Every graph occurs as an induced subgraph of some hypohamiltonian graph

Carol T. ZAMFIRESCU* and Tudor I. ZAMFIRESCU†

Abstract. We prove the titular statement. This settles a problem of Chvátal from 1973 and encompasses earlier results of Thomassen, who showed it for K_3, and Collier and Schmeichel, who proved it for bipartite graphs. We also show that for every outerplanar graph there exists a planar hypohamiltonian graph containing it as an induced subgraph.

Keywords. hypohamiltonian, induced subgraph.

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1 Introduction

Consider a non-hamiltonian graph G. We call G *hypohamiltonian* if for every vertex v in G, the graph $G - v$ is hamiltonian. In similar spirit, G is said to be *almost hypohamiltonian* if there exists a vertex w in G, which we will call *exceptional*, such that $G - w$ is non-hamiltonian, and for any vertex $v \neq w$ in G, the graph $G - v$ is hamiltonian. For an overview of results on hypohamiltonicity till 1993, see the survey by Holton and Sheehan [7]. For newer material which also includes work on the recently introduced almost hypohamiltonian graphs, we refer the reader to [4, 5, 8, 15] and the references found therein.

In 1973, Chvátal [2] asked whether every graph may occur as the induced subgraph of some hypohamiltonian graph. As Thomassen writes in [10], an important partial answer was provided by Collier and Schmeichel [3] who proved that every bipartite graph is an induced subgraph of some hypohamiltonian graph. In [11], Thomassen constructs an infinite family of planar cubic hypohamiltonian graphs and shows that certain edges can be added to these graphs such that the resulting graphs are hypohamiltonian, as well. He uses this to give a simple proof of the aforementioned result of Collier and Schmeichel.

*Department of Applied Mathematics, Computer Science and Statistics, Ghent University, Krijgslaan 281 - S9, 9000 Ghent, Belgium. E-mail address: czamfirescu@gmail.com

†Fachbereich Mathematik, Universität Dortmund, 44221 Dortmund, Germany and "Simion Stoilow" Institute of Mathematics, Roumanian Academy, Bucharest, Roumania and College of Mathematics and Information Science, Hebei Normal University, 050024 Shijiazhuang, P.R. China. E-mail address: tudor.zamfirescu@mathematik.uni-dortmund.de

Earlier, Thomassen [9] had proven that hypohamiltonian graphs of girth 3 and 4 exist, i.e. that K_3 and C_4 can occur as induced subgraphs of hypohamiltonian graphs—see also [3]. This refuted the conjecture of Herz, Duby, and Vigué [6] that hypohamiltonian graphs have girth at least 5. Thomassen emphasises in [10] that even for K_4 the answer to Chvátal's problem is unknown.

In this note, we prove that *any* graph can appear as an induced subgraph of some hypohamiltonian graph.

2 Auxiliary results

Consider a planar almost hypohamiltonian graph with a cubic exceptional vertex, for example the graph F of order 36 (discovered by Goedgebeur and the first author [5], and independently by Wiener [13, 14]), with exceptional vertex v. Note that F is the smallest known planar graph fit for the construction to come—however, there might be smaller graphs usable which have not been found yet. In fact, there is a smaller planar almost hypohamiltonian graph known (found by Wiener [14] and of order 31), but it does not contain a cubic exceptional vertex, which is needed for the method to work.

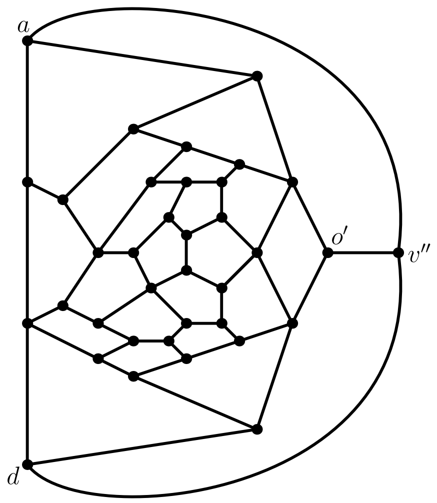

Fig. 1: The copy F'' of F. The exceptional vertex is marked v''.

Take two disjoint copies F', F'' of F, with $v' \in V(F')$ and $v'' \in V(F'')$ corresponding to v. Put $N(v') = \{o, b, c\}$ and $N(v'') = \{o', d, a\}$. For an illustration of F'', see Fig. 1. Take $F' - v'$, $F'' - v''$, identify o with o', and add the edges ab and cd. The neighbours a, b, c, d are chosen such that we obtain the graph H depicted in Fig. 2. (The "half-edges" shown in Figs. 2 and 3 end in vertices outside H.) Already Thomassen used such a construction in [9]. In what follows, we see $F' - v'$ and $F'' - v''$ as subgraphs of H. In particular, a, b, c, d denote vertices in H, as well.

Suppose an arbitrary but fixed graph W has a hamiltonian cycle Λ, and let W include the graph H such that among the vertices of H only a, b, c, d are connected by edges with vertices in $W - H$.

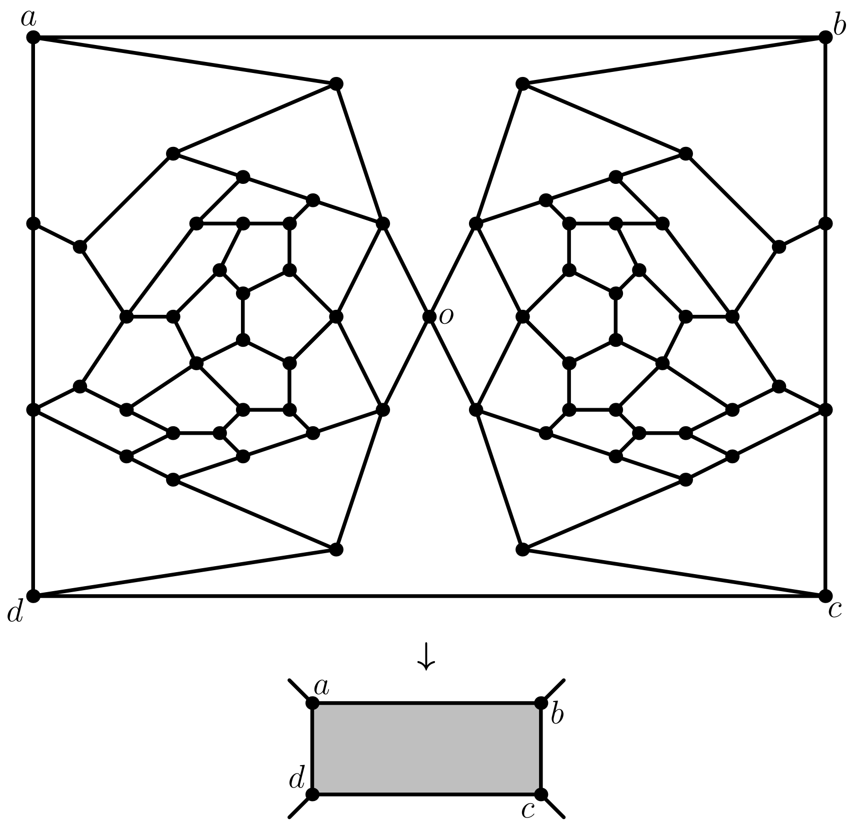

Fig. 2: We depict diagrammatically the graph H shown above with the rectangle shown below.

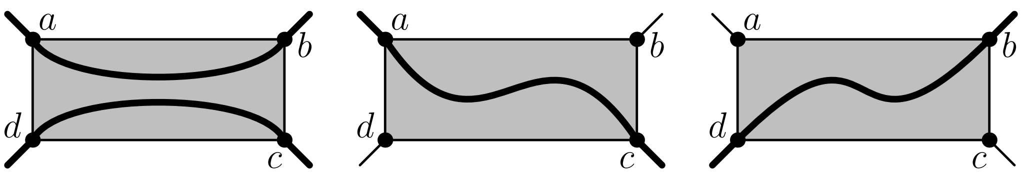

Fig. 3: There are two essentially different ways to traverse the rectangle $abcd$: in the first situation ($ab \cup cd$) we traverse disconnectedly, while in the latter two (ac or bd) we traverse diagonally.

Lemma 1. *Either $H \cap \Lambda$ is (i) the union of two disjoint paths, one from a to b and the other from c to d, or (ii) a path from a to c, or (iii) a path from b to d.*

Proof. There is no hamiltonian path $\mathfrak{p}$ in H between a and b, since $\mathfrak{p}$ would have to use the edge dc, yielding a hamiltonian path between a and d in $F'' - v''$ or a hamiltonian path between c and b in $F' - v'$ (depending on when $\mathfrak{p}$ picks up o). Adding the path $av''d$ or $bv'c$, respectively, we get a hamiltonian cycle in F and a contradiction is obtained. Now assume there is a hamiltonian path $\mathfrak{q}$ in H between a and d. Then there exists a hamiltonian path between o and b in F' or a hamiltonian path between o and c in F'. As above, we are led to a contradiction since F is non-hamiltonian.

If $H \cap \Lambda$ is the union of two disjoint paths, one from b to c, the other from d to a, then one of these paths must contain o. Assume w.l.o.g. the former to be that path. Considering it in F' and adding to it the path $bv'c$, we obtain a hamiltonian cycle in F', a contradiction since F' is almost hypohamiltonian. □

In case (i) we say that H is *disconnectedly traversed*, while if case (ii) or (iii) occurs H is called *diagonally traversed*. In cases (ii) and (iii) of Lemma 1, when H is diagonally traversed, we say more precisely that H is $a - c$ and $b - d$ traversed, respectively.

Consider the graph G^* of Fig. 4. There, A, B, C, ..., Q are graphs isomorphic to H. The length of the cycle $\Gamma = wyy^*...y'$ equals the number of copies of H used to construct G^*, i.e. $|\{A, B, C, ..., Q\}| = |V(\Gamma)|$. Γ is included in an arbitrary hamiltonian graph Z with the hamiltonian cycle Γ.

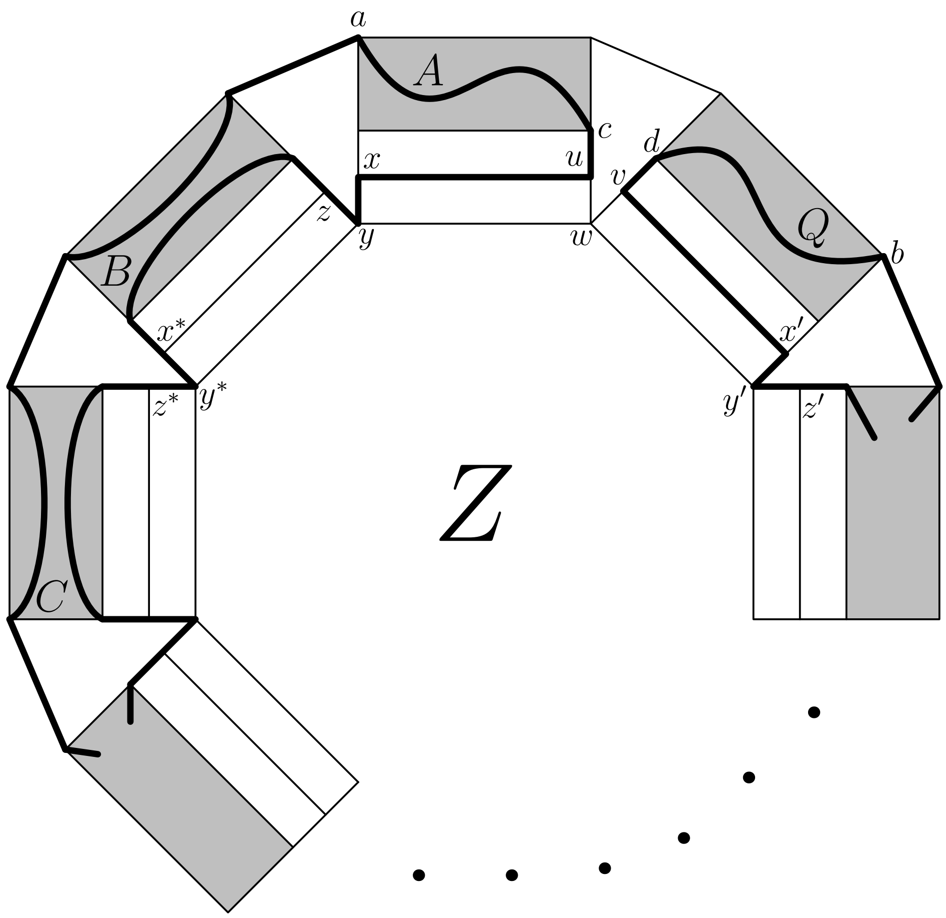

Fig. 4: The graph G^*.

Lemma 2. *The graph G^* is not hamiltonian.*

Proof. Suppose G^* has a hamiltonian cycle Λ. Obviously, not all copies of H are disconnectedly traversed. Suppose A is $a - c$ traversed. Then Λ quickly visits y, as it must continue with the path $cuxy$. If B is also diagonally traversed, then analogously Λ quickly visits y, and Λ is not hamiltonian. Hence, by Lemma 1, B is disconnectedly traversed. Thus $xyz \subset \Lambda$.

Necessarily, Q is $b - d$ traversed. So, analogously, $x'y'z' \subset \Lambda$. Continuing this reasoning, we see that no edge of Z is in Λ. Indeed, all other copies of H are disconnectedly traversed, so $x^*y^*z^* \subset \Lambda$ etc. Thus $w \notin \Lambda$, and a contradiction is obtained. □

3 Main results

A graph is *outerplanar* if it possesses a planar embedding in which every vertex belongs to the unbounded face. Note that a graph is outerplanar if and only if it does not contain a graph homeomorphic to K_4 or $K_{2,3}$, see [1]. We now present our main theorem.

Theorem 1. *Every graph is contained in some hypohamiltonian graph as an induced subgraph.*

Proof. Let G be an arbitrary graph. In the remainder of this proof, all notation refers to notions introduced in Section 2. Choose Z to be Γ to which G is added in such a way that the finite sequence of its vertices is placed at every second vertex of Γ. (So, the length of Γ is $2|V(G)|$.) By Lemma 2, G^* is not hamiltonian. It remains to provide a hamiltonian cycle in $G^* - s$, for every vertex s in G^*.

A hamiltonian cycle of $G^* - w$ is shown in Fig. 4. By changing uxy into uwy we get a hamiltonian cycle in $G^* - x$.

Due to the symmetries, it remains to show that $G^* - s$ is hamiltonian for any $s \in V(F')$. Consider $s \in V(F')$. There is a hamiltonian path in $F' - s$ joining b to c, or c to o, or o to b. In the second case (c to o), we change the route of Λ inside the subgraph spanned by $V(A) \cup \{x, y, u, w\}$ as shown in Fig. 5 (a), and in the third case (o to b), we change the route as depicted in Fig. 5 (b). In the first case, if s is the central vertex o of A, we obtain a hamiltonian cycle of $G^* - o$ as pictured in Fig. 6. For other positions of $s \in V(F')$, see Fig. 7. □

From the above proof, we immediately obtain the following.

Theorem 2. *If G is an outerplanar graph, then there exists a planar hypohamiltonian graph containing G as an induced subgraph.*

Theorem 2 cannot be extended to include *all* planar graphs due to an elegant argument of Thomassen, who proves in [11] that by Whitney's Theorem [12]—which states that planar triangulations without separating triangles are hamiltonian—, a planar triangulation cannot be an induced subgraph of any planar hypohamiltonian

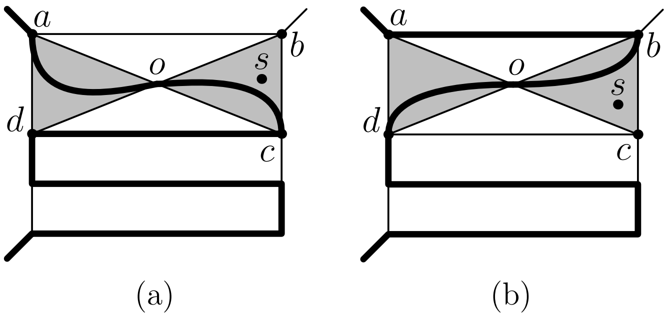

Fig. 5: The route of Λ inside the subgraph spanned by $V(A) \cup \{x, y, u, w\}$.

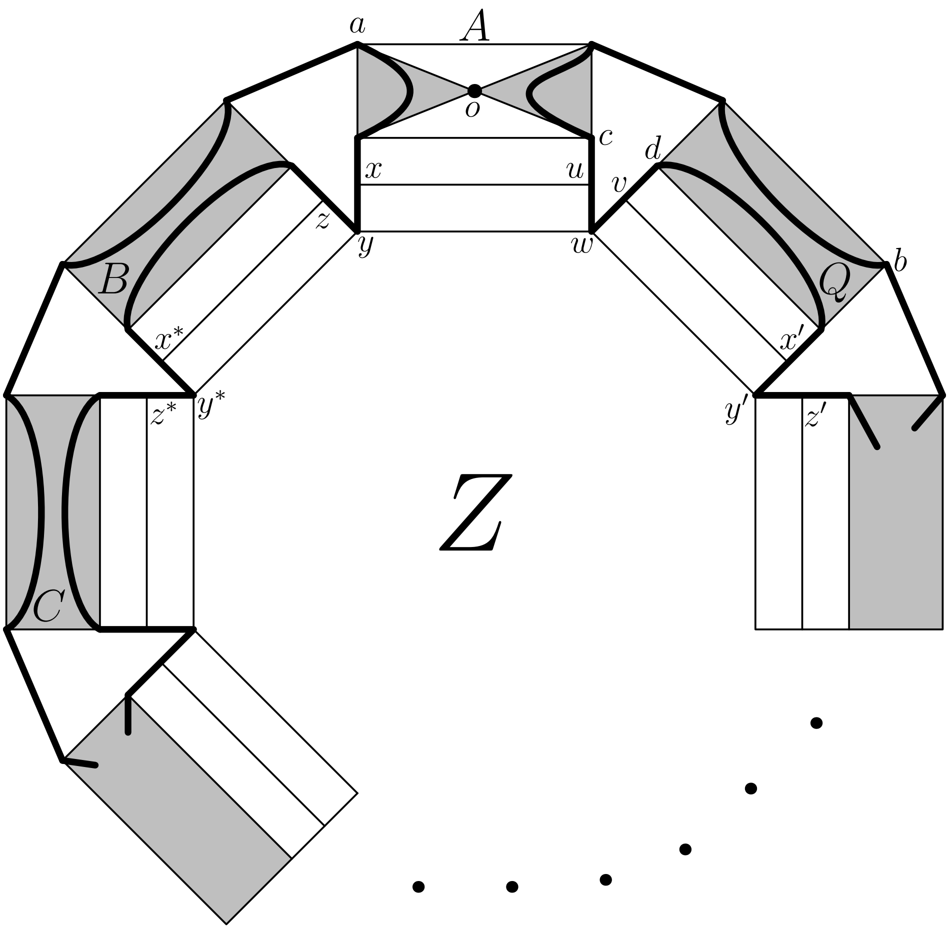

Fig. 6: A hamiltonian cycle of $G^* - o$.

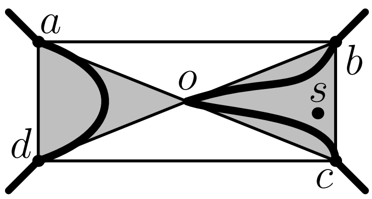

Fig. 7: A hamiltonian cycle of $G^* - s$.

graph. However, by Theorem 1, any planar triangulation is very well an induced subgraph of some (necessarily non-planar) hypohamiltonian graph.

Thomassen's results from [11] can be used to describe certain bipartite planar graphs which can be subgraphs of planar hypohamiltonian graphs. He goes on to write: "Maybe this is the case for every bipartite planar graph". This question remains unresolved, as for instance $K_{2,3}$ is planar yet neither a planar triangulation, nor among Thomassen's aforementioned planar bipartite graphs, nor an outerplanar graph.

Theorem 3.

(i) *There exists a hypohamiltonian graph of order* $20n$ *containing* K_n *as an induced subgraph. In particular, there exists a hypohamiltonian graph of order* 80 *containing* K_4.

(ii) *There exists a planar hypohamiltonian graph of girth* 3 *and order* 216.

(iii) *For an outerplanar graph* G *of order* n *there exists a planar hypohamiltonian graph of order* $144n$ *containing* G *as an induced subgraph.*

Proof. For (i), we may modify the construction from the proof of Theorem 1 by using every vertex of Γ (since G is in this case a complete graph). Thus, here the length of Γ is $|V(K_n)| = n$. Furthermore, we use the Petersen graph instead of the graph from Fig. 1. It is now easy to verify that the order of G^* is indeed $20n$.

The proof of (ii) is the same as the proof of (i), but we replace Petersen's graph by the plane almost hypohamiltonian graph shown in Fig. 1. Here, $n = 3$.

In (iii), the length of Γ is $2n$. We use $2n$ copies of the graph from Fig. 2, which is of order 69. We obtain a graph of order $((69 + 4) \cdot 2n) - 2n = 144n$. □

Part (ii) improves a bound given in [4, Corollary 3.4]. We end this note with the following.

Problem. *Characterise those planar graphs which occur as induced subgraphs of planar hypohamiltonian graphs.*

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