 \in E(T_4)$, such that e_1, e_2, e_3, e_4 are contained in C.*

The actual lemma they prove is stronger since it states that such a hamiltonian cycle exists if two of the four triangles are contained in a so-called outerplanar disc. This property is stronger than merely sharing a vertex, but introducing it here would be out of scope—for details, see [10].

There are already several papers that discuss the existence of hamiltonian cycles through prescribed edges in planar graphs and/or triangulations for certain values of parameters such as connectedness and the distance between separating triangles [7, 9]. None of these however solve the two conjectures stated above.

As was already observed by Jackson and Yu in [40], the approach using decomposition trees cannot directly be generalized to arbitrary polyhedra. A similar decomposition can be defined by connecting vertices in 3-cuts to form separating triangles until all 3-cuts are separating triangles. One drawback of this approach is that the decomposition need no longer be unique. The main drawback, however, is that the theorem by Jackson and Yu is false for these decomposition trees. The Herschel graph forms a counterexample. Nevertheless, the strength of this approach motivates us to pose the following two questions.

Problem 6.9. *Is there a non-trivial subclass of polyhedra—greater than triangulations—in which the technique of decomposition trees can be used?*

We note that a trivial class is the family of all polyhedra in which each 3-cut forms a separating triangle. A different approach might be to generalize the concept of decomposition trees.

Problem 6.10. *Is there a generalization of the concept of decomposition trees which can be applied to a subclass of polyhedra—greater than triangulations—in order to obtain results similar to Theorem 5.5?*

While Nishizeki [53] proved the existence of a 1-tough non-hamiltonian triangulation with twelve separating triangles, in the light of the fact that the constructions from Theorem 3.1 are not 1-tough, we ask:

Problem 6.11. *Describe* 1*-tough non-hamiltonian triangulations with few separating triangles.*

Related to the above problem, Owens asked whether non-hamiltonian triangulations with toughness exactly 3/2 exist [54]. It follows from Theorem 5.1 that a non-hamiltonian triangulation has toughness at most 3/2. In [54] Owens constructed non-hamiltonian triangulations with toughness greater than $3/2 - \delta$ for any positive δ.

The same question as Problem 6.11 can be asked for polyhedra in general, especially in the light of Theorem 4.19. However, in polyhedra the relative position of 3-cuts can be far more complicated, making the number of 3-cuts rise quickly and difficult to control.

With respect to Theorem 5.15 we can generalize Problem 6.11 to decomposition trees:

Problem 6.12. *Describe* 1*-tough non-hamiltonian triangulations with a decomposition tree with a low maximum degree and few vertices of degree* 4 *or* 5*.*

Note again that all non-hamiltonian triangulations for specific decomposition trees constructed above are not 1-tough, and that all non-hamiltonian 1-tough triangulations constructed in [53, 31, 54] have decomposition trees with either a maximum degree which is significantly larger than 5, or have many vertices of degree 4 or 5.

Dillencourt [20] extends Whitney's Theorem 5.1 by studying planar graphs in which every face with one exception is a triangle—let us call such a graph a *triangulation of the disk*—and which do not have separating triangles. (These graphs need not be 3-connected.) He proves that every triangulation of the disk is hamiltonian if the chords satisfy a certain condition and that a hamiltonian cycle in a graph satisfying this condition can be found in linear time. He also establishes the existence of a 1-tough, non-hamiltonian triangulation of the disk with no separating triangles.

Further intriguing questions and conjectures can be found in Jackson and Whitehead's note [39].

Finally, we remark that Helden's PhD Thesis [32], on which the articles [34] and [35] are based, contains numerous mistakes and should be cited only with much care and after examining the proofs. An (incomplete) erratum of [32] is available, see [33]. Furthermore, although some of Helden's results are correct, many of these were obtained earlier by other authors, as we have discussed in the previous sections.

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