Non-hamiltonian triangulations with distant separating triangles

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Abstract. In 1996 Böhme, Harant, and Tkáč asked whether there exists a non-hamiltonian triangulation with the property that any two of its separating triangles lie at distance at least 1. Two years later, Böhme and Harant answered this in the affirmative, showing that for any non-negative integer d there exists a non-hamiltonian triangulation with seven separating triangles every two of which lie at distance at least d. In this note we prove that the result holds if we replace seven with six, remarking that no non-hamiltonian triangulation with fewer than six separating triangles is known.

Key words. Triangulation, separating triangle, non-hamiltonian.

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1 Introduction

A *triangulation* shall be a plane graph, i.e. a planar graph embedded in the plane, on at least four vertices, all of whose faces—including the unbounded face—are triangles. Triangulations are also known as maximal planar graphs. Every triangulation is 3-connected, so by a theorem of Whitney [10], its embedding is unique. By another beautiful result of Whitney [9] dating back to 1931, every 4-connected triangulation is hamiltonian, i.e. has a spanning cycle. In the past eighty years a plethora of stronger versions of this theorem have appeared. Perhaps the most famous one is Tutte's result [8] that planar 4-connected graphs are hamiltonian, as well! But we shall focus here on triangulations.

Consider a triangulation G. A triangle Δ in G is *separating* if $G - \Delta$ has more than one component. For triangles Δ and Δ' in G call the *distance* between Δ and Δ' the number of edges of a shortest path in G between $v \in V(\Delta)$ and $v' \in V(\Delta')$ for all possible combinations of v and v'.

In 1996 Böhme, Harant, and Tkáč [2], using deep structural results of Thomas and Yu [7], showed that (i) every non-hamiltonian triangulation has at least three

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separating triangles—this strengthens Whitney's theorem—, (ii) if a non-hamiltonian triangulation has exactly three separating triangles, then any two of them have distance at least 2, and (iii) there exists a non-hamiltonian triangulation with exactly six separating triangles such that no two of them have an edge in common (while some of these separating triangles do have vertices in common).

One of the most far-reaching extensions of Whitney's theorem is the result of Jackson and Yu [5] from 2002 that even if the triangulation has up to three separating triangles, it is still guaranteed to be hamiltonian. (In fact, they prove a far more general result, using the notion of a decomposition tree of the triangulation which encodes the relative position of the separating triangles, but for our purposes the aforementioned statement gives the strongest result they obtained.) This strengthens (i) and renders (ii) obsolete. Concerning (iii), two natural questions come to mind, explicitly raised in the paper of Böhme, Harant, and Tkáč (see [2, Remarks 1 and 2]): are there non-hamiltonian triangulations (a) with fewer than six separating triangles, or (b) in which any two separating triangles lie at distance at least 1?

While no progress has been made on (a) (and this seems to be a very challenging problem), Böhme and Harant [1, Theorem 1.4] showed that for any integer $d \geq 0$ there exists a non-hamiltonian triangulation with exactly seven separating triangles every two of which lie at distance at least d. In the following, we shall prove that the word "seven" in the statement of Böhme and Harant can be replaced with "six". It is worth mentioning that every known non-hamiltonian triangulation contains at least six separating triangles.

2 Result

Theorem. *For every $d \geq 0$ there exists a non-hamiltonian triangulation with six separating triangles, any two of which lie at distance at least d.*

Proof. As stated above, Harant, Böhme, and Tkáč [2] have proven the statement for $d = 0$, so it remains to be shown for $d \geq 1$. Consider the plane 3-connected graph G shown in Figure 1. G has six pairwise disjoint triangles, while all other faces of G are quadrangles. The distance between any two facial triangles in G is at least 1.

The graph $S(G)$ is obtained by taking each face F of G, inserting a new vertex v into the interior of F, and connecting v to all vertices in $V(F)$. Clearly, $S(G)$ is a triangulation, and $S(G)$ has exactly as many separating triangles as G had triangles, i.e. six. Let us colour all vertices of G black and all vertices in $S(G) - G$ white. Furthermore, denote the order of G with n. Since G has six pairwise disjoint triangles and all other faces are quadrangles, the size of G is equal to $(6 \cdot 3 + (f - 6) \cdot 4)/2$, where f is the number of faces in G. Thus, by Euler's formula, we have that f exceeds the order of G by exactly one, so $S(G)$ contains $n + 1$ white and n black vertices, and the set of white vertices forms an independent set. Thus removing from $S(G)$ all n black vertices we obtain $n + 1$ components, whence $S(G)$ is not 1-tough, so it cannot be hamiltonian [4]. We have proven the statement for $d = 1$.

We now apply the following transformation to G. As depicted in Figure 2, we draw three grey cycles crossing each quadrangle such that the cycle intersects no edges belonging to a triangle. Considering the intersections with the edges as new vertices, we obtain a new graph G'. Note that the new graph G' has six triangles, which are at

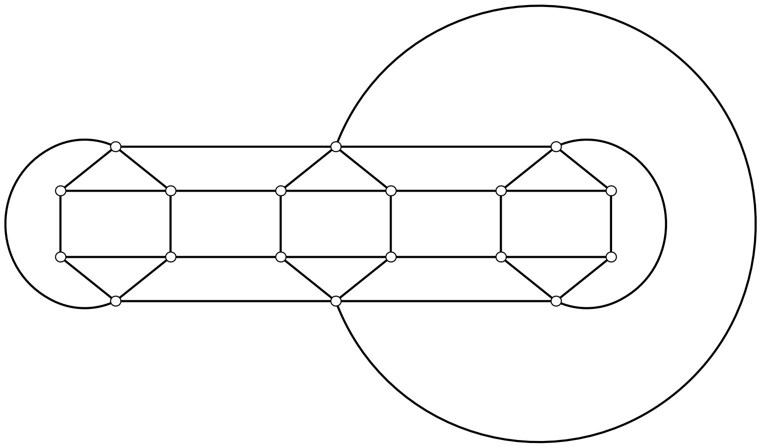

Figure 1: The plane 3-connected graph G in which six pairwise disjoint faces are triangles, and all other faces are quadrangles.

pairwise distance at least 2, and all other faces of G' are quadrangles. Thus, as above, we see that $S(G')$ is not 1-tough, and hence non-hamiltonian. This shows the case $d = 2$.

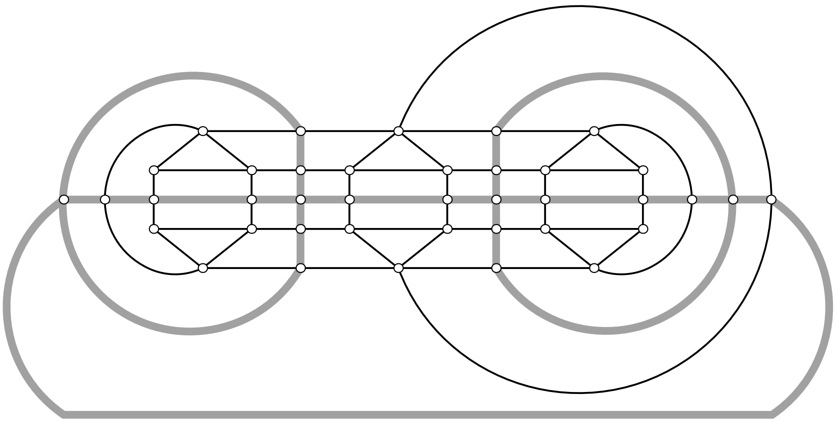

Figure 2: The plane 3-connected graph G' in which exactly six triangular faces occur, every two of which lie at distance at least 2, and all other faces are quadrangles.

Similarly, if we replace each grey cycle in Figure 2 with $d - 1$ parallel cycles, we guarantee that the distance between any two separating triangles is at least d. This completes the proof of the theorem. □

We have shown that there are non-hamiltonian triangulations with disjoint separating triangles. Our smallest example has 37 vertices and six separating triangles—every known triangulation with fewer separating triangles is hamiltonian. Positive counterparts to our theorem, i.e. conditions involving connectedness and the distance between separating triangles guaranteeing hamiltonicity, were studied in [1, 3].

Our statement presents a construction with six separating triangles, while Böhme and Harant showed it for seven [1]. With the ideas discussed above it is not difficult to generalise our result to t separating triangles for any $t \geq 6$. Unfortunately, our approach fails for $t < 6$ (a direct consequence of Euler's formula and our toughness argument).

We conclude with the following.

Remark. Toughness plays an essential role in our proof. Nishizeki [6] proved that there exist 1-tough non-hamiltonian triangulations—however, there exist non-disjoint separating triangles in his construction. We raise the following two questions.

(a) Are there 1-tough non-hamiltonian triangulations in which every two separating triangles are disjoint?

(b) Is there an integer d such that a 1-tough triangulation in which any two separating triangles lie at distance at least d must be hamiltonian?

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