Infinitely many planar cubic hypohamiltonian graphs of girth 5

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In loving memory of Ella.

Abstract. A graph G is *hypohamiltonian* if G is non-hamiltonian and for every vertex v in G, the graph $G - v$ is hamiltonian. McKay asked in [*J. Graph Theory* **85** (2017) 7–11] whether infinitely many planar cubic hypohamiltonian graphs of girth 5 exist. We settle this question affirmatively.

Keywords. Hamiltonian, hypohamiltonian, planar, cubic, dot product

1 Introduction

Every graph in this paper will be undirected, finite, connected, and contain neither loops nor multiple edges, unless stated otherwise. A graph is *hamiltonian* (*traceable*) if it contains a cycle (path) visiting every vertex of the graph—such a cycle or path is called *hamiltonian*. A graph G is *hypohamiltonian* (*hypotraceable*) if G itself is non-hamiltonian (non-traceable), but for every vertex v in G, the graph $G - v$ is hamiltonian (traceable).

The study of hypohamiltonicity goes back to a paper of Sousselier [15] from the early sixties. The 1993 survey of Holton and Sheehan [8] provides a good overview. For the latest developments we refer the reader to the following articles and the references found therein: the papers [6, 7] by the two authors of this work, and articles by Jooyandeh, McKay, Östergård, Pettersson, and the second author [11], McKay [12], and the second author [18]. Ozeki and Vrána [13] recently used hypohamiltonicity to show that there exist infinitely many graphs which are 2-hamiltonian but not 2-edge-hamiltonian-connected, while Wiener [17] makes heavy use of hypohamiltonian graphs in his study of leaf-stable and leaf-critical graphs.

We call a vertex *cubic* if it has degree 3, and a graph *cubic* if all of its vertices are cubic. Consider a graph G. Two edges of G are *independent* if they have no common vertices. The *girth* of a graph is the length of its shortest cycle.

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2 Main result

In [12] McKay constructed three planar cubic hypohamiltonian graphs of girth 5 and raised the natural question whether infinitely many such graphs exist. We will now prove that this question has a positive answer.

Let G be a graph containing vertices v, w, x, y. Following Chvátal [4], the pair of vertices (v, w) is *good* in G if there exists a hamiltonian path in G with end-vertices v and w. Two pairs of vertices $((v, w), (x, y))$ are *good* in G if there exist two disjoint paths which together span G, and which have end-vertices v and w, and x and y, respectively.

Let H and H' be cubic graphs on at least six vertices. Consider $J = H - \{ab, cd\}$, where ab and cd are independent edges in H, $J' = H' - \{x, y\}$, where x and y are adjacent vertices in H', and let a', b' and c', d' be the other neighbours of x and y in H', respectively. Then the *dot product* $H \cdot H'$ is defined as the graph

$$(V(H) \cup V(J'), E(J) \cup E(J') \cup \{aa', bb', cc', dd'\}).$$

(Note that under above conditions, the dot product may be disconnected. The dot product was defined by Isaacs in [10]. In fact, following Skupień [14], it seems that this operation was invented earlier by Adel'son-Vel'skiĭ and Titov [1].)

Theorem 1. *Let H and H' be cubic graphs, H non-hamiltonian and H' hypohamiltonian. Consider independent edges $ab, cd \in E(H)$. If each of*

$$(a, c),\ (a, d),\ (b, c),\ (b, d),\ \text{and}\ ((a, b), (c, d))$$

is good in $H - \{ab, cd\}$, and for each vertex $t \in V(H)$, at least one of (a, b) or (c, d) is good in $H - \{t, ab, cd\}$, then $H \cdot H'$ is a cubic hypohamiltonian graph as well. If H and H' are planar, and ab and cd lie on the same facial cycle, then the dot product can be applied such that $H \cdot H'$ is planar as well. If g (g') is the girth of H (H'), then the girth of $H \cdot H'$ is at least $\min\{g, g'\}$.

Proof. Put $G = H \cdot H'$ and denote by x, y the (adjacent) vertices which were deleted from H' to form the dot product. G is obviously cubic. Put $N(x) = \{a', b', y\}$ and $N(y) = \{c', d', x\}$ such that the unique neighbour of a' (b', c', d') in H is a (b, c, d). We treat $H - \{ab, cd\} = J$ and $H' - \{x, y\} = J'$ as subgraphs of G. W.l.o.g. we may assume that the aforementioned labeling of vertices satisfies that if H and H' are planar, then G is also planar.

We first show that G is non-hamiltonian. Assume there exists a hamiltonian cycle $\mathfrak{h}$ in G. If $\mathfrak{h} \cap J$ consists of two components P and P', then either $\mathfrak{h} \cap J + ab + cd$ is a hamiltonian cycle in H, a contradiction since H is non-hamiltonian, or P is a path with end-vertices a and b, and P' is a path with end-vertices c and d. But then $\mathfrak{h} \cap J'$ consists of two disjoint paths which together with $a'xb'$ and $c'yd'$ form a hamiltonian cycle in H', a contradiction since H' is hypohamiltonian. So $\mathfrak{h} \cap J$ consists of one component. If $\mathfrak{h} \cap J$ has end-vertices a, b or c, d, then we immediately obtain a contradiction (by adding the edge ab or cd) to the non-hamiltonicity of H, so we may assume that $\mathfrak{h} \cap J'$ is a hamiltonian path P in J' with end-vertices

$p \in \{a', b'\}$ and $q \in \{c', d'\}$. Then $P \cup pxyq$ is a hamiltonian cycle in H', once again a contradiction.

It remains to show that $G - s$ is hamiltonian for every $s \in V(G)$. Let $s \in V(J')$. Since H' is hypohamiltonian, there exists a hamiltonian cycle $\mathfrak{h}$ in $H' - s$. Put $Q = a'xb'$ and $Q' = c'yd'$. Note that if exactly one of Q and Q' lies in $\mathfrak{h}$, then necessarily $s \in \{x, y\}$, which is impossible, as neither x nor y lie in J'.

Case 1: Neither Q nor Q' lie in $\mathfrak{h}$. In this situation, $xy \in E(\mathfrak{h})$, and say a' and d' are the end-vertices of the path $P' = \mathfrak{h} - \{x, y\}$, which is a hamiltonian path in $J' - s$. (a, d) is good in J, i.e. there exists a hamiltonian path P in J with end-vertices a, d, so we obtain the hamiltonian cycle $P' \cup P + aa' + dd'$ in $G - s$ as desired. (If P' has end-vertices other than a', d', the argument is very similar. Likewise if $s \in \{a', b', c', d'\}$.)

Case 2: Both Q and Q' lie in $\mathfrak{h}$. $((a, b), (c, d))$ is good in H, so there exist disjoint paths P_1, P_2 in H which together span H, with end-vertices a, b and c, d, respectively. Denote by P_1', P_2' the components of $\mathfrak{h} - \{x, y\}$. Then $P_1 \cup P_2 \cup P_1' \cup P_2' + aa' + bb' + cc' + dd'$ is a hamiltonian cycle in $G - s$.

Now consider $s \in V(H)$. We know that at least one of (a, b) and (c, d) is good in $H - \{s, ab, cd\}$, say (a, b). So there exists a hamiltonian path P in $H - \{s, ab, cd\}$ with end-vertices a and b. Since H' is hypohamiltonian, $H' - y$ is hamiltonian, which implies that there exists a hamiltonian path P' in J' with end-vertices a' and b'. Now $P \cup P' + aa' + bb'$ is a hamiltonian cycle in $G - s$ (here it is crucial that $cd \notin E(P)$, as $cd \in E(H)$, but $cd \notin E(G)$). If (c, d) is good in $H - \{s, ab, cd\}$ and not (a, b), we use the fact that $H' - x$ is hamiltonian.

Finally, let g be the girth of H and g' be the girth of H'. Assume the girth of $H \cdot H'$ is $k < \min\{g, g'\}$, and let C be a cycle of length k in $H \cdot H'$. Clearly, C contains (at least) two edges $e_1, e_2 \in \{aa', bb', cc', dd'\}$ with $e_1 = aa'$ w.l.o.g. If $e_2 = bb'$, then $(C \cap J') \cup a'xb'$ is a cycle of length at most $k - 2$ in H'. If $e_2 = cc'$, then $(C \cap J') \cup a'xyc'$ is a cycle of length at most k in H'. If $k < g'$, in both cases we have a contradiction, so $k \geq g'$. But this is impossible, as $k < \min\{g, g'\}$. We have shown that the girth of $H \cdot H'$ is at least $\min\{g, g'\}$. □

The above result is inspired by a theorem of Fiorini [5], who showed that gluing two hypohamiltonian snarks using the dot product yields a new hypohamiltonian snark. Note that in Fiorini's statement the hypotheses he formulates are too weak to prove the result he advertises; Cavicchioli et al. [3] have identified this omission, but we believe their version is also not entirely correct. A more detailed account of this can be found in [7].

In the following, we will call the pair of edges ab, cd from the statement of Theorem 1 *suitable*. Let H be the graph from Figure 1. It is one of the three planar cubic hypohamiltonian graphs of girth 5 on 76 vertices found by McKay [12]. Using a computer, we found several suitable pairs of edges ab, cd in this graph. Since we will use double-digit numbers to label the vertices, we shall denote the edge vw by (v, w) for better readability.

Lemma 1. *Let H be the graph from Figure 1. The edge pair $(20, 28), (52, 62)$ is suitable in H.*

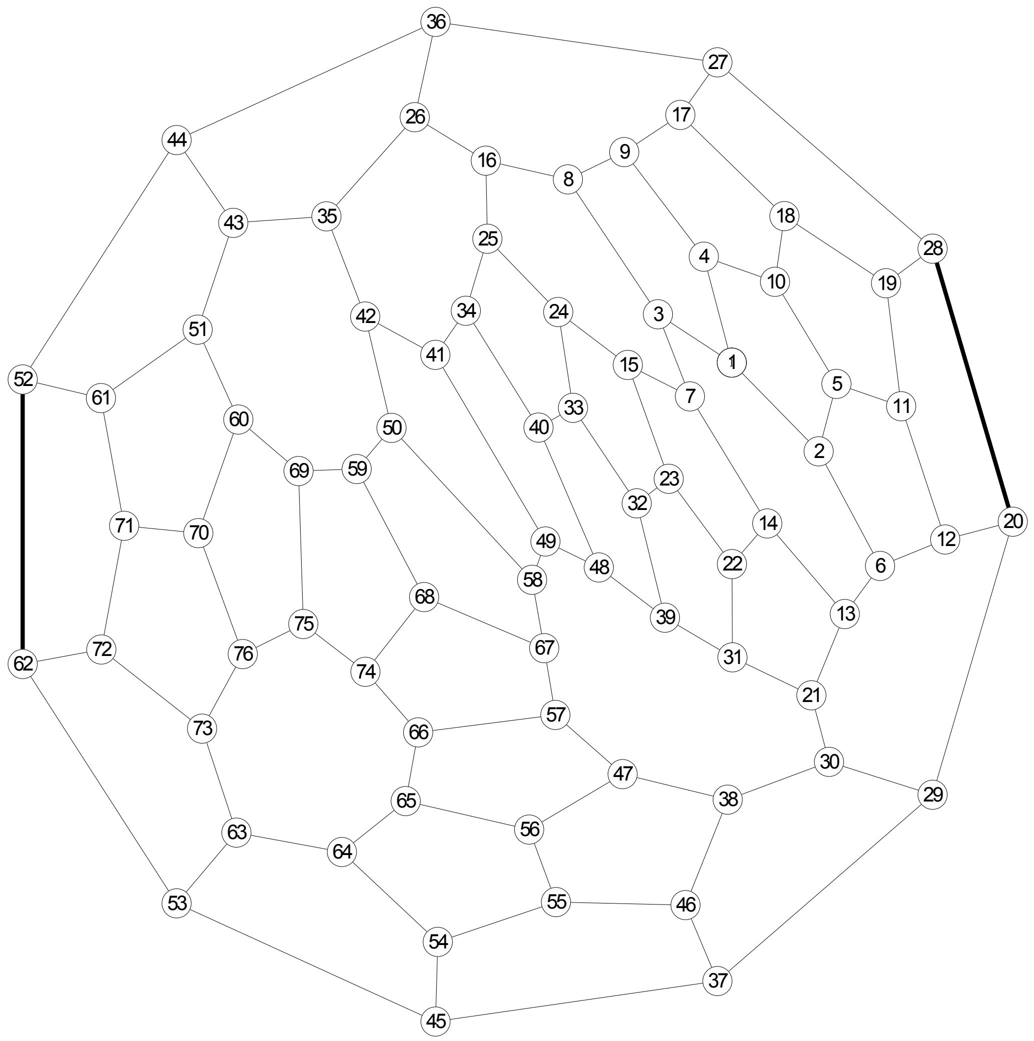

Figure 1: The planar cubic hypohamiltonian graph H. It has girth 5 and order 76. H is one of the three smallest planar cubic hypohamiltonian graphs of girth 5, and was found by McKay [12]. The suitable edge pair $(20, 28), (52, 62)$ is marked in bold.

Proof. It is clear from Figure 1 that the edges $(20, 28)$ and $(52, 62)$ are on the same facial cycle. (This guarantees that the dot product is planar.)

The pairs $(20, 52), (20, 62), (28, 52)$ and $(28, 62)$ are good in $H - \{(20, 28), (52, 62)\}$ due to the following hamiltonian paths, respectively:

- 52, 44, 36, 26, 16, 8, 3, 7, 14, 22, 23, 15, 24, 25, 34, 41, 42, 35, 43, 51, 61, 71, 72, 62, 53, 45, 37, 29, 30, 38, 46, 55, 54, 64, 63, 73, 76, 70, 60, 69, 75, 74, 66, 65, 56, 47, 57, 67, 68, 59, 50, 58, 49, 48, 40, 33, 32, 39, 31, 21, 13, 6, 2, 1, 4, 9, 17, 27, 28, 19, 18, 10, 5, 11, 12, 20
- 62, 53, 45, 37, 29, 30, 38, 46, 55, 54, 64, 63, 73, 72, 71, 61, 52, 44, 36, 26, 35, 43, 51, 60, 70, 76, 75, 69, 59, 68, 74, 66, 65, 56, 47, 57, 67, 58, 50, 42, 41, 49, 48, 39, 32, 23, 15, 24, 33, 40, 34, 25, 16, 8, 3, 7, 14, 22, 31, 21, 13, 6, 2, 1, 4, 9, 17, 27, 28, 19, 18, 10, 5, 11, 12, 20
- 52, 44, 36, 26, 16, 25, 24, 15, 7, 14, 13, 21, 30, 38, 46, 55, 54, 64, 63, 73, 76,

70, 60, 69, 75, 74, 66, 65, 56, 47, 57, 67, 68, 59, 50, 58, 49, 48, 39, 31, 22, 23, 32, 33, 40, 34, 41, 42, 35, 43, 51, 61, 71, 72, 62, 53, 45, 37, 29, 20, 12, 6, 2, 1, 3, 8, 9, 4, 10, 5, 11, 19, 18, 17, 27, 28

- 62, 53, 45, 37, 29, 20, 12, 6, 13, 14, 7, 15, 23, 22, 31, 21, 30, 38, 46, 55, 54, 64, 63, 73, 72, 71, 70, 76, 75, 69, 60, 51, 61, 52, 44, 43, 35, 42, 50, 59, 68, 74, 66, 65, 56, 47, 57, 67, 58, 49, 41, 34, 40, 48, 39, 32, 33, 24, 25, 16, 26, 36, 27, 17, 18, 10, 4, 9, 8, 3, 1, 2, 5, 11, 19, 28

Note that $((20,28),(52,62))$ is good in $H-\{(20,28),(52,62)\}$ due to the following two disjoint paths with end-vertices 20 and 28, and 52 and 62, respectively, which together span H.

- 28, 19, 11, 12, 20

- 62, 53, 45, 37, 29, 30, 21, 13, 6, 2, 5, 10, 18, 17, 27, 36, 26, 35, 42, 41, 34, 25, 16, 8, 9, 4, 1, 3, 7, 14, 22, 31, 39, 32, 23, 15, 24, 33, 40, 48, 49, 58, 50, 59, 69, 60, 70, 76, 75, 74, 68, 67, 57, 66, 65, 56, 47, 38, 46, 55, 54, 64, 63, 73, 72, 71, 61, 51, 43, 44, 52

The fact that at least one of $(20,28)$ or $(52,62)$ is good in $H-\{t,(20,28),(52,62)\}$ for every $t \in V(H)$ was shown by computer. We do not include the 76 hamiltonian paths here due to space constraints—76 paths are necessary since H has trivial automorphism group [12], but in each case we verified that the path found by the computer is indeed a valid hamiltonian path in the graph. □

As in Lemma 1, H will denote in the following the 76-vertex graph from Figure 1. Consider a copy H' of H. Using the pair of suitable edges lying on the same facial cycle in H we have found in Lemma 1, and any pair of adjacent vertices in H', we form the graph $H \cdot H'$. By Theorem 1, $H \cdot H'$ is a planar cubic hypohamiltonian graph of girth 5. (Theorem 1 states that the girth of $H \cdot H'$ is at least 5, and a planar 3-connected graph cannot have girth greater than 5.) Iterating this process (i.e., for a copy H'' of H, forming the dot product $H'' \cdot (H \cdot H')$, etc.) we obtain the following answer to McKay's question [12]. The second part of the statement was obtained by using the 78-vertex planar cubic hypohamiltonian graph of girth 5 constructed in [6] as H' in Theorem 1.

Theorem 2. *There are infinitely many planar cubic hypohamiltonian graphs of girth* 5*. More specifically, there exist such graphs of order n for every $n = 74k + a$, where $k \geq 1$ and $a \in \{2,4\}$.*

We remark that Theorem 1 may in fact be applied to show that there are smaller graphs which could be used in the above construction, since in its statement we only ask of H to be non-hamiltonian, not hypohamiltonian. However, our focus here was to solve McKay's problem—future work may include a more careful analysis of the graphs which may take the place of H.

In 1993, Holton and Sheehan [8] asked if there exists an integer n_0 such that a planar cubic hypotraceable graph exists for every even integer $\geq n_0$. Araya and

Wiener [2] settled the question affirmatively with $n_0 = 356$ by applying a method invented by Horton [9] and generalised by Thomassen [16] which combines five hypohamiltonian graphs into a hypotraceable one—in fact, the hypotraceable graphs of Araya and Wiener are not only planar, but 3-connected as well! They also showed that there exists a planar 3-connected cubic hypotraceable graph on 340 vertices. No smaller such graph is known. Applying Thomassen's construction method to the hypohamiltonian graphs described in [18, Theorem 8], we obtain planar 3-connected cubic hypotraceable graphs on n vertices for every even $n \geq 344$, thus improving the n_0 from the question of Holton and Sheehan.

We end this paper with an application of our result. We recall that before McKay's recent result from [12], all known planar cubic hypohamiltonian graphs had girth 4. With the above approach—knowing that the hypohamiltonian graphs used contain exactly one quadrilateral face—it is not difficult to verify that the girth of the resulting hypotraceable graph T depends on the selected pair of vertices x_i, y_i when applying Thomassen's construction method [16]: if for each chosen pair x_i, y_i at least one of x_i or y_i lies on the respective quadrilateral face, then T has girth 5, while if at least one pair is disjoint from the quadrilateral face, then T has girth 4. In contrast, when applying our Theorem 2, the choice of *any* pair x_i, y_i will yield a planar 3-connected cubic hypotraceable graph of girth 5.

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