

Gallai’s question and constructions of almost hypotraceable graphs

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Abstract. Consider a graph G in which the longest path has order $|V(G)| - 1$. We denote the number of vertices v in G such that $G - v$ is non-traceable with \mathfrak{t}_G . Gallai asked in 1966 whether, in a connected graph, the intersection of all longest paths is non-empty. Walther showed that, in general, this is not true. In a graph G in which the longest path has $|V(G)| - 1$ vertices, the answer to Gallai’s question is positive iff $\mathfrak{t}_G \neq 0$. In this article we study *almost hypotraceable* graphs, which constitute the extremal case $\mathfrak{t}_G = 1$. We give structural properties of these graphs, establish construction methods for connectivities 1 through 4, show that there exists a cubic 3-connected such graph of order 28, and draw connections to works of Thomassen and Gargano et al.

Keywords. Gallai’s problem, traceable, hypotraceable, almost hypotraceable.

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1 Introduction

Throughout this paper all graphs are undirected, finite, connected, and contain neither loops nor multiple edges, unless explicitly stated otherwise. A graph G is *hypohamiltonian* (*hypotraceable*) if G does not contain a hamiltonian cycle (hamiltonian path) but for any vertex v in G the graph $G - v$ does contain a hamiltonian cycle (hamiltonian path). The study of hypohamiltonian graphs was initiated in the early sixties by Sousselier [22]. Many important results were obtained by Thomassen [24–28].

Kapoor, Kronk, and Lick [15] asked in 1968 whether hypotraceable graphs exist—in [17], Kronk stated that he “strongly feels” that they do not exist. This was refuted when a hypotraceable graph was subsequently found by Horton [13].

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Thomassen [24, 26] showed that there exists a hypotraceable graph with n vertices for $n \in \{34, 37\}$ and every $n \geq 39$, but we emphasise that Horton’s graph has connectivity 3, whereas some of Thomassen’s graphs have connectivity 2, others 3, depending on the construction method. (No 4-connected hypotraceable or hypohamiltonian graphs are known.) Since 1976, this list has been neither expanded—in particular, no hypotraceable graph of order smaller than 34 is known—, nor has it been shown to be complete.

Chvátal [4] asked whether *planar* hypohamiltonian graphs exist. Thomassen answered this question in the affirmative [26]. Based thereupon, he proved that planar hypotraceable graphs exist as well. In their survey on hypohamiltonian graphs, Holton and Sheehan [12] asked whether there is an order n' such that for every $n \geq n'$ there exists a planar hypohamiltonian graph on n vertices. We denote with n_0 the smallest such n' . Holton and Sheehan’s question was settled by Araya and the first author [35] who showed that $n_0 \leq 76$. We now know that $23 \leq n_0 \leq 42$, see [9] and [14], respectively. Araya and the first author [35] also showed that there exists a planar hypotraceable graph on n vertices for every $n \geq 180$, which was improved to 156 in [14]. They also proved that there exists a planar hypotraceable graph on 162 vertices, improving the previous bound of 186 from [38]. This was then lowered to 154 by Jooyandeh et al. [14]. Currently, the smallest planar hypotraceable graph (of order 138) is due to the first author [33, 34], who used a new approach to construct hypotraceable graphs, explained at the end of the following paragraph.

In a further article, Araya and the first author [1] showed that planar cubic hypotraceable graphs—in fact, these graphs are polyhedral, i.e., planar and 3-connected—on n vertices exist for $n = 340$ (this is the smallest example we know of) and every even $n \geq 356$, settling affirmatively an open question of Holton and Sheehan [12]. The number 356 was lowered to 344 by the second author [37].

For a graph G , we denote with $V(G)$ its vertex set and with $E(G)$ its edge set. In a graph G in which a longest cycle has length $|V(G)| - 1$, let $W \subset V(G)$ be the set of vertices w such that the graph $G - w$ is non-hamiltonian (and thus, for all $v \in V(G) \setminus W$, the graph $G - v$ is hamiltonian). We call $|W|$ the *hypohamiltonicity* of G , denote it by $\mathfrak{h}(G) = \mathfrak{h}_G$, and say that G is \mathfrak{h}_G -*hypohamiltonian*. A vertex from W is called *exceptional*. Until recently, all constructions of hypotraceable graphs relied on hypohamiltonian graphs as “building blocks”. However, the first author showed [33, 34] that 1-hypohamiltonian (also known as *almost hypohamiltonian*) graphs in which the exceptional vertex is cubic can be used as such building blocks, as well. Using this fact, he constructed the aforementioned smallest known planar hypotraceable graph.

As Kapoor, Kronk, and Lick [15], for a graph G we denote with $\partial(G)$ the length of a longest path in G . A graph G is *traceable* if it contains a hamiltonian path, i.e., $\partial(G) = |V(G)| - 1$, and for $v \in V(G)$, G is *v -traceable* if it contains a hamiltonian path with end-vertex v . In analogy to the definition given for cycles, consider a graph G with $\partial(G) = |V(G)| - 2$ and let $W \subset V(G)$ be the set of all vertices w such that the graph $G - w$ is non-traceable (and thus, for all $v \in V(G) \setminus W$, the graph $G - v$ is traceable). We call $|W|$ the *hypotraceability* $\mathfrak{t}(G) = \mathfrak{t}_G$ of G and say that G is \mathfrak{t}_G -*hypotraceable*.

It is easy to see that in any graph, two longest paths meet. Gallai [7] asked in

1966 whether *all* longest paths intersect. (Which is reminiscent of *Helly's property*: a collection of sets satisfies it, if any sub-collection of pairwise intersecting sets has a nonempty intersection.) We follow Chen et al. [6], and call a vertex present in all longest paths of a given graph a *Gallai vertex*, and the set of all Gallai vertices the *Gallai set*.

It turns out that, in general, the answer to Gallai's question is negative. It was Walther [29] who first showed that there exists a graph in which the intersection of all longest paths is empty, i.e., a graph with empty Gallai set. A few years later, a significantly smaller example—of order 12—was independently found by Walther and T. Zamfirescu, see [11, 30, 39]. It is shown in Fig. 1 (a). Brinkmann and Van Cleemput [3] proved (using computers) that there is no smaller example.

The smallest known example of a *planar* graph in which all longest paths have empty intersection has order 17 and is due to Schmitz [20], see Fig. 1 (b). More variants of Gallai's problem were discussed, for instance by asking arbitrary *pairs* of vertices to be missed, by demanding higher connectivity, or by posing the question for graphs which can be embedded in various lattices. For a survey, see [21].

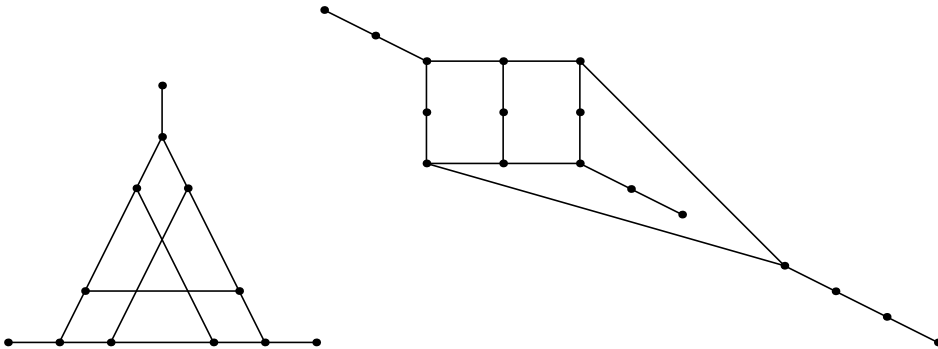


Fig. 1: (a) The Walther-Zamfirescu graph; (b) Schmitz' graph.

Let us emphasise the connection between hypotraceability and Gallai vertices: consider a graph G with $\partial(G) = |V(G)| - 2$. This is extremal in the sense that it is the greatest length of a longest path for which Gallai's question is interesting. Then the hypotraceability of G is precisely the cardinality of the Gallai set of G . Hence, in the (extremal) family of graphs G with $\partial(G) = |V(G)| - 2$, the answer to Gallai's question is positive if and only if $\mathfrak{t}_G \neq 0$.

Gallai's question has drawn much attention; we give here only a small selection of results. For an overview, see [21]. One central direction of research was—since with Walther's result, in general, Gallai's question has a negative answer—to study in which families of graphs Gallai's question had a *positive* answer. We call such graphs, ad hoc, *good*. Trees, for instance, abide. Klavžar and Petkovšek [16] proved that if every block of G is *hamiltonian-connected*, i.e., any two vertices are the end-vertices of a hamiltonian path, then G is good. Balister, Győri, Lehel, and Schelp [2] showed that circular arc graphs—a graph G is a *circular arc graph* if there exists a mapping α of $V(G)$ into a collection of arcs of a circle such that, for every $v, w \in V(G)$, there is an edge between v and w if and only if $\alpha(v) \cap \alpha(w) \neq \emptyset$ —are good, as well. (Note that this includes interval graphs. According to Rautenbach and Sereni [19], there is a gap in the proof of Balister et al. For details, see [19].)

In [36] and [10], almost hypohamiltonian graphs were studied, i.e., graphs of hypohamiltonicity 1. In analogy thereto, we here define a graph G to be *almost hypotraceable* if $t_G = 1$, i.e., if G is non-traceable, there exists a vertex w such that $G - w$ is non-traceable, and for any vertex $v \neq w$ the graph $G - v$ is traceable. (Observe that an almost hypohamiltonian graph G cannot be almost hypotraceable, since for every non-exceptional vertex $v \in V(G)$ we have that $G - v$ contains a hamiltonian cycle, and this immediately yields a hamiltonian path in G .) As before, we call w the *exceptional vertex* of G . For graphs G, H , we will denote by $G + H$ the join of G and H . An edge between vertices v and w will be denoted with vw , while for $n \geq 3$ a path $(\{v_1, \dots, v_n\}, \{v_i v_{i+1}\}_{i=1}^{n-1})$ will appear as $v_1 \dots v_n$. For the set of neighbours of a vertex v we write $N(v)$ and put $N[v] = N(v) \cup \{v\}$.

This article is devoted to the family of good graphs G which are “extremal” in two senses: first, $\partial(G) = |V(G)| - 2$ (the greatest length of a longest path for which Gallai’s question is interesting), and second, $t_G = 1$ (the smallest hypotraceability for which the answer to Gallai’s question is positive). In Sections 2 through 4, we study almost hypotraceable graphs of connectivity at most 4. Cuts of various sizes play a crucial role in the study of hypohamiltonicity and hypotraceability and this is also true for almost hypohamiltonicity and almost hypotraceability. We emphasise that while the existence of hypotraceable graphs of connectivity 4 is unknown, we here present 4-connected *almost* hypotraceable graphs. The article ends with Section 5, in which we give a brief overview of potential future research.

2 Connectivity 1

It is not difficult to see that hypotraceable graphs have no vertices of degree 1 or 2. For an almost hypotraceable graph G the latter holds, see Proposition 2, while the former is not necessarily true: G may contain a vertex v of degree 1, but then v must be a neighbour of the exceptional vertex. Do such graphs exist? One of the graphs answering this question is the smallest almost hypotraceable graph and surprisingly minute: the “claw” $K_{1,3}$. (While all hypotraceable graphs are easily seen to be 2-connected and 3-edge-connected, this is not the case for almost hypotraceable graphs, even if we exclude the claw.) But the claw plays a special role, which is discussed in the next proposition—its proof is a consequence of the paragraph following it, which treats the issue more generally.

Proposition 1. *With the exception of $K_{1,3}$, every almost hypotraceable graph contains at most one vertex of degree 1.*

We now discuss almost hypotraceable graphs of connectivity 1. Let G be such a graph, and consider $w \in V(G)$. Vertex w is the exceptional vertex if and only if $G - w$ is disconnected. Hence, G contains exactly one 1-cut, namely $\{w\}$. As for all $v \in V(G) \setminus \{w\}$ there exists a hamiltonian path in $G - v$, and such a path has exactly two end-vertices, we have that $G - w$ consists of at most three (connected) components. If $G - w$ consists of exactly three components, then each of these components must be K_1 —assume to the contrary that there is a component $X \neq K_1$, and let $v \in V(X)$. Then a hamiltonian path in $G - v$ would have to traverse w (at least) twice, absurd. So in this case we have $G = K_{1,3}$. Graph $G - w$ cannot consist

of exactly two components $X_1 \neq K_1$ and $X_2 \neq K_1$: for $x_i \in V(X_i)$ there exists a path \mathbf{p}_i visiting all vertices in $G - x_i$. But then

$$(\mathbf{p}_1 \cap G[V(X_2) \cup \{w\}]) \cup (\mathbf{p}_2 \cap G[V(X_1) \cup \{w\}])$$

is a hamiltonian path in G , a contradiction. Also $X_1 = X_2 = K_1$ is obviously impossible, so if $G - w$ consists of exactly two components X_1, X_2 , then $X_1 \neq K_1$ and $X_2 = K_1$.

Let us address the natural question whether, in addition to the claw, almost hypotraceable graphs exist. Let G be a non-complete graph with connectivity k , X be a (vertex-)cut of G of cardinality k , and H be a component of $G - X$. Then $G[V(H) \cup X]$ is called a k -fragment of G , and X is called the set of *vertices of attachment* of H . We now present two ways in which infinite families of almost hypotraceable graphs can be constructed—both rely heavily on results concerning hypotraceable graphs.

Theorem 1. *Let G be a 2-fragment of a hypotraceable graph with vertices of attachment a, b . Then for vertices $v, w \notin V(G)$,*

$$G' = (V(G) \cup \{v, w\}, E(G) \cup \{wa, wb, vw\})$$

is an almost hypotraceable graph with exceptional vertex w .

For the proof, we need a lemma of Thomassen characterising hypotraceable 2-fragments [26].

Lemma 1 (Thomassen, 1976). *A graph G is a 2-fragment of a hypotraceable graph with vertices of attachment a, b if and only if G is not a -traceable and not b -traceable, but for any $u \in V(G)$ we have that $G - u$ is either a -traceable or b -traceable.*

Proof of Theorem 1. If \mathbf{p} is a hamiltonian path of G' , then v is one of the end-vertices of \mathbf{p} , the edge vw is included in \mathbf{p} , and exactly one of the edges wa and wb is included in \mathbf{p} . Therefore $G \cap \mathbf{p}$ is a hamiltonian path of G starting at either a or b , a contradiction to Lemma 1 which shows that G' is not traceable. By deleting w from G' we obtain a non-traceable graph, since $G' - w$ is disconnected. It remains to show that if $w \neq u \in V(G)$ then $G' - u$ is traceable. Let first $u = v$. By Lemma 1, there exists a hamiltonian path \mathbf{q} in $G - a$ starting at b . Now $\mathbf{q} \cup bwa$ is a hamiltonian path in $G' - v$. Suppose now that $u \neq v$. Again by Lemma 1, there exists a hamiltonian path \mathbf{q}' in $G - u$ starting at either a or b , let us suppose w.l.o.g. that \mathbf{q}' starts at a . Then $\mathbf{q}' \cup awv$ is a hamiltonian path in $G' - u$, finishing the proof. \square

Corollary 1. *Almost hypotraceable graphs containing a vertex of degree 1 and of order n exist for $n = 4, 20, 23, 25, 26$ and all $n \geq 28$.*

Proof. For order 4, consider $K_{1,3}$. Thomassen [24] showed that hypotraceable 2-fragments of order $n_1 + n_2 - 2$ can be obtained from hypohamiltonian graphs of order n_1 and n_2 containing a cubic vertex. In [24] he also proved that hypohamiltonian graphs of order n containing a cubic vertex exist for $n = 10, 13, 15, 16$ and every $n \geq 18$, from which the corollary easily follows. \square

Almost hypotraceable graphs of connectivity 1 can be constructed in other ways, as well:

Theorem 2. *Let G be a hypotraceable graph. Then for vertices $v, w \notin V(G)$,*

$$G' = (V(G) \cup \{v, w\}, E(G) \cup \{xw : x \in V(G) \cup \{v\}\})$$

is an almost hypotraceable graph with exceptional vertex w . Furthermore, there exists an almost hypotraceable graph with maximum degree d for $d = 3, 35, 38$ and every $d \geq 40$.

Proof. By [36, Lemma 1], $G' - v$ is almost hypohamiltonian with exceptional vertex w . First assume that G' contains a hamiltonian path \mathbf{p} . Then $G \cap \mathbf{p}$, where we consider G to be a subgraph of G' , is a hamiltonian path in G , which contradicts the fact that G is hypotraceable. Hence G' is non-traceable. Furthermore, since $G' - w$ is disconnected, it is trivially non-traceable.

Consider a vertex $u \in V(G)$. Then there exists a hamiltonian path \mathbf{p}' in $G - u = G' - \{u, v, w\}$, since G is hypotraceable, and let z be an end-vertex of \mathbf{p}' . Now $\mathbf{p}' \cup zwv$ is a hamiltonian path in $G' - u$. Similarly, $\mathbf{p}' \cup zwu$ is a hamiltonian path in $G' - v$. Therefore G is almost hypotraceable with exceptional vertex w . Using two theorems of Thomassen [24, 26] stating that there exists a hypotraceable graph of order n for $n \in \{34, 37\}$ and every $n \geq 39$, the proof is complete. \square

However, if the exceptional vertex of the almost hypotraceable graph is to be cubic, then by the following theorem the construction of Theorem 1 is the only possibility.

Theorem 3. *Let G be an almost hypotraceable graph of connectivity 1 with a cubic exceptional vertex x , whose neighbours are a, b, y , such that y has degree 1. Then $G - \{x, y\}$ is a hypotraceable 2-fragment with vertices of attachment a, b .*

Proof. By Lemma 1, it suffices to show that $G' = G - \{x, y\}$ is not a -traceable and not b -traceable, but for any $v \in V(G')$: $G' - v$ is either a -traceable or b -traceable. It is obvious that G' is not a -traceable and not b -traceable, since if \mathbf{p} is a hamiltonian path of G' starting at (say) a , then $\mathbf{p} \cup axy$ is a hamiltonian path of G , a contradiction. Now let $v \in V(G')$. Since G is almost hypotraceable with exceptional vertex x , the graph $G - v$ has a hamiltonian path \mathbf{q} . Then y is one of the end-vertices of \mathbf{q} , $xy \in E(\mathbf{q})$ and either $xa \in E(\mathbf{q})$ or $xb \in E(\mathbf{q})$. Then clearly $\mathbf{q} \cap (V(G') \setminus \{v\})$ is a hamiltonian path of $G' - v$ starting at either a or b . \square

The constructions of Theorems 1 and 2 rely heavily on the existence of a cut-vertex, so it is natural to ask whether almost hypotraceable graphs of higher connectivity exist. In Section 3, we describe such graphs of connectivity 2 and 3, while in Section 4 we even give a construction of connectivity 4. We highlight the importance of the 4-connected case at the beginning of that section. Proposition 2, which follows, also holds for hypotraceable, hypohamiltonian, and almost hypohamiltonian graphs. For hypohamiltonian graphs this was already observed by Bondy—see Chvátal's paper [4]—but first published in [5].

Proposition 2. *An almost hypotraceable graph G does not contain vertices of degree 2. Furthermore, the vertices of each triangle in G have degree at least 4.*

Proof. Let G be an almost hypotraceable graph with exceptional vertex w , and let $v \in V(G)$ be a vertex of degree 2 with neighbours v' and v'' , one of which—say v' —is not w . There exists a hamiltonian path \mathbf{p} in $G - v'$, so $\mathbf{p} \cup vv'$ is a hamiltonian path in G , a contradiction.

For the second part, consider a triangle T in G with $V(T) = \{v_1, v_2, v_3\}$, where v_3 shall be cubic. Since G is almost hypotraceable, at least one of $G - v_1$ and $G - v_2$ must be traceable, say $G - v_1$. Let \mathbf{p} be a hamiltonian path in $G - v_1$. If v_3 is an end-vertex of \mathbf{p} , then $\mathbf{p} \cup v_3v_1$ is a hamiltonian path in G , a contradiction. If v_3 is not an end-vertex of \mathbf{p} , then $v_3v_2 \in E(\mathbf{p})$. By replacing v_3v_2 with $v_3v_1v_2$, we obtain a hamiltonian path in G , once more a contradiction. \square

3 Connectivity 2 and 3

We turn our attention to 2-connected almost hypotraceable graphs. These graphs are of special interest because they all belong to the class of arachnoid graphs, defined (and studied) in 2002 by Gargano, Hammar, Hell, Stacho, and Vaccaro [8] as follows. A tree is a *spider* if it has at most one vertex of degree at least 3. A spider is *centred* at the vertex of degree at least 3 if there is such a vertex, and centred at any other vertex otherwise. A graph G is said to be *arachnoid* if for any vertex v of G , there exists a spanning spider of G centred at v .

Arachnoid graphs are natural generalisations of traceable graphs. Gargano et al. observed that all hypotraceable graphs are arachnoid, but were unable to find other non-traceable arachnoid graphs and therefore raised the question whether such graphs exist. This was answered affirmatively by the first author [31, 32]—the smallest example has order 73 and all examples contain a vertex of high degree (more than $\frac{32}{33}n$, where n is the order of the graph). Here we give, among other results, the first cubic examples. Furthermore, our smallest construction—which happens to be cubic—has only 28 vertices.

Proposition 3. *Every 2-connected almost hypotraceable graph is arachnoid.*

Proof. Let w be the exceptional vertex of the 2-connected almost hypotraceable graph G and let $v \in V(G)$ be an arbitrary vertex. Since G is 2-connected, v has a neighbour x different from w . As $G - x$ is traceable, in order to obtain a spanning spider centred at v , it suffices to add the vertex x and the edge vx to a hamiltonian path of $G - x$. \square

In the above proof we just used that all vertices have degree at least 2—but we have seen that almost hypotraceable graphs that are not 2-connected have a vertex of degree 1.

Since almost hypotraceable graphs are obviously neither traceable, nor hypotraceable, every 2-connected example provides an answer to the question of Gargano et al. We present a method to construct such graphs by adapting a technique of Thomassen [24]. Thomassen’s approach provides the smallest known hypotraceable

graph, which has order 34, by using four copies of the Petersen graph—contrasting this, we shall discuss in Theorem 4 a method which provides smaller almost hypotraceable graphs with higher connectivity than the ones derivable from Proposition 4.

Proposition 4. *Let G_1, G_2, G_3, G_4 be pairwise disjoint graphs, G_1 be an almost hypohamiltonian graph with exceptional vertex w , and G_2, G_3, G_4 be hypohamiltonian graphs. Assume furthermore that each G_i contains a cubic vertex x_i with $N(x_i) = \{x_i^1, x_i^2, x_i^3\}$, $1 \leq i \leq 4$, such that $w \notin N[x_1]$. Consider the graphs $G_i - x_i$, $1 \leq i \leq 4$, and identify x_1^3, x_2^3 into a vertex y_1 and x_3^3, x_4^3 into a vertex y_2 . Also add the edges $x_1^1 x_3^1, x_1^2 x_3^2, x_2^1 x_4^1, x_2^2 x_4^2$. The graph G we obtain is almost hypotraceable with exceptional vertex w .*

Proof. We have to show that G and $G - w$ are not traceable, but for any $v \in V(G)$, $v \neq w$, we have that $G - v$ is traceable. Let H_1 be the subgraph of G spanned by $V(G_1) \cup V(G_3)$ and H_2 be the subgraph of G spanned by $V(G_2) \cup V(G_4)$. Had we chosen G_1 to be hypohamiltonian instead of almost hypohamiltonian, we would have that G is hypotraceable by Thomassen's Lemma 3.1 of [26] and therefore H_2 would be a hypotraceable 2-fragment with vertices of attachment y_1 and y_2 . Since a change of G_1 does not affect H_2 , H_2 is actually a hypotraceable 2-fragment in our construction as well. By Lemma 5.1 of [26] there is no hamiltonian path of H_2 starting at y_1 or y_2 . Now let us assume that there exists a hamiltonian path \mathbf{p} of G ($G - w$). If at most one of the end-vertices of \mathbf{p} lies in H_2 , then $\mathbf{p} \cap V(H_2)$ would be a hamiltonian path of H_2 starting at y_1 or y_2 , which is impossible, thus both end-vertices of \mathbf{p} are in H_2 . This means that $\mathbf{p} \cap V(H_1)$ ($\mathbf{p} \cap V(H_1 - w)$) is a hamiltonian path of H_1 ($H_1 - w$) between y_1 and y_2 , thus $\mathbf{p} \cap V(G_3 - x_3)$ is a hamiltonian path of $G_3 - x_3$ between x_3^3 and either x_3^1 or x_3^2 , in which case G_3 would be hamiltonian, a contradiction that shows that neither G , nor $G - w$ are traceable.

Now let $v \in V(G)$, $v \neq w$ and let us prove that $G - v$ is traceable. For this, we only use the following properties (A) and (B) of the subgraphs $G_i - x_i$ of G .

(A) For any $z \in V(G_i - x_i)$ (in case of $i = 1$, $z \neq w$) there exists a hamiltonian path of $G_i - x_i - z$ between two of the vertices x_i^1, x_i^2, x_i^3 .

(B) There exists a hamiltonian path of $G_i - x_i$ starting at x_i^j for $j = 1, 2, 3$.

Both (A) and (B) are immediate consequences of the fact that G_2, G_3, G_4 are hypohamiltonian and G_1 is almost hypohamiltonian with exceptional vertex w , where w is none of x_1, x_1^1, x_1^2, x_1^3 . Since we only use the properties (A) and (B) and $v \neq w$, by symmetry reasons it suffices to prove that $G - v$ is traceable for (say) $v \in V(G_1 - x_1)$. Now we distinguish two cases.

Case 1. There is a hamiltonian path \mathbf{p}_1 between x_1^1 and x_1^2 in $G_1 - x_1 - v$. By (A), there exists a hamiltonian path \mathbf{p}_3 between x_3^1 and x_3^3 in $G_3 - x_3 - x_3^2$, a hamiltonian path \mathbf{p}_4 between x_4^1 and x_4^3 in $G_4 - x_4 - x_4^2$, and a hamiltonian path \mathbf{p}_2 between x_2^1 and x_2^2 in $G_2 - x_2 - x_2^3$. Now

$$x_3^2 x_1^2 \cup \mathbf{p}_1 \cup x_1^1 x_3^1 \cup \mathbf{p}_3 \cup \mathbf{p}_4 \cup x_4^1 x_2^1 \cup \mathbf{p}_2 \cup x_2^2 x_4^2$$

is a hamiltonian path of $G - v$.

Case 2. There is no hamiltonian path between x_1^1 and x_1^2 in $G_1 - x_1 - v$, therefore there is a hamiltonian path \mathbf{p}_1 between x_1^3 and either x_1^1 or x_1^2 in $G_1 - x_1 - v$. W.l.o.g.

we may suppose that the end-vertices of \mathbf{p}_1 are x_1^3 and x_1^1 . By (B), there exists a hamiltonian path \mathbf{p}_3 of $G_3 - x_3$ starting at x_3^1 . By (A), there exists a hamiltonian path \mathbf{p}_2 between x_2^3 and x_2^1 in $G_2 - x_2 - x_2^2$, and a hamiltonian path \mathbf{p}_4 between x_4^1 and x_4^2 in $G_4 - x_4 - x_4^3$. Now

$$\mathbf{p}_3 \cup x_3^1 x_1^1 \cup \mathbf{p}_1 \cup \mathbf{p}_2 \cup x_2^1 x_4^1 \cup \mathbf{p}_4 \cup x_4^2 x_2^2$$

is a hamiltonian path of $G - v$. □

Let H and G be graphs each containing a vertex of degree k , say v and w , respectively. We say that we *replace v with $G - w$* if we delete v (and all incident edges) from H and connect the neighbours of v in H to the neighbours of w in $G - w$ using a bijection. The next theorem provides a powerful tool to construct 3-connected almost hypotraceable graphs, allowing for planar and cubic constructions as well depending on the input graphs. An example of its application is given in Fig. 2.

Theorem 4. *Let G_1, G_2, G_3 be pairwise disjoint hypohamiltonian or almost hypohamiltonian graphs, each containing a cubic vertex x_1, x_2, x_3 , respectively, where, in case G_i is almost hypohamiltonian, x_i must be its exceptional vertex. Consider K_4 and put $V(K_4) = \{v_1, \dots, v_4\}$. By replacing v_i with $G_i - x_i$, $1 \leq i \leq 3$, a 3-connected almost hypotraceable graph G is obtained.*

Proof. We will treat each $G_i - x_i = H_i$ as a subgraph of G , and v_4 as a vertex in G . First we show that G is 3-connected, by showing that three pairwise internally disjoint paths (p.i.d.p.'s) exist between any two vertices u and v in G . Let the neighbours of x_i in G_i be a_i, b_i, c_i . Let us assume first that u and v are in the same subgraph H_i . Since each G_i is hypohamiltonian or almost hypohamiltonian, they are 3-connected, thus three p.i.d.p.'s between u and v exist in G_i . If none of these contains x_i , they are all present in G and we are done. If one of them contains x_i , we may suppose w.l.o.g. that the neighbours of x_i in the path are a_i and b_i . Now using a path in G between a_i and b_i outside H_i (which clearly exists) we can easily construct the third path in G . Let us suppose now that $u \in V(H_1)$ and $v \in V(H_2)$. By the 3-connectivity of G_1 and G_2 , there exist three p.i.d.p.'s between u and x_1 in G_1 and also between v and x_2 in G_2 . Deleting x_1 (x_2) from these paths we obtain three paths from u to a_1, b_1, c_1 (from v to a_2, b_2, c_2). It is easy to see that these paths can be joined in G to form three p.i.d.p.'s between u and v . Finally, the case when $u = v_4$ and v is in (say) H_1 can be dealt with similarly as the previous one.

Assume G contains a hamiltonian path \mathbf{p} . Since \mathbf{p} has two end-vertices, there exists a G_i , say G_1 , such that \mathbf{p} contains a subpath \mathbf{q} which has end-vertices $y, z \in N(x_1) \subset V(G_1)$, and is a hamiltonian path in $G_1 - x_1$. Thus $\mathbf{q} \cup yx_1z$ is a hamiltonian cycle in G_1 , a contradiction. The same argument yields that $G - v_4$ is non-traceable.

Consider now $v \in V(G) \setminus \{v_4\}$. W.l.o.g. we may assume that $v \in V(H_1)$. Since G_1 is hypohamiltonian or almost hypohamiltonian with exceptional vertex $x_1 \notin V(H_1)$, there exists a path \mathbf{q}_1 in $H_1 - v$ with end-vertices $y, z \in N(x_1)$ which visits every vertex in $H_1 - v$.

Case 1: $\{y, z\} \cap N(v_4) = \emptyset$. W.l.o.g. let the neighbour of y (z) not lying in G_1 lie in H_2 (H_3). We denote this vertex by y_1 (z_1). Put $\{y_2\} = N(v_4) \cap V(H_2)$,

$\{z_2\} = N(v_4) \cap V(H_3)$, $N(x_2) = \{y_1, y_2, y_3\}$, and $N(x_3) = \{z_1, z_2, z_3\}$. Since G_2 (G_3) is hypohamiltonian or almost hypohamiltonian with exceptional vertex $x_2 \notin V(H_2)$ ($x_3 \notin V(H_3)$), there is a path \mathfrak{q}_2 (\mathfrak{q}_3) in $H_2 - y_2$ ($H_3 - z_3$) with end-vertices y_1 and y_3 (z_1 and z_2) which visits all vertices in $H_2 - y_2$ ($H_3 - z_3$). Now

$$y_2v_4z_2 \cup \mathfrak{q}_3 \cup z_1z \cup \mathfrak{q}_1 \cup yy_1 \cup \mathfrak{q}_2 \cup y_3z_3$$

is a hamiltonian path in $G - v$.

Case 2: $\{y, z\} \cap N(v_4) \neq \emptyset$. W.l.o.g. let $y \in N(v_4)$. We need the following result.

Claim. Let G be a hypohamiltonian or almost hypohamiltonian graph with a cubic exceptional vertex, and consider $v \in V(G)$. Then, for every $x \in N(v)$, there exists a hamiltonian path in $G - v$ which has x as an end-vertex.

Proof of the Claim. If G is hypohamiltonian, let \mathfrak{h} be a hamiltonian cycle in $G - v$ and $xy \in E(\mathfrak{h})$. Then $\mathfrak{h} - xy$ is the desired hamiltonian path. Now consider an almost hypohamiltonian graph G' . The argument is the same for all but the exceptional vertex. Denote this vertex by w and consider $x' \in N(w)$. Furthermore, let $y' \in N(x') \setminus \{w\}$. Vertex y' exists since almost hypohamiltonian graphs are 3-connected, so $\deg(x') \geq 3$, and such graphs have order at least 17, see [10]. Note that $y' \notin N(w)$, as triangles in almost hypohamiltonian graphs do not contain cubic vertices [10, Lemma 8]. As $y' \neq w$, there exists a hamiltonian cycle \mathfrak{h}' in $G - y'$. Then $(\mathfrak{h}' - w) \cup x'y'$ is a hamiltonian path with the desired properties. This completes the proof of the claim. \square

W.l.o.g. let the neighbour of z which does not lie in G_1 lie in G_3 . We denote this vertex by z_1 . Let $\{y_2\} = N(v_4) \cap V(G_2)$. By the above claim, since G_2 (G_3) is hypohamiltonian or almost hypohamiltonian, there is a path \mathfrak{q}'_2 (\mathfrak{q}'_3) in H_2 (H_3) which has y_2 (z_1) as an end-vertex and which visits all vertices in H_2 (H_3). Now

$$\mathfrak{q}'_2 \cup y_2v_4y \cup \mathfrak{q}_1 \cup zz_1 \cup \mathfrak{q}'_3$$

is a hamiltonian path in $G - v$. \square

If, in Theorem 4, each G_i is planar (cubic), then the resulting graph is planar (cubic), as well. The graph from Fig. 2 is the smallest known non-traceable arachnoid graph, since the smallest known hypotraceable graph has order 34. Furthermore, it exhibits interesting properties in the context of snarks, explained in detail in a paper of Steffen, see [23, Theorem 2.4].

Corollary 2.

- (i) *There is a cubic 3-connected almost hypotraceable graph on 28 vertices.*
- (ii) *3-connected almost hypotraceable graphs on n vertices exist for $n = 28, 31$ and every $n \geq 33$.*
- (iii) *Cubic 3-connected almost hypotraceable graphs on n vertices exist for $n = 28$ and every even $n \geq 36$.*
- (iv) *Planar 3-connected almost hypotraceable graphs on n vertices exist for $n = 106, 110$ and every $n \geq 112$.*

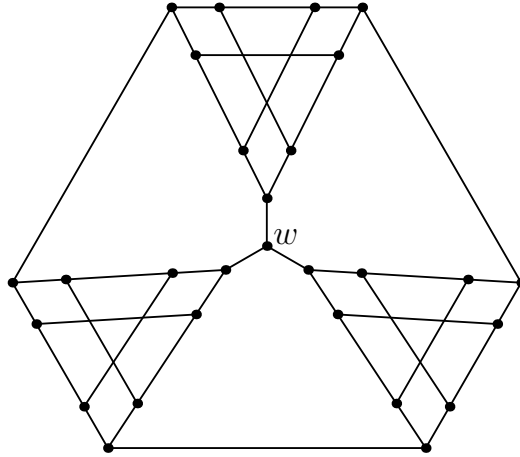


Fig. 2: By letting G_1, G_2, G_3 each be the Petersen graph, we obtain the graph above when applying Theorem 4. It is a 3-connected cubic almost hypotractable graph of order 28.

Its exceptional vertex is w .

- (v) *Planar cubic 3-connected almost hypotractable graphs on n vertices exist for every even $n \geq 202$.*

Proof. (i) Using in Theorem 4 for each G_i the Petersen graph, we obtain that the 28-vertex graph shown in Fig. 2 is almost hypotractable.

(ii) Theorem 4 implies that if there exist graphs which are hypohamiltonian or almost hypohamiltonian of order n_1, n_2, n_3 , each containing a cubic vertex (in case the graph is almost hypohamiltonian, the exceptional vertex must be cubic), then there exists a 3-connected almost hypotractable graph of order $n_1 + n_2 + n_3 - 2$. Thomassen [24] proved that a hypohamiltonian graph of order n and containing a cubic vertex exists for $n = 10, 13, 15, 16$ and every $n \geq 18$, while the second author [37, p. 56] showed that there exists an almost hypohamiltonian graph with cubic exceptional vertex on 17 vertices.

(iii) Cubic hypohamiltonian graphs of order n exist if and only if $n = 10$ or $n \geq 18$ is even [12]. We proceed exactly as in (ii), using the aforementioned fact that if each G_i is cubic, then the resulting graph is cubic, too.

(iv) In [14] it was shown that there exists a planar hypohamiltonian graph of order 40 and of order n for every $n \geq 42$; denote this family of graphs by \mathcal{F} . In contrast to the general case, for which it is unknown whether hypohamiltonian graphs of minimum degree greater than 3 exist, Thomassen showed that every planar hypohamiltonian graph contains a cubic vertex [27]. It was proven independently by the first author, and Goedgebeur and the second author, that a planar almost hypohamiltonian graph G of order 36 with cubic exceptional vertex exists [10, 33, 34]. By applying Theorem 4 to three copies of G , we obtain a planar 3-connected almost hypotractable graph of order 106. The full statement follows by applying Theorem 4 to all triples of graphs from $\mathcal{F} \cup \{G\}$, as above.

(v) There exists a planar cubic hypohamiltonian graph of order 70 due to a result of Araya and the first author [1] and there are such graphs of order n for every even $n \geq 74$, as shown by the second author [36]. Furthermore, there exists a planar cubic almost hypohamiltonian graph on 68 vertices, constructed independently by

McKay, and Goedgebeur and the second author [10]. By applying Theorem 4, we obtain the advertised statement. \square

McKay [18] has shown computationally that all bridgeless cubic graphs up to 26 vertices are traceable. Recently, he generated all 40,157,414,176 bridgeless cubic graphs of order 28 and found that ten of them are non-traceable. Of these ten, only one is 3-connected: it is the graph shown in Fig. 2. However, before stating this as a theorem, we await an independent verification of McKay’s computations.

4 4-connected constructions

It is a long-standing open question whether 4-connected hypotraceable graphs exist. Even the simpler questions whether (i) hypotraceable graphs without cubic vertices and (ii) 4-connected graphs in which every vertex is avoided by a longest path exist, remain open. Moreover, dramatic difficulties are encountered when studying problems on longest paths and longest cycles on 4-connected graphs, see [39] as well as Problems 1 and 3 in Section 5. Thus, deciding on the existence of 4-connected almost hypotraceable graphs is of special interest. The aforementioned difficulties are also very much linked to the fact that hypohamiltonian graphs play a central role in many solutions—unfortunately, Thomassen’s question whether 4-connected such graphs exist [27] remains unanswered. Although almost hypohamiltonian graphs may take the role of hypohamiltonian graphs in certain applications (and 4-connected examples of such graphs *are* known), and in some cases even yield stronger results than their hypohamiltonian counterparts (see e.g. [33, 34]), to date none of them satisfy the needed properties to solve problems on longest paths and longest cycles in 4-connected graphs.

On the other hand, some of the 4-connected arachnoid graphs appearing in [32] are almost hypotraceable. Here we extend the method presented in [32] to obtain such graphs by defining so-called path-critical graphs. For a graph G , the *path-covering number* $\mu(G)$ of G is the minimum number of vertex-disjoint paths that cover all vertices of G , where a path may consist of just one vertex. A (possibly disconnected) graph is called *μ -path-critical* if $\mu(G) = \mu$ and for each $v \in V(G)$ we have $\mu(G - v) = \mu - 1$. Path-critical graphs are natural extensions of hypotraceable graphs: 2-path-critical graphs and hypotraceable graphs are the same.

It was shown in [36] that 4-connected almost hypohamiltonian graphs exist. The blunt approach to achieve this is to consider a 3-connected hypotraceable graph T (for instance the graph of Horton [13]), a vertex w disjoint from T , and to connect w to every $v \in V(T)$. This yields the join of T and K_1 , i.e., $T + K_1$. Graph $T + K_1$ is almost hypohamiltonian, as proven in [36, Lemma 1]. What would be the analogous procedure for almost hypotraceable graphs, i.e., which properties must a (not necessarily connected) graph H have in order for the join of K_1 and H to be almost hypotraceable? Two conditions must be satisfied: at least three pairwise disjoint paths are necessary to span H and for any $v \in V(H)$, at most two disjoint paths suffice to span $H - v$. That is,

$$(\star) \mu(H) = 3 \quad \text{and} \quad (\star\star) \mu(H - v) = 2 \text{ for every } v \in V(H),$$

i.e., precisely 3-path-critical graphs. Therefore $H + K_1$ is almost hypotraceable if and only if H is 3-path-critical. Clearly, if the connectivity of a 3-path-critical graph G is $k \geq 0$, then the connectivity of the almost hypotraceable graph $G + K_1$ is $k + 1$. One might be inclined to think that no 3-path-critical graphs exist, but this would be false: take for instance $3K_1$, i.e., the disjoint union of three isolated vertices. Now $3K_1 + K_1 = K_{1,3}$, which is almost hypotraceable. But this is a very special case, since $3K_1$ is the only 3-path-critical graph consisting of three components. For otherwise, let $A \neq K_1$ be a component of such a graph. Then for every $v \in V(A)$ we have $\mu(A - v) \geq 1$, so $\mu(G - v) \geq 3$, which contradicts $(\star\star)$. Due to $(\star\star)$, it is also impossible that a 3-path-critical graph contains more than three components.

What if we consider two components A and B ? Clearly, at least one of them, say A , must be different from K_1 . If $B = K_1$, then $\mu(A) = 2$ and for any vertex $v \in V(A)$, the graph $A - v$ must be traceable and thus A is hypotraceable. (Since there are infinitely many hypotraceable graphs A , as shown by Thomassen [24], there exist infinitely many 3-path-critical graphs $A + K_1$ consisting of two components.) Now let both A and B contain more than one vertex. Due to $(\star\star)$, since $\mu(A - v) \geq 1$ for $v \in V(A)$, component B must be traceable. The same holds for A , so both A and B must be traceable. But then (\star) is not satisfied.

It remains to study connected 3-path-critical graphs. Due to very similar arguments as the ones presented above, if we consider two disjoint hypotraceable graphs T, T' and $v \in V(T)$, $v' \in V(T')$, identifying v with v' yields a connected 3-path-critical graph. As above, we may construct in this manner infinitely many connected 3-path-critical graphs, which yields infinitely many almost hypotraceable graphs of connectivity 2.

A construction of path-critical graphs of connectivity 3 appears implicitly in the paper [32] of the first author—these are essential ingredients in our approach towards finding 4-connected almost hypotraceable graphs. (Observe that even the construction of 2-connected path-critical graphs is far from trivial.) Here we describe these graphs explicitly (while a description also appears in the extended abstract [31] of the first author). For any $k \geq 1$ let G_k be the 3-connected graph defined in [32].

Proposition 5. *For every $k \geq 2$, the graph G_{4k-3} is 3-connected and k -path-critical.*

The proof of Proposition 5 essentially appears in [32] (and is given in the extended abstract [31], see Theorem 2.6), so we omit it here. Note that Proposition 5 is an extension of a theorem of Thomassen [24], who showed that G_5 is hypotraceable, i.e., 2-path-critical.

Theorem 5. *Let G' be a graph obtained from a 3-path-critical graph G by adding a new vertex w connected to each vertex of G . Then G' is an almost hypotraceable graph with exceptional vertex w .*

Proof. Graph G' is not traceable: if \mathbf{p} is a hamiltonian path of G' , then $\mathbf{p} - w$ consists of at most two disjoint paths that cover $V(G)$, a contradiction. It is obvious that $G' - w = G$ is also non-traceable. Thus, it remains to show that for any $w \neq v \in V(G')$ the graph $G' - v$ is traceable: since $\mu(G - v) = 2$, there exist two disjoint paths P and Q that cover $V(G) \setminus \{v\}$. Let p and q be end-vertices of P and Q , respectively. Now $P \cup Q \cup pwq$ is a hamiltonian path of $G' - v$. \square

By Proposition 5 we know that 3-connected 3-path-critical graphs exist, and thus, with Theorem 5, we obtain that 4-connected almost hypotraceable graphs also exist.

5 Discussion

1. Recently, the second author showed [36] that infinitely many 4-connected almost hypohamiltonian graphs exist, while Thomassen's question [27] whether 4-connected hypohamiltonian graphs exist remains open. (Even his simpler question [27] whether hypohamiltonian graphs with minimum degree at least 4 exist remains unanswered.) We ask here whether 5-connected almost hypotraceable graphs exist.

In the same vein, we ask whether there is a 4-connected hypotraceable graph. Even the weaker question whether there exists a 4-connected graph in which every vertex is missed by a longest path, raised by T. Zamfirescu in 1976 (see [39]) is still open.

2. In 1969, Walther [29] asked whether there exists a natural number k , such that every graph G contains k vertices, such that every longest path in G visits at least one of them. If there is such a k , how small can it be? T. Zamfirescu [39] showed that there exist graphs in which any two vertices are missed by some longest path. It is unknown whether there exists a graph in which any *three* vertices are missed by a longest path.

Related to this, Grünbaum defines [11] the set $\Pi(j, k)$ to be the family of all graphs G in which the difference between the order of the graph and the length of a longest path is $k + 1$ and for every $S \subset V(G)$ with $|S| = j$ there exists a longest path P such that $V(P) \cap S = \emptyset$. Clearly, $j \leq k$. Grünbaum [11] conjectured in 1974 that $\Pi(j, j) = \emptyset$ for every $j \geq 2$.

Let G be a connected graph of order n . Rautenbach and Sereni [19] defined $\text{lpt}(G)$ to be the minimum cardinality of a set $X \subset V(G)$ such that X intersects every longest path in G . They showed that $\text{lpt}(G) \leq \left\lceil \frac{n}{4} - \frac{n^{2/3}}{90} \right\rceil$.

3. With the approach given in Proposition 5 and Theorem 5, the smallest 4-connected almost hypotraceable graph we can obtain has order 73 (since we construct a graph G by taking nine copies of the Petersen graph minus two adjacent vertices, and then consider $G + K_1$). Are there smaller examples?

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