A cut locus for finite graphs and the farthest point mapping

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Abstract. We reflect upon an analogue of the cut locus, a notion classically studied in Differential Geometry, for finite graphs. The *cut locus* $C(x)$ of a vertex x shall be the graph induced by the set of all vertices y with the property that no shortest path between x and z, $z \neq y$, contains y. The cut locus coincides with the graph induced by the vertices realizing the local maxima of the distance function. The function F mapping a vertex x to $F(x)$, the set of global maxima of the distance function from x, is the *farthest point mapping*. Among other things, we observe that if, for a vertex x, $C(x)$ is connected, then $C(x)$ is the graph induced by $F(x)$, and prove that the farthest point mapping has period 2. Elaborating on the analogy with Geometry, we study graphs satisfying *Steinhaus' condition*, i.e. graphs for which the farthest point mapping is single-valued and involutive.

Keywords. Graph distance function; cut locus; farthest point mapping; diameter; injectivity radius.

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1 Introduction

The main goal of this paper is to study the class of graphs for which the function which maps a vertex x to the set of vertices farthest from x (with respect to the metric defined via shortest paths), the so-called *farthest point mapping*, is both single-valued and involutive. (Note that in more established graph-theoretical terms, the farthest vertices from a fixed vertex x are just the vertices whose distance to x coincides with the eccentricity of x.) For an overview of results concerning distance in graphs, see the monograph by Buckley and Harary [9], and the surveys by Goddard and Oellermann [19], and Chartrand and Zhang [14]. The following two aspects will be reflected upon:

(i) We will work with the notion of "cut locus" from Differential Geometry (see [31], and for a "panoramic" view [4]; its connection to the famous optimal transport problem can be found in [38]), and analyze consequences of its application in Graph Theory. The geometric origins of the cut locus, in their most general setting, lie in a metric space (X, ρ), the metric of which is intrinsic. This motivates the graph theoretical definition, which mimics—as far as possible—its geometric analogue. Consider $x \in X$. $y \in X$ is a *cut point* of x if for any

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segment γ from x to y there exists no other segment $\sigma \supsetneq \gamma$ starting at x. The set of all cut points of x is called the *cut locus* of x. As it is of central importance, let us emphasize what we understand by *segment*: a shortest path between x and y, i.e. an arc of length $\rho(x, y)$. Explicitly, the geometric concept of the cut locus was introduced by Poincaré [37] in 1905, who used the term "ligne de partage", whereas implicit use occurred at least as early as 1881, in a paper by von Mangoldt [32]. Poincaré's work on surfaces was extended to Riemannian manifolds in the 1930's by Whitehead [40] (who coined the term "cut locus") and Myers [33, 34].

(ii) We also investigate graphs for which the farthest point mapping need not be single-valued, but the set of vertices realizing the local maxima and the set of vertices realizing the global maxima of the distance function coincide for every vertex of the graph. The underlying motivation for studying these graphs is a natural consequence of the involutivity of the farthest point mapping, the results arising from (i), and the study of the *injectivity radius* (i.e. the minimum over all distances between a vertex and its cut locus), another concept borrowed from Differential Geometry and relevant to our pursuits.

Recently, O'Rourke asked for translating the idea of the cut locus to Graph Theory [35]. Itoh and Vîlcu also connected the concepts of cut locus and graphs, but in a different sense: One of their main results is that for every graph G there exists a surface S and a point in S whose cut locus is isomorphic to G; rephrasing, every graph can be realized as a cut locus. This, and related results, can be found in a series of four papers [23–26].

1.1 Preliminaries

Throughout this paper all graphs will be simple, i.e. without loops or multiple edges[1], undirected, and finite, and have order at least 3, unless explicitly mentioned otherwise. For the inclusion relation between sets we will use the following notation. The symbol $\subset$ shall designate inclusion where equality may hold, and $\subsetneq$ inclusion where equality is forbidden.

Let $G = (V(G), E(G))$ be a connected graph, where $V(G)$ $(E(G))$ denotes its vertex (edge) set. For a set S we will denote by $|S|$ its cardinality. The edge between two vertices u and v will be denoted by uv. For vertex sets X, Y put $E(X, Y) = \{xy : x \in X, y \in Y\}$ and for subgraphs X, Y put $E(X, Y) = \{xy : x \in V(X), y \in V(Y)\}$. For a set of vertices or a (possibly disconnected) subgraph $H \subset G$, we shall use $\langle H \rangle$ for the graph induced by H. We also need the (set-theoretical) *complement* $\complement H = \langle V(G - H) \rangle$. We say that H *dominates* G if every vertex of $V(G) \setminus V(H)$ is adjacent with a vertex of H. Abusing notation, we will make no distinction between a vertex x, the graph $(\{x\}, \emptyset) \cong K_1$ and the set $\{x\}$. For two graphs G and G' we denote by $G \square G'$ the Cartesian product of G and G'. $S \subset V(G)$ is a *separator* (of G) if $\langle V(G) \setminus S \rangle$ is not connected. We denote by $\kappa(G)$ the connectivity of G, namely the minimum cardinality of a separator of G (if G is not a complete graph).

Consider now $H \subset J \subset G$, $Y \subset J \subset G$ with $H \cap Y = \emptyset$, $x \in V(H)$, and $y \in V(Y)$. An *xy-path* P in J is a path in J connecting x and y, while the *length of the path* P is equal to $|E(P)|$. Furthermore, let the *distance* between H and Y in J be defined as

$$d^J(H, Y) = \min_{x \in V(H),\, y \in V(Y)} \{|E(P)| : P \text{ } xy\text{-path in } J\}.$$

When there exists no path between H and Y in J, we put $d^J(H, Y) = \infty$. When $J = G$ we write $d^G(H, Y) = d(H, Y) = d_H(Y)$ for the distance between H and Y, and call d_H the *distance function* (with respect to H)—in this paper, we will use both the second $(d(H, Y))$ and third $(d_H(Y))$ notation, depending on which seems more intuitive in a given context.

The *eccentricity* of a vertex x, and the *radius* and *diameter* of G are defined as

$$\varepsilon(x) = \max_{y \in V(G)} d(x, y), \quad r(G) = \min_{x \in V(G)} \varepsilon(x), \quad \text{and} \quad \operatorname{diam}(G) = \max_{x \in V(G)} \varepsilon(x),$$

respectively. For a survey on diameters of graphs, see [15]. An important article on radii, diameters and minimum degree is [17]. The *centre* of a graph is $Z(G) = \{x \in V(G) : \varepsilon(x) = r(G)\}$. A graph is *self-centred* (also *equi-eccentric*), if $r(G) = \operatorname{diam}(G)$. Denote the set of all self-centred graphs by $\mathcal{C}$. For a survey on the family of graphs $\mathcal{C}$, see [7].

[1] Extending the results to multigraphs would result in additional technicalities in some proofs

We recall [27] that a finite metric space (X, ρ) is isometric to some graph if and only if (i) for any two points $x, y \in X$ we have $\rho(x, y) \in \mathbb{N}_0$, and (ii) if $\rho(x, y) \geq 2$ then there exists a third point z such that $\rho(x, z) + \rho(z, y) = \rho(x, y)$.

A shortest path σ between x and y, i.e. an xy-path of length $d(x, y)$, will be called an *xy-segment*, and its length denoted by $|\sigma|$. (Note that many graph theorists call the shortest path between two vertices "geodesic", but we opine that, especially when writing a paper concerning the cut locus, a notion from Differential Geometry, it is acceptable to deviate from standard graph-theoretical nomenclature.) In Geometry, this corresponds to the term "segment", or "geodesic segment", or "minimizing geodesic", i.e. a globally distance-minimizing arc. A graph theoretical notion corresponding to a "geodesic" (i.e. locally distance minimizing) can easily be defined, for instance by asking a path to admit no chord between any two of its vertices at distance 2 on the path, but will not be used in this paper.

Consider a subgraph $H \subset G$ and a vertex $y \in V(G)$. An *Hy-segment* is a path between some $x \in V(H)$ and y of length $d_H(y)$. Put

$$N_i(H) = \{y \in V(G) : d_H(y) = i\} \quad \text{and} \quad N_i[H] = \{y \in V(G) : d_H(y) \leq i\}, \quad i \geq 0.$$

$N_1(H) = N(H)$ ($N_1[H] = N[H]$) is the *neighbourhood* (*closed neighbourhood*) of H in G. When two vertices are at distance 1, we say that they are *neighbours*. For a vertex $x \in V(H)$ define its *degree* (with respect to H) as $\deg_H(x) = |\{y \in V(H) : d^H(x, y) = 1\}|$, and put $\deg_G(x) = \deg(x)$. Furthermore, let us write $V_1(H) = \{x \in V(H) : \deg(x) = 1\}$. Define

$$F(H) = \left\langle \left\{ x \in V(G) : d_H(x) = \max_{y \in V(G)} d_H(y) \right\} \right\rangle,$$

which is the graph induced by the vertices realizing global maxima of the distance function d_H, i.e. the graph induced by the *farthest vertices from H*. Two points $x, y \in V(G)$ are *antipodal* (each is the antipode of the other) if $d(x, y) = \operatorname{diam}(G)$. The function F is the *farthest point mapping*. Notice that for a vertex x we have $\varepsilon(x) = d(x, F(x))$. For a local version, put

$$M(H) = \langle \{ x \in V(\complement H) : d_H(x) \geq d_H(y) \ \forall y \in N(x) \} \rangle \supset F(H)$$

and, for $i \geq 1$,

$$M_i(H) = \{v \in N_i(H) : E(v, N_{i+1}(H)) = \emptyset\}.$$

Lemma 1.1. *(a)* $M(H) = \langle \bigcup_i M_i(H) \rangle$ *and*

(b) $M_i(H)$ *and* $M_j(H)$ *lie in different components of* $M(H)$ *for all* i, j, $i \neq j$.

Proof. By definition, all neighbours of a vertex in $M_i(H)$ are in $N_i(H) \cup N_{i-1}(H)$, therefore every vertex in $M_i(H)$ is a local maximum of the distance function. Vice versa, suppose v is a local maximum of the distance function: say $v \in N_j(H)$, for some j. We have that v cannot be incident with $N_{j+1}(H)$, therefore $v \in M_j(H)$.

Part (b) follows from the fact that, for $i \neq j$, $E(M_i(H), M_j(H)) = \emptyset$, given that $M_i(H)$ is incident only with $N_i(H) \cup (N_{i-1}(H) \setminus M_{i-1}(H))$. □

Continuing our thoughts from the introduction, we define the *cut locus* $C(x)$ of a vertex $x \in V(G)$ as the graph induced by the vertices $y \neq x$ with the property that for any xy-segment σ' there exists no other segment $\sigma \supsetneq \sigma'$ starting at x. Equivalently, $C(x)$ is the graph induced by the vertices y with the property that no xz-segment with $z \neq y$ contains y.

More generally, let us define the *cut locus* $C(H)$ of a subgraph $H \subsetneq G$ as the graph induced by the vertices $y \notin V(H)$ such that for no Hy-segment σ there exists a Hy'-segment σ' with $\sigma' \supsetneq \sigma$. Again, this may be reformulated as: $C(H)$ is the graph induced by the vertices $y \notin V(H)$ such that no Hz-segment with $z \neq y$ contains y. Put $C^0(H) = H$ and $C^k(H) = C(C^{k-1}(H))$ for $k \geq 2$.

The following Lemma provides a powerful yet easy to prove tool for studying the cut locus.

Lemma 1.2. *For any connected graph G and any subgraph H of G, the graph induced by the vertices realizing the local maxima of the distance function d_H coincides with the cut locus of H.*

Proof. First, we show $M(H) \subset C(H)$. Consider $x \in V(M(H))$. By Lemma 1.1, $x \in M_j(H)$ with $j = d_H(x)$. Let σ be an Hx-segment. Suppose by contradiction that there exists a segment $\sigma' \supsetneq \sigma$ starting at H and let $x' \in V(\sigma' \setminus \sigma) \cap N(x)$. For σ' to be a segment we should have $x' \in N_{j+1}(H)$, but this is not possible because x does not have neighbours in $N_{j+1}(H)$.

Second, we prove $C(H) \subset M(H)$. Consider $x \in V(C(H))$, an Hx-segment σ and, w.l.o.g., consider $x' \in N(x) \setminus V(\sigma)$. By definition, there exists no Hx'-segment σ' containing σ. Therefore, $d_H(x') \leq d_H(x)$, which implies $x \in V(M(H))$. □

We will implicitly use this lemma and refer to the graph induced by the local maxima of the distance function as $C(x)$.

The *boundary* of a graph G, denoted by $\partial(G)$, is defined [12] as follows.

$$\partial(G) = \{v \in V(G) : \exists u \in V(G) \text{ s.t. } \forall w \in N(v) \text{ we have } d_u(w) \leq d_u(v)\}.$$

Clearly, we have

$$\bigcup_{x \in V(G)} C(x) = \partial(G) \quad \text{and} \quad C(x) \subset \langle \partial(G) - x \rangle \ \forall x \in V(G).$$

A graph is *self-bounded* if $\partial(G) = G$.
The following result characterizes the graphs of the form $\bigcup_{x \in V(G)} C(x)$, for some G.

Theorem 1.3 (Chartrand, Erwin, Johns, and Zhang [12])**.** *A graph G is the boundary of some connected graph if and only if G does not have exactly one vertex with eccentricity* 1.

We require further concepts inspired by their analogue in Geometry. In a metric space we say that $F(x)$ is the *set of farthest points from x* (or global maxima of the distance function d_x which measures how far a given point is from x). Then the (geometric) cut locus $C(x)$ contains $F(x)$. As we have seen, this is also the case for graphs, where, for a graph G we even have $F(H) \subset C(H)$ for any subgraph $H \subset G$. The same holds in Geometry.

In light of geometrical results (see for instance Vîlcu's treatment of two conjectures of Steinhaus [39]) it seems natural to discuss the situations in which $x = C^2(x)$. We may ask for the *cut locus function* $C_G : \mathcal{G} \to \mathcal{G}$ with $H \mapsto C_G(H) = C(H)$ to satisfy $C_G \circ C_G \equiv \text{id}$, where $\mathcal{G}$ is the set of all (possibly disconnected) subgraphs of G.

Consider a function $f : \mathcal{G} \to \mathcal{G}$, and $H \in \mathcal{G}$. The sequence $\mathfrak{o}_{H,f} = (f^j(H))_{j \geq 0}$, where $f^0(H) = H$, is the *orbit of H under f*. $\mathfrak{o}_{H,f}$ is (k,p)-*periodic*, if $f^k(H) = f^{k+p}(H)$, and both k and p are minimal; a short argument shows that this is well defined. The number k is the *delay* and p is the *period* of $\mathfrak{o}_{H,f}$. (Whenever confusion is impossible, we shall say that k is the delay and p the period of f.) As G is finite, so is $\mathcal{G}$, whence so are both k and p. The *orbit of f*, $\mathfrak{o}_f = \{\mathfrak{o}_{H,f}\}_{H \in \mathcal{G}}$, is (ℓ, q)-*periodic*, if ℓ is the maximum delay among all $\mathfrak{o}_{H,f}$, and q is the least common multiple of all periods of the $\mathfrak{o}_{H,f}$.

1.2 Adjacent concepts

1.2.1 Tessellations

We reproduce a result from [3]. An infinite planar graph $\mathcal{T}$ is called *tessellating*, if the following conditions are satisfied.

(i) Any edge is the side of precisely two different faces.

(ii) Any two distinct faces have at most one vertex or at most one side in common.

(iii) Any face is a polygon with finitely many sides.

(iv) Every vertex has finite degree.

Furthermore, only *locally finite* tessellations will be considered: for every $p \in \mathbb{R}^2$ there is an open set containing p which meets only finitely many faces of the tessellation. We say that $\mathcal{T}$ has *no cut locus* if, for any fixed vertex $v \in V(\mathcal{T})$, the distance function with respect to v, $d_v^{\mathcal{T}}$, has no local maxima. We also require the notion of *combinatorial curvature* κ (see e.g. [22]) at a *corner* (v, f), where f is a face of $\mathcal{T}$ and $v \in V(\mathcal{T})$ lies on the boundary bd$f$ of f, which has $|\mathrm{bd} f|$ edges:

$$\kappa(v, f) = \frac{1}{\deg(v)} + \frac{1}{|\mathrm{bd} f|} - \frac{1}{2}.$$

We now present the main result from [3], a combinatorial analogue of the Hadamard-Cartan Theorem from Differential Geometry:

Theorem 1.4 (Baues and Peyerimhoff [3])**.** *Let $\mathcal{T}$ be a plane tessellating graph. If $\kappa(v, f) \leq 0$ for all corners (v, f), then the metric space $(\mathcal{T}, d^{\mathcal{T}})$ has no cut locus.*

Further results on this subject can be found in [2, 28, 30].

1.2.2 Geodetic sets

For two vertices $x, y \in V(G)$ we call the union of all xy-segments a *thick xy-segment* and denote it by $I[x, y]$ (this is also called "geodesic interval"). For $S \subset V(G)$, define the *geodetic closure* as

$$I[S] = \bigcup_{x,y \in S} I[x, y].$$

If $\langle I[S] \rangle = G$, then S is called a *geodetic set* for G. The *geodetic number* is the minimum cardinality of a geodetic set (see [10, 13, 21] for results on geodetic sets in graphs).

2 The cut locus and the farthest point mapping

2.1 Properties of the cut locus

The following remarks will be used tacitly; the proofs are easy or follow quickly from previous observations, and are left to the reader. Let $H \neq G$ be a non-empty (possibly disconnected) subgraph of G.

(R1) $\emptyset \neq F(H) \subset M(H) = C(H) \subset \complement H$.
(R2) $C(H) = \complement H$ *if and only if* $V(H)$ *dominates* G.
(R3) $V_1(G) \setminus V(H) \subset V(C^p(H))$ *for odd* p *and* $(V_1(G) \cap V(H)) \subset V(C^p(H))$ *for even* p.
(R4) *Each component of* $\complement H$ *contains a vertex of* $C(H)$*, and similarly:*
Each component of $\complement C(H)$ *contains a vertex of* H*.*
(R5) *If* $N[x] \subset V(H)$*, then* $C_G(H) = C_{G-x}(H)$*, and similarly:*
Every vertex in the cut locus has a neighbour which is not in the cut locus.
(R6) *If* $C(H)$ *is connected, then* $C(H) = F(H)$*. The inverse need not hold.*

Proposition 2.1. *If G is a graph and $H \subsetneq G$ a non-empty connected subgraph, no separator is fully contained in $C(H)$.*

Proof. First, consider a connected subgraph $H \subsetneq G$ and suppose the separator S is contained in $C(H)$. Then H has empty intersection with S. By definition, the deletion of S yields at least two connected components. Let X be the component of $G \setminus S$ in which H resides, and $Y \neq X$ a non-empty component of $G \setminus S$. Choose $y \in V(Y)$. An Hy-segment σ must intersect S, say in z. Thus, $z \notin V(C(H))$, as we may extend any Hz-segment to σ, whence, there exist vertices of S not lying in $C(H)$. □

Note that the above result does not hold for disconnected subgraphs. Take for instance $K_3 \Box P_3 = G$, and choose $H \subset G$ to be the two 3-cycles which have only cubic vertices. Then $C(H) \cong C_3$, which is a separating cycle in G. See Fig. 1.

The cut locus of a point on a Riemannian surface homeomorphic to the sphere is, topologically, a tree. On surfaces of higher genus, cycles do appear as subsets of the cut locus, but they are not null-homotopic.

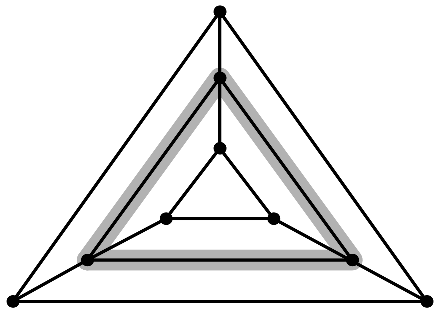

Fig. 1: The graph $G = K_3 \Box P_3$. Let $H \subset G$ be the two 3-cycles which have only cubic vertices. Then $C(H)$ is the highlighted 3-cycle.

Proposition 2.2. *For every $H \subset G$*

$$\left\langle \bigcap_{x \in V(H)} C(x) \right\rangle \subset C(H), \tag{1}$$

and we have equality in (1) for every $H \subset G$ if and only if G is complete.

Proof. Put $\left\langle \bigcap_{x \in V(H)} C(x) \right\rangle = Y_H$. For any $y \in V(Y_H)$ there must exist an $x_0 \in V(H)$ such that $d_y(H) = d_y(x_0)$. Clearly, for all $x \in V(H)$ we have $Y_H \subset C(x)$, so d_{x_0} has a maximum in y, and, as for all $z \in N(y)$ we have $d_z(x_0) \geq d_z(H)$, d_H has a maximum in y, whereby $y \in C(H)$. Therefore, $Y_H \subset C(H)$.

We now prove that if $G \cong K_n$, then $Y_H = C(H)$ for all $H \subset G$. As any subset S of V dominates G, we may use (R2) and obtain

$$\left\langle \bigcap_{x \in V(H)} C(x) \right\rangle = \left\langle \bigcap_{x \in V(H)} \complement x \right\rangle = \complement H = C(H).$$

It remains to show that if $Y_H = C(H)$ for all $H \subset G$, then G must be complete. Assume the contrary: then $\operatorname{diam}(G) \geq 2$. This means there exist $x, y \in V(G)$ at distance 2, so we have some $z \in N(x) \cap N(y)$. Putting $H = \langle V(G) \setminus \{z\} \rangle$, we obtain $C(H) = z$, but $Y_H = \emptyset$, a contradiction. □

Somewhat intriguingly,

$$C(H) \subset \left\langle \bigcup_{x \in V(H)} C(x) \right\rangle$$

need not hold. Consider a graph G with $C^2(x) = x$ for all $x \in V(G)$ (such graphs exist; take for instance even cycles, or the more general class of Blaschke-Steinhaus graphs, discussed in Section 3.1), and given a vertex x, let $H = \langle V(C(x)) \cup \{x\} \rangle$. In this case we even have

$$C(H) \cap \left\langle \bigcup_{x \in V(H)} C(x) \right\rangle = \emptyset.$$

Proposition 2.3. *Any graph G, connected or not, can arise as the cut locus of a vertex.*

Proof. Construct a graph G' as follows. Take G, add a new vertex x, and connect every vertex of G with x. Now for all $v \in V(G)$ we have $d_{G'}(x, v) = 1$, so by Lemma 1.2, we have $C(x) = G$. □

As mentioned in the Introduction, Proposition 2.3 mirrors the situation in Geometry in a certain sense, since by results of Itoh and Vîlcu for every graph G there exists a surface S and a point in S whose cut locus is isomorphic to G.

Theorem 2.4. *Consider a graph G and $H \subsetneq V(G)$. Then any $y \in V(G)$ lies on some HY-segment, for some connected component Y of $C(H)$.*

Proof. Among the segments starting at H and containing y, let σ be one of maximum length. Let z be the endpoint of σ that is not in H. We have $z \in N_i(H)$, where $i = |\sigma| = d_H(z)$.

Due to the maximality of σ, z has no neighbour in $N_{i+1}(H)$, thus by Lemma 1.1, $z \in V(C(x))$. Let Y be the connected component of $C(x)$ containing z, using again Lemma 1.1, we get $V(Y) \subset M_i(H) \subset N_i(H)$ and thus σ is the required HY-segment. □

It is easy to see that in Riemannian Geometry, above phenomenon (i.e. the union of all segments between a point x and its cut locus $C(x)$ covers the entire surface) holds as well.

Corollary 2.5. *Consider a graph G and $x \in V(G)$. $C(x) = F(x)$ holds if and only if for all $y \in V(G)$ there exists an $xF(x)$-segment σ such that $y \in V(\sigma)$.*

Proof. One direction follows from Theorem 2.4. We now show the converse. Assume by contradiction that there exists a $y \in (C(x) \setminus F(x)) \cap \sigma$, where σ is some $xF(x)$-segment. For σ to be a segment one of the two neighbours of y in σ must belong to $N_{j+1}(x)$, where $j = d_x(y)$. But this cannot be, since by Lemma 1.1 we have $y \in M_j(x)$. □

Corollary 2.6. *We have* $\left\langle \bigcup_{y \in C(x)} I[x,y] \right\rangle = G$ *for all* $x \in V(G)$.

2.2 A first example with interesting properties

One might believe that the cut locus and its associated function behave nicely in certain families of graphs—trees, for instance. As we shall see, this is not the case. In Fig. 2, a vertex y carries the number j if and only if $y \in V(C^j(x))$. Fig. 2 shows a graph G in which the orbit of the cut locus function is $(6,2)$-periodic for x. The highlighted subgraph is $C^6(x)$ (which dominates G). The graph G is a (counter)example showing the two following facts.

(i) *Consider a graph G. For fixed $x \in V(G)$ and p moving through the odd natural numbers, the functions $p \longmapsto |C^p(x)|$ and $p \longmapsto |C^{p-1}(x)|$ are not necessarily monotonically increasing.*

(ii) $\bigcup_{p \text{ even}} V(C^p(x)) \cap \bigcup_{q \text{ odd}} V(C^q(x))$ *is not necessarily empty.*

The first part of Observation (i) follows from $|C^3(x)| = 4 > 3 = |C^5(x)|$, while Observation (ii) is implied by $v \in V(C^3(x)) \cap V(C^6(x))$.

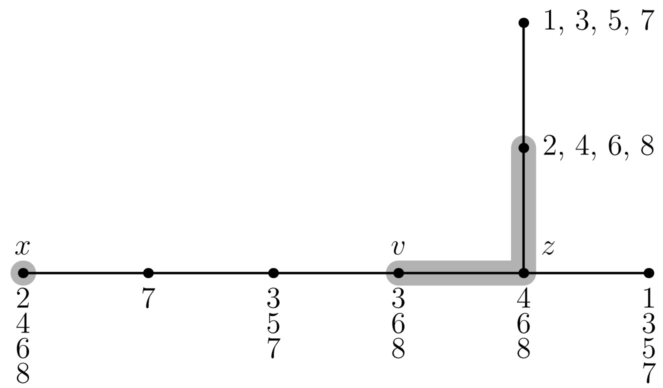

Fig. 2: The graph G_5 and $x, C(x), C^2(x), ..., C^8(x)$; $C^6(x)$ is highlighted.

Now consider a graph similar to G, where the number of vertices of the xz-path (including x and z) is six instead of five; we denote the graph obtained in this way by G_6. The second part of Observation (i) follows from the sequence $\{|V(C^k(x))|\}_{k=0}^{\infty}$ in G_6.

2.3 Iterating the cut locus function and the farthest point mapping

We write $F^k(H)$ for the graph induced by the vertices obtained when applying the farthest point mapping k times to a set of vertices or a subgraph H, and put $F^0(H) = H$.

Our treatment of the farthest point mapping differs from Buckley's distance dependent mapping [8], which assigns to any graph its "eccentric digraph" (for details, see [5, 8, 18]).

An easy corollary of (R3):

Proposition 2.7. *Let G be a graph with $V_1(G) \neq \emptyset$ and $\mathfrak{o}_{x,C_G}$ (k,p)-periodic. Then p is even for all $x \in V(G)$.*

As before, consider a graph G and let $\mathcal{G}$ be the set of all (possibly disconnected) subgraphs of G. Now assume that the orbit of the cut locus function is (k,p)-periodic for x. Trivially, both k and p are bounded above by $|\mathcal{G}|$.

Theorem 2.8. *Let (X,d) be a finite metric space. The farthest point mapping $F : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ has period 2, where $\mathcal{P}(X)$ is the power set of X.*

Proof. It is enough to prove that there exists an n_0 such that for every $H \subset X$, $F^{n_0+1}(H) \supset F^{n_0-1}(H)$. Indeed

$$F^{n_0+3}(H) = F^{n_0+1}(F^2(H)) \supset F^{n_0-1}(F^2(H)) = F^{n_0+1}(H).$$

thus, the sequence $F^{n_0-1}(H) \subset F^{n_0+1}(H) \subset F^{n_0+3}(H) \subset ... \subset X$ must become eventually constant and so there must exist, for every H, an integer k_H such that $F^{k_H+2}(H) = F^{k_H}(H)$. Now, for every integer $k \geq k_H$, $F^{k+2}(H) = F^k(H)$, therefore $k_0 := \max_{H \subset X} k_H$, is such that $F^{k_0+2}(\cdot) = F^{k_0}(\cdot)$. Consider $H \subset X$. We have

$$d(H, F(H)) = d(F(H), H) \leq d(F(H), F^2(H)) \leq \max_{x,y \in X} d(x,y),$$

where the first inequality follows from the definition of the farthest point mapping applied to $F(H)$. Now $\delta(i) := d(F^i(H), F^{i+1}(H))$ is an increasing function of i and has only a finite number of pairwise different values, it follows that there exists an n_H such that δ is constant from n_H on. Namely, for every integer $k \geq 0$

$$d(F^{n_H+k}(H), F^{n_H+k+1}(H)) = d(F^{n_H}(H), F^{n_H+1}(H)).$$

We can conclude that $n_0 := 1 + \max\limits_{H \subset X} n_H$ is such that for every H

$$d(F^{n_0-1}(H), F^{n_0}(H)) = d(F^{n_0}(H), F^{n_0+1}(H)).$$

Therefore $F^{n_0-1}(H)$ has the same distance from $F^{n_0}(H)$ as the set of farthest points from $F^{n_0}(H)$, namely $F^{n_0+1}(H)$. This implies $F^{n_0+1}(H) \supset F^{n_0-1}(H)$. □

In particular we have the following

Corollary 2.9. *The farthest point mapping on graphs has period 2.*

For a geometric point of view, consider a convex body in $\mathbb{R}^2$ with a long ellipse S as boundary. Let $a, b \in S$ be the two points at maximum distance (endpoints of the long axis). Now take a point $x \in S \setminus \{a, b\}$ such that $|F(x)| = 1$. Now we have $\{F(x)^k\} \cap \{a, b\} = \emptyset$ for all $k \in \mathbb{N}$, and there is no periodicity. Yet, a and b are the limit points of $\{F^k(x)\}_{k=1}^{\infty}$ and periodicity occurs, with period 2, for $x \in \{a, b\}$.

In contrast, the delay of the farthest point mapping in graphs is upper bounded by $|V(G)|$, since the value of n_0 defined in the proof of Theorem 2.8 is at most $|V(G)|$. (We conjecture that the delay of the farthest point mapping can be arbitrarily large, see Conjecture (C1) in the last section.

3 On the families $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{S}^*$

Consider a function $f : \mathcal{G} \longrightarrow \mathcal{G}$, and $x \in V(G)$. If the orbit of f is $(0,2)$-periodic, then f is an *involution*. In this section, we will be especially interested in graphs for which C_G, the cut locus function, is a single-valued involution. Asking less, a graph G is called *Steinhaus* if it satisfies *Steinhaus' condition*: $|F(x)| = 1$ and $F^2(x) = x$ for all $x \in V(G)$. (Asking even less, note that graphs satisfying $|F(x)| = 1$ for all $x \in V(G)$ were treated in the literature as "unique eccentric point graphs", see for instance [36].) Moreover, call G *Blaschke-Steinhaus* if additionally $C(x) = F(x)$ for all $x \in V(G)$. (The motivation for this name comes in a moment.)

Let the *injectivity radius* of a graph G be defined as

$$i(G) = \min_{x \in V(G)} d(x, C(x)).$$

In Geometry, the definition coincides with the above, except for the range of x and that the minimum must be replaced by an infimum. For more on the injectivity radius in Geometry, see [4]. In analogy to manifolds (see e.g. [29]), we call a graph G *Blaschke* if its injectivity radius and diameter are equal. Denote the set of all Blaschke graphs by $\mathcal{B}$, and the set of all (Blaschke-Steinhaus) Steinhaus graphs by $(\mathcal{S}^*)$ $\mathcal{S}$.

Steinhaus conjectured [16, p. 44] that if for a convex surface the farthest point mapping is single-valued and involutive—i.e. satisfies Steinhaus' condition—then the surface is a sphere. This was disproved by Vîlcu [39]. As Vîlcu did in Geometry, we want to explore which graphs satisfy Steinhaus' condition.

We begin with a few easy observations.

(R7) *If G is Blaschke, then for all $x \in V(G)$ we have $C(x) = F(x)$. Thus, for all $y \in V(G)$ there exists an $xF(x)$-segment containing y.*

The second part of (R7) follows from Corollary 2.5. Note that in a Steinhaus graph G, and for $x \in V(G)$, it is not imperative that for any $y \in V(G)$ there exists an $xF(x)$-segment σ such that $y \in V(\sigma)$: Consider I', the icosahedral graph minus one edge (this example is due to J. Itoh). For an appropriate vertex $x \in V(I')$, one easily finds that $y \in V(C(x) \setminus F(x))$ does not lie on any $xF(x)$-segment, as $d_y(x) = d_y(F(x)) = 2$ but $\mathrm{diam}(I') = 3$.

(R8) *Steinhaus graphs have even order.*

(R9) *Steinhaus graphs have geodetic number* 2.

Remark (R9) follows directly from Theorem 2.4, (R7), and the definition of the geodetic number (see 1.2.2).

Lemma 3.1. *If $C(x) = F(x)$ for all $x \in V(G)$, then $C^2(x) = F^2(x) = x$.*

Proof. By Corollary 2.5, every vertex lies on an $xF(x)$-segment. Therefore, for every $y \neq x$ we have

$$d(F(x), y) = d(y, F(x)) < d(x, F(x)) = d(F(x), x),$$

namely $x = F^2(x)$. ☐

Corollary 3.2. *A graph G is Blaschke-Steinhaus if and only if for every $x \in V(G)$ we have $|C(x)| = 1$.*

Using (R7) and Lemma 3.1 we get

Corollary 3.3. *If $G \in \mathcal{S} \cup \mathcal{B}$, then $F^2(x) = x$ for every $x \in V(G)$.*

Lemma 3.4. *For $G \in \mathcal{S} \cup \mathcal{B}$ we have $d(x, F(x)) = \mathrm{diam}(G)$ for all $x \in V(G)$.*

Proof. The statement is true by definition for Blaschke graphs, so it remains to be proven for Steinhaus graphs. Assume by contradiction the statement is not true and consider $x, y \in V(G)$ such that $d(x, F(x)) = \mathrm{diam}(G) =: d$ and $y \in N(x)$, with $d(y, F(y)) = k \leq d-1$ (such a pair exists because the graph is connected). We have $d(y, F(x)) = d-1$, so $k = d-1$ and $F(x) \subset F(y)$, whence, $F(y) = F(x)$, but $F^2(y) = x \neq y$, which is a contradiction. ☐

Corollary 3.5. *If $G \in \mathcal{S} \cup \mathcal{B}$, then G is self-centred and thus, self-bounded.*

In the context of Lemma 3.4, let us mention the following related result. For vertices $x, y \in V(G)$ denote by $D(x, y)$ the length of the longest xy-path.

Theorem 3.6 (Chartrand, Escuadro, and Zhang [11])**.** *Let G be a 2-connected graph. If x and y are two vertices of G for which $D(x, y) = d(x, y)$, then x and y are antipodal.*

For the next result we follow an argument of Bacher [1]. Call a vertex permutation ρ of a graph G an *antipodal map* if $d(v, \rho(v)) = \operatorname{diam}(G)$ for every $v \in V(G)$. Notice that in Blaschke-Steinhaus graphs, a function is a farthest point mapping if and only if it is an antipodal map. Given $S \subset V(G)$, put $\mathcal{A}(S) = \{v \in V(G) : \exists w \in S \text{ s.t. } d(v, w) = \operatorname{diam}(G)\}$. We now have the following.

Proposition 3.7 (Bacher [1])**.** *A graph G admits an antipodal map if and only if $|\mathcal{A}(S)| \geq |S|$ for every $S \subset V(G)$.*

Recall that $Z(G)$ is the set of points whose eccentricity equals the radius of G.

Proposition 3.8. *If G is such that for any vertex x we have $F^2(x) = x$ and $|Z(G)| \geq 2$, then $\kappa(G) = 2$.*

Proof. Suppose not and let $\{x\}$ be a separator. Take $z \in Z(G)$, with $z \neq x$ and let Y be the component of $G - x$ containing z. Let

$$d_1 = \max_{v \in V(Y)} d(x, v) \qquad \text{and} \qquad d_2 = \max_{v \in V(G) \setminus V(Y)} d(x, v).$$

Observe that if $d_1 \neq d_2$, then $F(x) \subset V(Y)$ or $F(x) \subset V(G) \setminus V(Y)$, violating in both cases the condition $F^2(x) = x$. It follows that $d_1 = d_2 = r(G)$, being x in the centre of G. On the other hand,

$$\max_{v \in V(G) \setminus V(Y)} d(z, v) \geq d_2 + 1 = r(G) + 1,$$

which is a contradiction, because $z \in Z(G)$. □

Corollary 3.9. *For $G \in \mathcal{S} \cup \mathcal{B}$ we have $\kappa(G) \geq 2$.*

Proposition 3.10. *We have*

$$\mathcal{S}^* \subsetneq \mathcal{S}, \quad \mathcal{S}^* \subsetneq \mathcal{B}, \quad \mathcal{S} \cap \mathcal{B} = \mathcal{S}^*, \quad \textit{and} \quad \mathcal{S} \cup \mathcal{B} \subsetneq \mathcal{C}.$$

Proof. Evidently, $\mathcal{S}^* \subset \mathcal{S}$, but they do not coincide, as the 1-skeleton of the icosahedron from which we delete one edge satisfies Steinhaus' condition but is not a Blaschke-Steinhaus graph. Furthermore, Blaschke-Steinhaus graphs are Blaschke, but not vice versa. If G is Blaschke-Steinhaus, then $x = C^2(x)$ and $C(x) = F(x)$ for all $x \in V(G)$. From Lemma 3.4 we know that $d(x, F(x)) = d(x, C(x)) = \operatorname{diam}(G)$ for all $x \in V(G)$, so we have for the injectivity radius $i(G) = \operatorname{diam}(G)$, whence, G is Blaschke. The converse is not true: simply consider an odd cycle.

From (R7) we deduce that if a graph is both Blaschke and Steinhaus, then it is Blaschke-Steinhaus.

By Corollary 3.5, $\mathcal{S} \cup \mathcal{B} \subset \mathcal{C}$. Equality does not hold. Consider a 7-cycle $v_1 ... v_7 v_1$, add a vertex v_8, and the edges $v_5 v_8$ and $v_8 v_7$: one obtains a self-centred graph which is neither Blaschke nor Steinhaus. □

By keeping Corollary 3.5 in mind, we can extract from the following Theorem necessary conditions for a graph to be in $\mathcal{S} \cup \mathcal{B}$.

Theorem 3.11 (Buckley [6])**.** *Let G be a self-centred graph with n vertices, m edges, and diameter d.*
(1) If $d = 1$, then $m = \binom{n}{2}$,
(2) If $d = 2$ and $n = 4$, then $m = 4$,
(3) If $d \geq 2$ and $n \geq 2d \neq 4$, then

$$\left\lceil \frac{nd - 2d - 1}{d - 1} \right\rceil \leq m \leq \frac{n^2 - 4nd + 5n + 4d^2 - 6d}{2}.$$

All integer values of m in the range given in (3) are attained.

Theorem 3.12. $(\mathcal{S}, \Box)$ *is a commutative monoid.*

Proof. It is well-known that the Cartesian product between graphs is commutative and associative (modulo isomorphisms). Therefore we need only show that given $G, H \in \mathcal{S}$, we have $G \Box H \in \mathcal{S}$.

Observe that for all graphs G, H and all vertices $(u_1, u_2), (v_1, v_2) \in G \Box H$, we have

$$d_{G \Box H}((u_1, u_2), (v_1, v_2)) = d_G(u_1, v_1) + d_H(u_2, v_2)$$

Therefore, given $G, H \in \mathcal{S}$, for all $(u_1, u_2) \in G \Box H$ we have that

$$F_{G \Box H}(u_1, u_2) = (F_G(u_1), F_H(u_2))$$

and $F_{G \Box H}$ is single-valued and involutive. This proves the assertion. □

3.1 Properties of Blaschke-Steinhaus graphs

Two paths are *internally disjoint* if their intersection coincides with their respective end-vertices. In the following, put $\operatorname{diam}(G) = d$.

Lemma 3.13. *For $G \in \mathcal{S}^*$, $x \in V(G)$, and $y \in N(x)$, we have $F(y) \in N(F(x))$.*

Proof. By Corollary 2.5, there exists an $xF(x)$-segment σ with $F(y) \in V(\sigma)$. Moreover, $d(x, F(y)) = d - 1$, as x has a unique antipode, and $y \in N(x)$. So $d(F(y), F(x)) = 1$. □

As a direct consequence of the above, for a vertex x in a Blaschke-Steinhaus graph there must exist a bijection between $N(x)$ and $N(F(x))$, so we have the following.

Corollary 3.14. *In a Blaschke-Steinhaus graph, antipodal vertices have the same degree.*

Corollary 3.15. *For $G \in \mathcal{S}^*$ and $x \in V(G)$ there exist two internally disjoint $xF(x)$-segments.*

Proof. By Lemma 3.4, for any $x \in V(G)$ there exists an $xF(x)$-segment σ_1 with $\{x = v_0, ..., v_d = F(x)\} = V(\sigma_1)$, where $d_x(v_i) = i \in \{0, ..., d\}$. By Lemmas 3.4 and 3.13, $F(v_1) \in N(F(x)) \setminus V(\sigma_1)$. Furthermore, $F(v_2) \in N(F(v_1))$, but $F(v_2)$ cannot lie on σ_1, because this would contradict Lemma 3.4 (as $d(v_2, F(v_2)) < d$). We iterate this analogously—notice that in a Blaschke-Steinhaus graph the farthest point mapping is single-valued and bijective—up to $F(v_{d-1})$. This vertex lies in $N(x) \cap N(F(v_{d-2})) \setminus V(\sigma_1)$. As there exists an edge $F(v_i)F(v_{i+1})$ for all $i \in \{0, ..., d-1\}$, we have proven the existence of an $xF(x)$-segment σ_2 internally disjoint from σ_1. □

In $G \in \mathcal{S}^*$, consider two $xF(x)$-segments σ and σ' with the property that each vertex of σ has its antipode in σ' (and vice versa); the existence of such a pair of $xF(x)$-segments can be derived from the proof of Corollary 3.15. In this case, $\sigma \cup \sigma'$ forms a cycle of length $2d$. We shall refer to it as a *great cycle of* x and denote by Σ_x. Notice that we can choose σ_1 at will—so, by Corollary 2.5, for any $y \in V(G)$ we can choose σ_1 to contain y. Thus, we may restate Corollary 3.15 as

Proposition 3.16. *Let $G \in \mathcal{S}^*$, and $x, y \in V(G)$. Then there exists a great cycle of x containing y.*

Proposition 3.17. *In a Blaschke-Steinhaus graph G, we have $F \equiv C_G \in \operatorname{Aut}(G)$.*

Proof. Consider $x, y \in V(G)$. As $G \in \mathcal{S}^*$, $F \equiv C_G$. By Proposition 3.16, there exists a great cycle Σ_x of x containing y. Evidently, $F(y) \in V(\Sigma_x)$. It is now clear that $d(x, y) = d(F(x), F(y))$. □

Clearly, $i(G) \leq r(G)$, and if G is 2-connected we have $i(G) \geq \lfloor g(G)/2 \rfloor$, where $g(G)$ denotes the girth of G. Any upper bound on the radius is also an upper bound on the injectivity radius.

Theorem 3.18 (Erdős, Pach, Pollack, and Tuza [17]). *If* $\delta(G) \geq 2$*, then*

$$r(G) \leq \frac{3}{2} \cdot \frac{n-3}{\delta+1} + 5.$$

This is optimal up to the additive constant.

By Corollary 3.9 and the remark above, and by Theorem 3.18, we have for a graph $G \in \mathcal{S} \cup \mathcal{B}$

$$\lfloor g(G)/2 \rfloor \leq i(G) \leq \frac{3}{2} \cdot \frac{n-3}{\delta+1} + 5.$$

3.2 A second example with interesting properties

A priori, one might believe that Blaschke-Steinhaus graphs are regular. But, as A. Fruchard points out with the beautiful example shown in Fig. 3, this is not true.

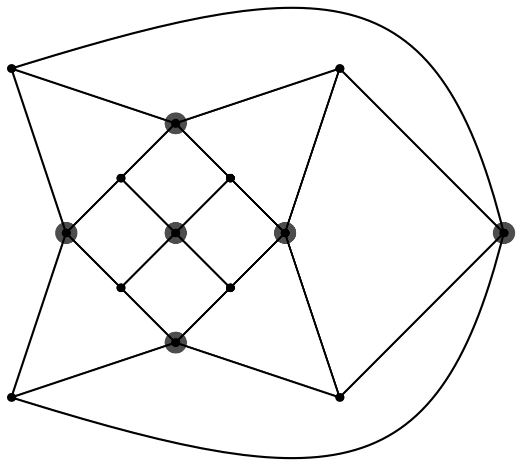

Fig. 3: The Fruchard graph.

The existence of the Fruchard graph proves that:

(i) *Blaschke-Steinhaus graphs need not be regular (whence, they need not be vertex-transitive) and*

(ii) *Blaschke-Steinhaus graphs need not be hamiltonian.*

Observation (i) is evident, Observation (ii) follows from the existence of an unbalanced bipartition of the vertex set of the Fruchard graph, shown in Fig. 3.

4 Final Remarks

Concerning infinite families of graphs in $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{S}^*$, notice that Blaschke-Steinhaus graphs can be very sparse (cycles of even length) or very dense (take a cycle $v_1v_2...v_{2k}v_1$ and add all possible edges except for v_iv_{i+k}). One may also consider a complete bipartite graph $K_{t,t}$, and delete a perfect matching: the resulting graph lies in $\mathcal{S} \cup \mathcal{B}$ (this example is due to K. Ozeki). Due to Theorem 3.12, we may easily construct infinitely many members of $\mathcal{S}$, for instance the d-dimensional cube, so we have an infinite family of Steinhaus graphs of arbitrarily high connectivity.

We conclude with observations, open questions, and conjectures.

1. By Theorem 2.8 we know that the period of the farthest point mapping is 2. (C1) We conjecture that its delay can be arbitrarily high. We also believe that (C2) the period of the cut locus function is 2 as well, and that (C3) its delay can be arbitrarily high, too.

2. Classifying Blaschke-Steinhaus graphs by connectivity seems an interesting endeavour in light of the results from Section 3.1, but even for connectivity 2 the problem is not trivial. We conjecture that (C4) a Blaschke-Steinhaus graph with connectivity 2 must be a cycle of even length—this resembles Steinhaus' original conjecture. Certainly, even more interesting would be structural answers concerning 3-connected Blaschke-Steinhaus graphs, for instance giving a complete list of Blaschke-Steinhaus graphs that are polyhedral; examples of such graphs are the 1-skeleta of the Platonic solids with the exception of the tetrahedron, as well as the Fruchard graph from Fig. 3.

In the same line of ideas, (C5) consider $G \in \mathcal{S}^*$ and a vertex x of degree k. We conjecture that if k is odd (even), there exist $k-1$ (exactly k) pairwise internally disjoint $xF(x)$-segments.

3. What follows is a discussion pertaining to Theorem 3.12. Initially, we tried to prove the result by introducing a new kind of operation between matrices, based on the Kronecker product—we believe that this new operation, which we were unable to find in the literature, may be of separate interest and therefore dedicate the following paragraphs to it.

Consider a graph G with $V(G) = \{v_1, ..., v_n\}$, and denote the *distance matrix* of G by D_G, i.e. the square matrix with non-negative integer entries, where entry (i,j) contains $d(v_i, v_j)$. We now give a characterization of a distance matrix via adjacency matrices, which is useful to recognize when a matrix is the distance matrix of a graph. (This is classical and can be found, for instance, in [20, p. 151]). A matrix $(d_{ij}) \in \mathbb{N}_0{}^{n\times n}$ is the distance matrix of a graph if and only if there exists an adjacency matrix $A \in \{0,1\}^{n\times n}$ such that

$$d_{ij} = \min\left\{k : a_{ij}^k \neq 0, i \neq j\right\} \quad \text{and} \quad d_{ii} = 0, \quad i,j \in \{1, ..., n\},$$

where a_{ij}^k is the (i,j)-entry of the matrix A^k. This connection between distance matrices and adjacency matrices is due to the following well-known fact. Let A be the adjacency matrix of a graph, and a_{ij}^k defined as above. Then a_{ij}^k represents the number of walks of length k between v_i and v_j. So, excluding the entries on the main diagonal, a_{ij}^k will be non-zero if and only if there exists a walk between v_i and v_j of length k. If we take the smallest value of k where a_{ij}^k, $i \neq j$, does not vanish, we obtain the shortest walk between v_i and v_j, which must be a v_iv_j-segment of length k. We observe that any symmetric matrix A with entries in $\{0,1\}$ and the property that no entry of the matrix

$$\sum_{k=1}^{\max\{2,\,\mathrm{diam}(G)\}} A^k$$

vanishes, is the adjacency matrix of some graph G. (The $\max\{2, \mathrm{diam}(G)\}$ in the sum is due to the following trivial case: If we simply write $\mathrm{diam}(G)$ and we have $\mathrm{diam}(G) = 1$, then the entries of a_{ii}^2 will be 0, because walks of length 2 (of the form $v_iv_jv_i$) are not taken into account, so the sum must have at least two members.)

Now let G and H be Steinhaus graphs of order n and p, respectively, and $D_G = (g_{ij})_{1\leq i,j\leq n}$, $D_H = (h_{k\ell})_{1\leq k,\ell\leq p}$ their associated (symmetric) distance matrices. Evidently, $D_G \in \{0, ..., \mathrm{diam}(G)\}^{n\times n}$ and $D_H \in \{0, ..., \mathrm{diam}(H)\}^{p\times p}$. As G and H are Steinhaus, D_G and D_H have special properties, which we summarize in the following statement, the proof of which is left to the reader.

Claim. *Consider a graph G. D_G is the distance matrix of a Steinhaus graph G if and only if (i) D_G is a distance matrix and (ii) every row and every column of D_G has exactly one maximum, which is the same for all rows (or columns). This maximum is precisely the diameter of G.*

Now define the following operation, denoted by $\boxplus$, between D_G and D_H.

$$D_G \boxplus D_H = D_G \otimes U_p + U_n \otimes D_H,$$

where $\otimes$ denotes the usual Kronecker product and U_k is the $k \times k$ matrix with all entries equal to 1. We have

$$\Gamma(D_G \boxplus D_H) = \Gamma(D_H \boxplus D_G) = G \square H = H \square G,$$

where $\Gamma(D)$ is the graph with distance matrix D. Note that in general $D_G \boxplus D_H \neq D_H \boxplus D_G$. In fact, we can get $D_G \boxplus D_H$ by permuting rows and columns of $D_H \boxplus D_G$, using the same permutation rule for rows and columns. This permutation defines the isomorphism between $G(D_G \boxplus D_H)$ and $G(D_H \boxplus D_G)$. The permutation is the one such that the $(n(i-1)+j)$-th column (row) becomes the $(p(j-1)+i)$-th column (row), for $i = 1, .., p$, $j = 1, ..., n$.

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