On hypohamiltonian and almost hypohamiltonian graphs

Carol T. ZAMFIRESCU

Dedicated to my father on the occasion of his 70th birthday.

Abstract. A graph G is *almost hypohamiltonian* if G is non-hamiltonian, there exists a vertex w such that $G - w$ is non-hamiltonian, and for any vertex $v \neq w$ the graph $G - v$ is hamiltonian. We prove the existence of an almost hypohamiltonian graph with 17 vertices and of a planar such graph with 39 vertices. Moreover, we find a 4-connected almost hypohamiltonian graph, while Thomassen's question whether 4-connected hypohamiltonian graphs exist remains open. We construct planar almost hypohamiltonian graphs of order n for every $n \geq 76$. During our investigation we draw connections between hypotraceable, hypohamiltonian and almost hypohamiltonian graphs, and discuss a natural extension of almost hypohamiltonicity. Finally, we give a short argument disproving a conjecture of Chvátal (originally disproved by Thomassen), strengthen a result of Araya and Wiener on planar cubic hypohamiltonian graphs, and mention open problems.

Key Words. Grinberg's Criterion, hypohamiltonian, planar.

MSC 2010. 05C10, 05C38, 05C45.

1 Introduction

Throughout this paper all graphs are undirected, finite, connected, and contain neither loops nor multiple edges. For undefined notions, please consult [29]. A graph G is *hypohamiltonian* if G does not contain a hamiltonian cycle but for any $v \in V(G)$ the graph $G - v$ does contain a hamiltonian cycle. Replacing in the preceding sentence "cycle" by "path", we obtain the definition of a *hypotraceable* graph. The study of hypohamiltonian graphs was initiated in the early sixties by Sousselier [21] and Gaudin, Herz, and Rossi [8]. Hypohamiltonian graphs were extensively studied by Thomassen [22–26]; other important contributors are Chvátal [4], and Collier and Schmeichel [6, 7], among others. For further details, see the survey by Holton and Sheehan [15]. Grötschel [10] discusses hypohamiltonian graphs in the context of the travelling salesman problem. Recent applications of concepts closely related to hypohamiltonicity can be found in [20].

In the early seventies, Chvátal [4] raised the problem whether there exists a planar hypohamiltonian graph, and offered \$5 for its solution [5, Problem 19]. Grünbaum conjectured that no such graph exists [11, p. 37]. In 1976, Thomassen [24] constructed infinitely many such graphs, the smallest among them having order 105. In 1979, Hatzel [12] found a smaller planar hypohamiltonian graph, having 57 vertices. This was improved to 48 by the author and Zamfirescu [32], to 42 by Araya and Wiener [30], and most recently to 40 vertices by Jooyandeh, McKay, Östergård, Pettersson, and the author [17]. The latter three graphs are shown in Fig. 1. The 40-vertex graph is the smallest example known so far, together with other 24 graphs of the same order [17].

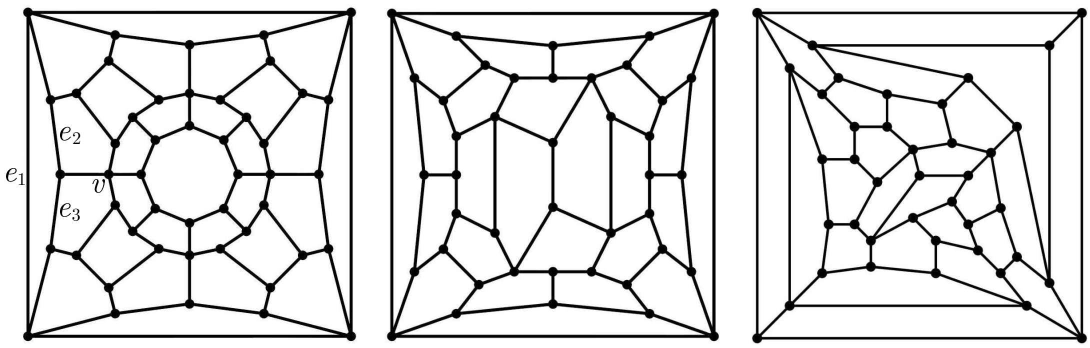


Fig. 1: From left to right: Planar hypohamiltonian graphs of order 48, 42, and 40, respectively. They held or hold the world record of smallest known planar hypohamiltonian graph for the periods 2007–9, 2009–12, and since 2012, respectively.

Kapoor, Kronk, and Lick [18] conjectured in 1968 that no hypotraceable graphs exist. This conjecture was refuted when a hypotraceable graph was subsequently found by Horton [16]. It has 40 vertices. Thomassen [22] showed that there exists a hypotraceable graph with n vertices for $n \in \{34, 37, 39, 40\}$ and every $n \geq 42$.

A graph G is *almost hypohamiltonian* if G is non-hamiltonian, and there exists a vertex w, which we will call *exceptional*, such that $G - w$ is non-hamiltonian, yet for any vertex $v \neq w$ the graph $G - v$ is hamiltonian. We denote the family of hypohamiltonian graphs by $\mathcal{H}$, and the family of almost hypohamiltonian graphs by $\mathcal{H}_1$. If these graphs are additionally planar, we denote them by $\overline{\mathcal{H}}$ and $\overline{\mathcal{H}}_1$, respectively. For a vertex v, we denote by $N(v)$ the set of vertices which are joined by an edge to v, and put $N[v] = N(v) \cup \{v\}$. A 4-cycle or a quadrilateral face is *cubic*, if all of its vertices are cubic. We will use the following.

Grinberg's Criterion [9]. *Given a plane graph with a hamiltonian cycle* $\mathfrak{h}$ *and exactly* f_i (f_i') *i-gons inside (outside) of* $\mathfrak{h}$, *we have*

$$\sum_{i \geq 3} (i-2)(f_i - f_i') = 0.$$

In this paper we study the almost hypohamiltonian graphs and their relationship to the hypohamiltonian graphs. Afterwards, we extend the notion to k-hypohamiltonicity for various k. For $k = 0$ we get hypohamiltonicity and for $k = 1$ almost hypohamiltonicity. In search for such graphs of minimal order, we notice the special position of the cases $k = 0$ and $k = 1$ among k-hypohamiltonian graphs, and that almost hypohamiltonian graphs play a special role among non-hamiltonian non-hypohamiltonian graphs.

2 Results

2.1 Almost hypohamiltonian graphs

Let G be a graph. Define G^w as G to which we add a vertex w and edges vw for all $v \in V(G)$. Furthermore, for a vertex u put $G - u = G_u$; still, we will sometimes write $G - u$ for emphasis. Let S be a subset of $V(G)$ defined as follows. For each $v \in V(G)$, there exists a hamiltonian path in $G - v$ the end-vertices of which lie in S. Call S a *set of ends*, and write $G^{w,S} = (V(G) \cup \{w\}, E(G) \cup \{vw : v \in S\})$. Thus $G^{w,V(G)} = G^w$ if $V(G)$ is a set of ends.

Lemma 1. *Let G be a hypotraceable graph, and S a set of ends. Then $G^{w,S}$ is almost hypohamiltonian with exceptional vertex w.*

Proof. G is hypotraceable and therefore non-traceable, so G and $G^{w,S}$ are non-hamiltonian. Consider $v \in V(G)$. In $G - v$ there exists a hamiltonian path $\mathfrak{p}$ the end-vertices u and u' of which belong to S. If we add to $\mathfrak{p}$ the edges uw and wu', we obtain a hamiltonian cycle in $G^{w,S} - v$. □

Thomassen [22] introduced the following method to construct hypotraceable graphs from known hypohamiltonian graphs. Consider four hypohamiltonian graphs $G_1, ..., G_4$, and cubic vertices $v_i \in V(G_i)$ with $N(v_i) = \{v_{i1}, v_{i2}, v_{i3}\}$. Take the four vertex-disjoint graphs $G_i - v_i$. Therein, identify v_{11} with v_{21} and v_{31} with v_{41}, and add the edges $v_{12}v_{32}, v_{22}v_{42}, v_{13}v_{33}, v_{23}v_{43}$. This operation preserves planarity. Thomassen [22] showed that the resulting graph is hypotraceable. Call T the 34-vertex hypotraceable graph constructed by Thomassen [22] by applying above method to four copies of the Petersen graph. As far as the author is aware, T is the smallest known hypotraceable graph.

Corollary 1. *T^w is an almost hypohamiltonian graph of order* 35.

Here, it is worth mentioning that Thomassen [25] asked whether hypohamiltonian graphs with minimum degree 4 or 4-connected such graphs exist (see Problem 6). T^w is almost hypohamiltonian and has minimum degree 4. But we obtain an even more surprising result if we apply Lemma 1 to Horton's hypotraceable graph H from [16] (with $S = V(H)$). As H is 3-connected, this yields a 4-connected almost hypohamiltonian graph of order 41, shown in Fig. 2. (Adding planarity as condition is futile due to Tutte's famous result [28].) After Horton's discovery, Thomassen [24] generalized his construction and showed that there exist infinitely many 3-connected hypotraceable graphs. This immediately yields infinitely many 4-connected almost hypohamiltonian graphs.

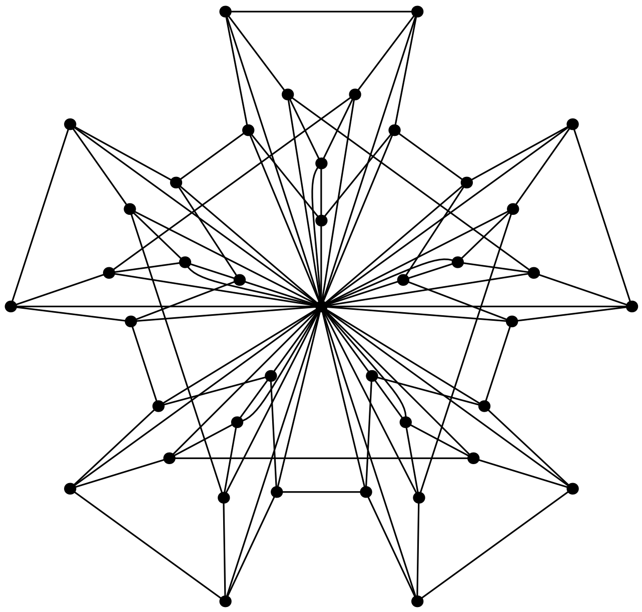


Fig. 2: A 4-connected almost hypohamiltonian graph.

In reverse order, if we take an almost hypohamiltonian graph and delete its exceptional vertex, we are only guaranteed to obtain a non-hamiltonian graph which is traceable if an arbitrary vertex is deleted – the family of such graphs might be of future interest (as it contains both the family of all hypotraceable graphs and the family of all hypohamiltonian graphs, but is not their union), but will not be discussed in this paper.

Lemma 2. *The* 39*-vertex graph shown in Fig. 3 is planar and almost hypohamiltonian.*

Proof. We denote the graph from Fig. 3 by G.

G is obviously planar. By Grinberg's Criterion, G is non-hamiltonian. Fig. 3 shows that for all $v \in V(G) \setminus \{w\}$ the graph G_v is hamiltonian. It remains to prove that G_w is non-hamiltonian. Assume the contrary, i.e. G_w contains a hamiltonian cycle $\mathfrak{h}$. Denote by f_5 (f_5') the number of pentagons inside (outside) of $\mathfrak{h}$. By Grinberg's Criterion, we have

$$\pm 2 + 3(f_5 - f_5') - 16 = 0,$$

which implies that the quadrilateral Q lies outside of $\mathfrak{h}$, i.e. on the same side as the unbounded face. Thus, the following holds.

Claim. *Consider a half-line ℓ emanating from a point in* $\mathrm{int}(Q)$ *such that each intersection of ℓ with $\mathfrak{h}$ is transversal. Then ℓ and $\mathfrak{h}$ have an even number of common points.*

Evidently, the bold edges in Fig. 4 (a) lie in $\mathfrak{h}$.

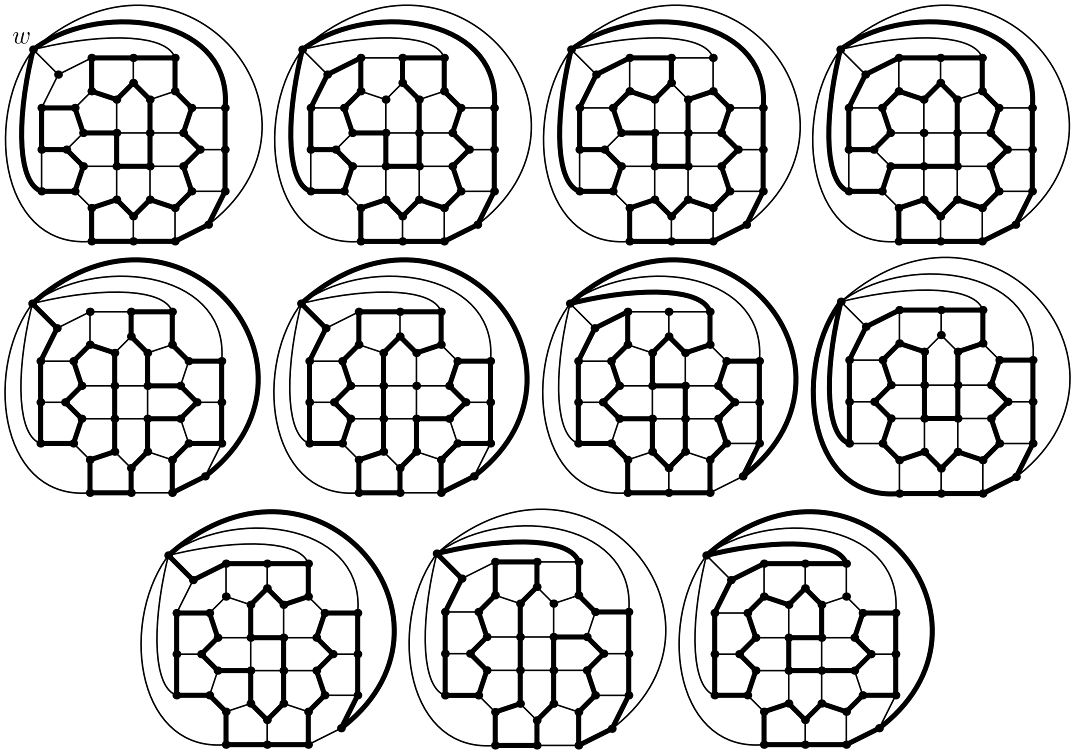


Fig. 3: Hamiltonian cycle in $G - v$, $v \neq w$.

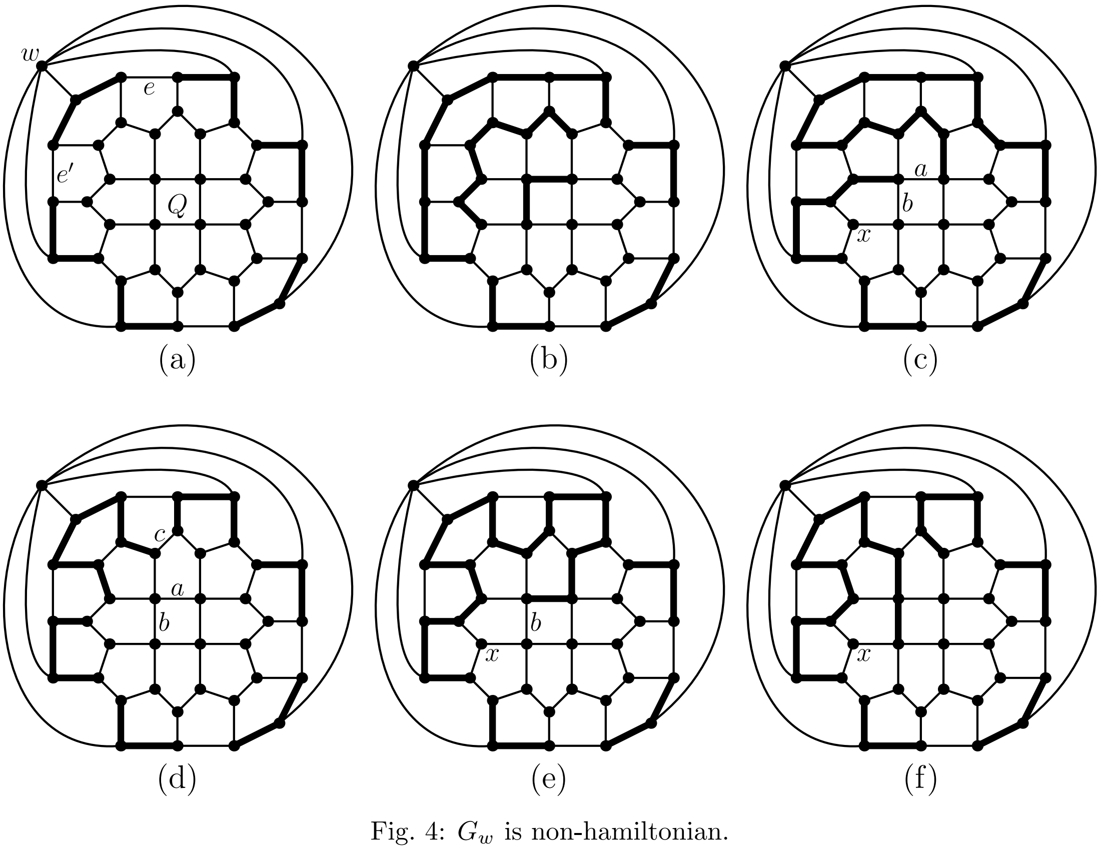


Fig. 4: G_w is non-hamiltonian.

Consider the edges e and e' from Fig. 4 (a). Due to symmetry, there are three cases to consider. (1) $\{e, e'\} \subset E(\mathfrak{h})$, (2) $e \in E(\mathfrak{h})$ and $e' \notin E(\mathfrak{h})$, and (3) $\{e, e'\} \cap E(\mathfrak{h}) = \emptyset$.

Case (1). Clearly, all edges drawn boldly in Fig. 4 (b) lie in $\mathfrak{h}$. Applying the Claim leads to contradiction.

Case (2). The bold edges from Fig. 4 (c) certainly lie in $\mathfrak{h}$. Due to the Claim, $a \notin E(\mathfrak{h})$, whence, $b \in E(\mathfrak{h})$. But now we lose the vertex x.

Case (3). The bold edges shown in Fig. 4 (d) lie in $\mathfrak{h}$. Consider the edges a and c from Fig. 4 (d). If $a \in E(\mathfrak{h})$, then $c \in E(\mathfrak{h})$ due to the Claim. So we are in the situation depicted in Fig. 4 (e). Now the edge b from Fig. 4 (e) lies in $\mathfrak{h}$ and we lose x. Hence, the assumption that $a \in E(\mathfrak{h})$ is false. In consequence, by the Claim, $c \notin E(\mathfrak{h})$, too. So $b \in E(\mathfrak{h})$. We are in the situation shown in Fig. 4 (f). But now clearly x is lost, so we have yet again obtained a contradiction. □

Consider graphs G and H, and the cubic vertices $x \in V(G)$ and $y \in V(H)$. Denote by G_xH_y one of the graphs obtained from G_x and H_y by identifying the vertices in $N(x)$ with those in $N(y)$ using a bijection. Thomassen [22] showed that if $G, H \in \mathcal{H}$, then $G_xH_y \in \mathcal{H}$. Note that if $G \in \mathcal{H}$, then G contains no triangle with a cubic vertex.

Lemma 3. *Let $G \in \mathcal{H}_1$ contain a cubic vertex x different from the exceptional vertex w of G, and $H \in \mathcal{H}$ contain a cubic vertex y. Then $G_xH_y \in \mathcal{H}_1$ with exceptional vertex w. If G and H are planar, then so is G_xH_y.*

Proof. We treat $G - x$ and $H - y$ as subgraphs of G_xH_y. Let $N(x) = N(y) = \{z_1, z_2, z_3\}$ in G_xH_y. Abusing notation, we also denote by z_i the corresponding vertices in G and H, i.e. by $\{z_1, z_2, z_3\}$ the set of neighbours of x in G, and by $\{z_1, z_2, z_3\}$ the set of neighbours of y in H. Assume G_xH_y contains a hamiltonian cycle $\mathfrak{h}$. If $\mathfrak{h} \cap G$ is connected, then it is a path. W.l.o.g. this path has end-vertices z_2, z_3. Then $(\mathfrak{h} \cap G) \cup z_2xz_3$ is a hamiltonian cycle in G, a contradiction, as $G \in \mathcal{H}_1$. If $\mathfrak{h} \cap G$ consists of two components, w.l.o.g. a path with end-vertices z_2, z_3 and the isolated vertex z_1, then $(\mathfrak{h} \cap H) \cup z_2yz_3$ is a hamiltonian cycle in H, again a contradiction.

Next, we prove that $G_xH_y - w$ is non-hamiltonian. Assume the contrary, and let $\mathfrak{h}$ be a hamiltonian cycle in $G_xH_y - w$. Suppose that $w \notin N(x)$. W.l.o.g. let z_1 be a vertex satisfying either $\{e \in E(\mathfrak{h}) : e \text{ is incident to } z_1\} \subset E(G)$ or $\{e \in E(\mathfrak{h}) : e \text{ is incident to } z_1\} \subset E(H)$. In the former case, by adding the edges xz_2 and xz_3 to $\mathfrak{h} \cap (G - x)$ we obtain a hamiltonian cycle in $G - w$, a contradiction, as w is an exceptional vertex of G. In the latter case, $(\mathfrak{h} \cap (H - y)) \cup z_2yz_3$ is a hamiltonian cycle in H, a contradiction. Now say $w = z_1$. Once more, let $\mathfrak{h}$ be a hamiltonian cycle in $G_xH_y - w$. By adding to $\mathfrak{h} \cap (G - x)$ the edges xz_2 and xz_3 we obtain a hamiltonian cycle in $G - w$, again a contradiction.

It remains to show that $G_xH_y - v$ is hamiltonian for all $v \neq w$. Let $v \in V(G) \setminus N(x)$. Then there exists a hamiltonian cycle $\mathfrak{h}$ in $G - v$. Assume w.l.o.g. that $z_2xz_3 \subset \mathfrak{h}$. Put $\mathfrak{p}_G = \mathfrak{h} - x$. Let $\mathfrak{p}_H$ be the path in H obtained by taking the hamiltonian cycle in $H - z_1$ minus y. $\mathfrak{p}_G \cup \mathfrak{p}_H$ is the desired hamiltonian cycle in $G_xH_y - v$. What if $v \in N(x)$, say $v = z_1$? Then certainly $z_2xz_3 \subset \mathfrak{h}$, $\mathfrak{h}$ being a hamiltonian cycle in $G - z_1$. We are once more in the situation discussed above. For $v \in V(H)$ the treatment is very similar. □

Let G be a graph containing a 4-cycle $v_1v_2v_3v_4v_1 = C$. We denote by $\mathrm{Th}(G_C)$ the graph obtained from G by deleting the edges v_1v_2 and v_3v_4 and adding a 4-cycle $v_1'v_2'v_3'v_4'v_1'$ disjoint from G, and the edges v_iv_i', $1 \leq i \leq 4$, to G. With Araya and Wiener [30], we call Th the *Thomassen operation*, since it was introduced by Thomassen [26] to show that there exist infinitely many planar cubic hypohamiltonian graphs. In the remainder of this article, we tacitly treat $G - \{v_1v_2, v_3v_4\}$ as a subgraph of $\mathrm{Th}(G_C)$. The statement of Lemma 4 is a slight modification of a claim of Thomassen, see [26]; we omit here its proof.

Lemma 4 [26]. *Let G be a planar non-hamiltonian graph containing a quadrilateral face bounded by the cycle C. Then* $\mathrm{Th}(G_C)$ *is planar and non-hamiltonian.*

In [26], Thomassen also showed that given a graph $G \in \overline{\mathcal{H}}$ which contains a cubic quadrilateral face bounded by the cycle C, we have $\mathrm{Th}(G_C) \in \overline{\mathcal{H}}$. We now prove a modified version of this result tailored to our needs.

Lemma 5. *Let $G \in \overline{\mathcal{H}}_1$ have exceptional vertex w and contain a cubic quadrilateral face bounded by the cycle C, $w \notin V(C)$. Then* $\mathrm{Th}(G_C) \in \overline{\mathcal{H}}_1$. *If G is cubic, then so is* $\mathrm{Th}(G_C)$.

Proof. Let $C = v_1v_2v_3v_4v_1$. By Lemma 4, both $\mathrm{Th}(G_C)$ and $\mathrm{Th}((G-w)_C) = \mathrm{Th}(G_C) - w$ are planar and non-hamiltonian. We first show that $\mathrm{Th}(G_C) - v_i'$ is hamiltonian. Let $\mathfrak{h}$ be the hamiltonian cycle in $G - v_2$. Clearly, $P = v_1v_4v_3 \subset \mathfrak{h}$. In $\mathrm{Th}(G_C)$, transform P into $v_1v_4v_4'v_3'v_2'v_2v_3$. We obtain a hamiltonian cycle in $\mathrm{Th}(G_C) - v_1'$. The cases v_2', v_3', and v_4' work in the same way. We construct the remaining hamiltonian cycles in $\mathrm{Th}(G_C) - v$, where $v \neq w$, by modifying cycles of length $|V(G)| - 1$ in G_v, see Fig. 5. □

Theorem 1. *There exists a planar almost hypohamiltonian graph of order $39 + 4k$ for every $k \geq 0$, as well as of order n for every $n \geq 76$.*

Proof. Consider the graph G from Fig. 3 and denote the vertices of the quadrilateral by v_1, v_2, v_3, v_4 in counter-clockwise order starting in the top left. Now consider $\mathrm{Th}(G_C)$ as in the paragraph above Lemma 4. By Lemma 4, $\mathrm{Th}(G_C)$ is planar and non-hamiltonian. Consider $v \in V(G) \setminus \{w\}$ and denote its corresponding vertex in $\mathrm{Th}(G_C)$ also by v. For every v, Fig. 3 depicts a hamiltonian cycle $\mathfrak{h}$ in $G - v$ which uses at least one edge of C. Thus, we can use Fig. 5 and transform $\mathfrak{h}$ into a hamiltonian cycle in $\mathrm{Th}(G_C) - v$. Fig. 3 shows a hamiltonian cycle in $G - v_4$ (the sixth graph). If we replace in this cycle v_1v_2 with $v_1v_4v_4'v_3'v_2'v_2$, we obtain a hamiltonian cycle in $\mathrm{Th}(G_C) - v_1'$. Fig. 3 shows a hamiltonian cycle in $G - v_1$ (the fourth graph). If we replace in this cycle v_3v_4 with $v_3v_3'v_2'v_1'v_1v_4$, we obtain a hamiltonian cycle in $\mathrm{Th}(G_C) - v_4'$. Hamiltonian cycles in $\mathrm{Th}(G_C) - v_2'$ and $\mathrm{Th}(G_C) - v_3'$ follow from the graph's symmetries. Assume $\mathrm{Th}(G_C) - w$ contains a hamiltonian cycle $\mathfrak{h}'$. Taking symmetry into account, $\mathrm{Th}(G_C)[\{v_i, v_i'\}_{i=1}^4] \cap \mathfrak{h}'$ must be shown (in bold) on the right side of one of the seven diagrams from Fig. 5. Going from right to left, we can transform $\mathfrak{h}'$ into a hamiltonian cycle in $G - w$, a contradiction.

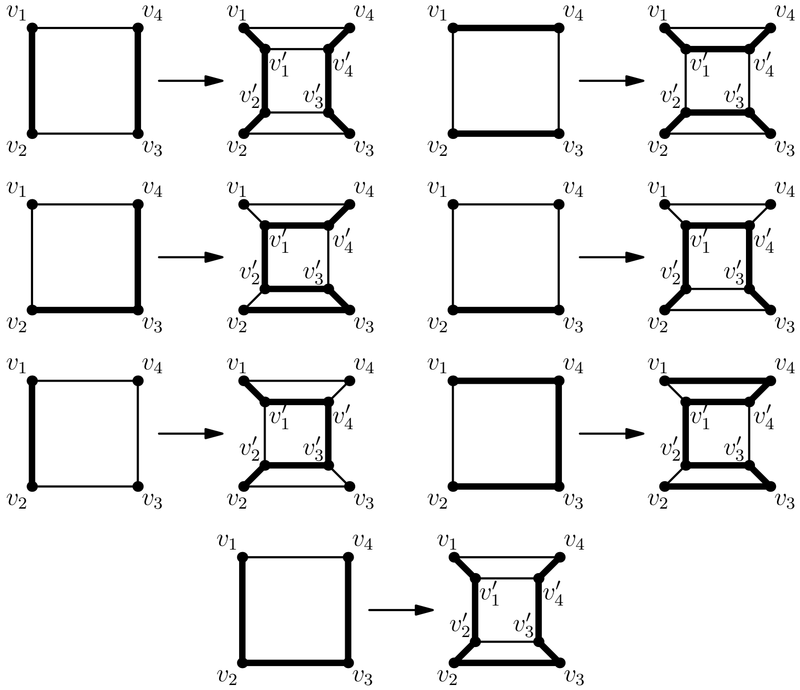


Fig. 5: On the left-hand side of each of the seven diagrams the bold edges show the subset of edges of the 4-cycle C contained in an $(n-1)$-cycle $\mathfrak{h}'$ in G; on the right-hand side it is shown what the modified $\mathfrak{h}'$ looks like in $\text{Th}(G_C)$.

We have shown that $\text{Th}(G_C)$ is a planar almost hypohamiltonian graph. $\text{Th}(G_C)$ has order 43 and contains the cubic quadrilateral $v_1'v_2'v_3'v_4'v_1'$. Consider $\text{Th}(G_C)$ and apply Lemma 5 as often as necessary to obtain the first statement of the theorem. For the second statement, we use a result from [17]: for every $n \geq 42$ there exists a planar hypohamiltonian graph of order n. As described in Lemma 3, we glue each of these graphs to G. □

Let $G \in \mathcal{H}_1$ contain a 4-cycle $v_1v_2v_3v_4v_1 = C$. We delete the edges v_1v_2 and v_3v_4, add two new vertices v_1' and v_4', and add the edges $v_1'v_4'$, v_1v_1', v_4v_4', $v_1'v_3$ and $v_4'v_2$. Denote the resulting graph by $G_C^\star$. The proof of Lemma 6 essentially coincides with the proof of Lemma 1 from the author's paper [31], and is omitted here.

Lemma 6. *Let $G \in \mathcal{H}_1$ have exceptional vertex w and include a cubic 4-cycle C not containing w. Then $G_C^\star \in \mathcal{H}_1$. If G is cubic, then so is $G_C^\star$.*

Notice that $\text{Th}(G_C) = (G_C^\star)_{C'}^\star$, where $C' = v_1v_1'v_4'v_4v_1$. (For the second iteration of $\star$ we delete v_1v_1' and v_4v_4'.) So in a certain sense, this describes "half" of a Thomassen operation.

In order to give a good upper bound on the smallest n_0 for which there exists an almost hypohamiltonian graph of order n for every $n \geq n_0$, we prove a simple yet useful gluing lemma that transforms two hypohamiltonian graphs into an almost hypohamiltonian one.

Lemma 7. *Let G and H be hypohamiltonian graphs containing cubic vertices $w \in V(G)$ and $w' \in V(H)$, and let $u, v \in N(w)$ and $u', v' \in N(w')$. If we identify u with u', v with v' and w with w', we obtain an almost hypohamiltonian graph Γ with exceptional vertex $w = w'$. If G and H are planar, then so is Γ.*

Proof. Note that as G and H are hypohamiltonian and w and w' are cubic, we have $uv \notin E(G)$ and $u'v' \notin E(H)$. Assume there exists a hamiltonian cycle $\mathfrak{h}$ in Γ. By abuse of notation u, v, w shall also denote the vertices in Γ obtained when identifying u with u', v with v' and w with w', respectively. Let $xvy \subset \mathfrak{h}$. There are three cases to study.

(a) $x, y \in V(G_w)$. Then $(\mathfrak{h} \cap G) \cup uw$ is a hamiltonian cycle in G, a contradiction.

(b) $x \in V(G_w)$ and $y \in V(H_w)$. Now $(\mathfrak{h} \cap G) \cup vw$ is a hamiltonian cycle in G, a contradiction.

(c) $x = w$ and (w.l.o.g.) $y \in V(G)$. Thus $vw \in E(\mathfrak{h})$. But then $(\mathfrak{h} \cap G) \cup uw$ is a hamiltonian cycle in G, once more a contradiction. Hence, Γ is non-hamiltonian.

We now show that $\Gamma - w$ is non-hamiltonian. Again, assume the contrary, and let $\mathfrak{h}$ be a hamiltonian cycle in G_w. Put $\{x, y\} = N(u) \cap V(\mathfrak{h})$ and $\{x', y'\} = N(v) \cap V(\mathfrak{h})$. We have $x, x' \in V(G_w)$. Then $(\mathfrak{h} \cap G) \cup vw \cup wu$ yields a hamiltonian cycle in G, a contradiction.

Finally, we prove that $\Gamma - x$ is hamiltonian for $x \neq w$. There are two cases.

(a) $x \in \{u, v\}$, say $x = u$. As w has degree 3 in G, a hamiltonian cycle $\mathfrak{h}$ in G_u contains the edge vw. Similarly, a hamiltonian cycle $\mathfrak{h}'$ in $H_{u'}$ uses the edge $v'w'$. Now $(\mathfrak{h} - vw) \cup (\mathfrak{h}' - v'w')$ yields a hamiltonian cycle in $\Gamma - u$.

(b) $x \notin \{u, v\}$. Let $x \in V(G)$. Consider a hamiltonian cycle $\mathfrak{h}$ in G_x. $\mathfrak{h}$ contains wu or vw (possibly both), say vw. Let $\mathfrak{h}'$ be a hamiltonian cycle in $H_{u'}$. As before, $\mathfrak{h}'$ contains $v'w'$. Now $(\mathfrak{h} - vw) \cup (\mathfrak{h}' - v'w')$ is a hamiltonian cycle in $\Gamma - x$. □

Theorem 2. *There exists an almost hypohamiltonian graph of order n for every $n \geq 17$ with the possible exception of $18, 19, 21$, and 24.*

Proof. It is known (see e.g. [1] for details) that there exist hypohamiltonian graphs of order n if and only if $n \in \{10, 13, 15, 16\}$ or $n \geq 18$, to which we apply Lemma 7. (Note that no hypohamiltonian graph with minimum degree at least 4 is known.) The equation $x + y - 3 = n$ has solutions $x, y \in \{10, 13, 15, 16, 18, 19, 20, ...\}$ for every $n \geq 17$ except $n = 18, 19, 21, 24$. □

Next, we present a method of transforming two almost hypohamiltonian graphs into a hypohamiltonian one.

Theorem 3. *Consider $G, H \in \mathcal{H}_1$ with cubic exceptional vertices w and w', respectively. Then $G_w H_{w'} \in \mathcal{H}$. If G and H are planar, then so is $G_w H_{w'}$.*

Proof. We denote by x, y, z the vertices in $G_w H_{w'}$ obtained when identifying $N(w)$ with $N(w')$. Abusing notation, we also write $N(w) = \{x, y, z\}$ in G and $N(w') = \{x, y, z\}$ in H, where x in $G_w H_{w'}$ is the vertex obtained when identifying x in G with x in H, and analogously for y and z.

First we show that $G_w H_{w'} - x$ is hamiltonian. Let $\mathfrak{h}_G$ be a hamiltonian cycle in G_x, and $\mathfrak{h}_H$ a hamiltonian cycle in H_x. By deleting from $\mathfrak{h}_G$ the edges yw and

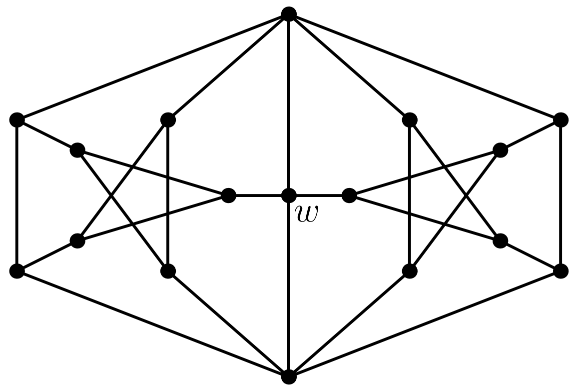


Fig. 6: The smallest known almost hypohamiltonian graph (of order 17) with exceptional vertex w; it is obtained by applying Lemma 7 to two copies of the Petersen graph.

wz we obtain a path $\mathfrak{p}_G$ in G which avoids x and w and has end-vertices y and z. From $\mathfrak{h}_H$ we delete yw' and $w'z$ and obtain a path $\mathfrak{p}_H$ which avoids x and w' and has end-vertices y and z. Now $\mathfrak{p}_G \cup \mathfrak{p}_H$ is a cycle of length $|V(G_w H_{w'})| - 1$ avoiding x, as wished.

Now we show that $G_w H_{w'} - v$ is hamiltonian, where $v \in V(G_w H_{w'}) \setminus N(w)$; w.l.o.g. $v \in V(G) \setminus \{w\}$. Consider a hamiltonian cycle $\mathfrak{h}_G$ in G_v. Assume w.l.o.g. that $yw, wz \in E(\mathfrak{h}_G)$. Now consider a hamiltonian cycle $\mathfrak{h}_H$ in H_x. Delete from $\mathfrak{h}_G$ the edges yw and wz, thus obtaining a path $\mathfrak{p}_G$, and delete from $\mathfrak{h}_H$ the edges yw' and $w'z$, thereby obtaining a path $\mathfrak{p}_H$. Now $\mathfrak{p}_G \cup \mathfrak{p}_H$ yields the desired cycle.

Finally, we prove that $G_w H_{w'}$ is not hamiltonian. Indeed, if $G_w H_{w'}$ is hamiltonian, either G_w or $H_{w'}$ has a hamiltonian path joining two vertices in $N(w)$, which can be immediately extended to a hamiltonian cycle in G or H, contrary to the hypothesis. □

Theorem 3 warrants the question whether there exist almost hypohamiltonian graphs whose exceptional vertex is cubic. Although we are not able to provide an almost hypohamiltonian graph which is cubic, we can answer the previous question affirmatively, even if planarity is added as condition.

Theorem 4. *There exists a planar almost hypohamiltonian graph of order n whose exceptional vertex is cubic for $n = 47$ and for every $n \geq 84$.*

Proof. Consider the graph G of order 47 from Fig. 7, and denote by w the (unique) cubic vertex surrounded by quadrilaterals. By Grinberg's Criterion, G and $G - w$ are non-hamiltonian. That indeed for every $v \in V(G) \setminus \{w\}$ the graph $G - v$ is hamiltonian we skip here; the proof thereof can be found in the Appendix. In order to obtain an infinite family, as in the proof of Theorem 1, we use the following result from [17]. For every $n \geq 42$ there exists a planar hypohamiltonian graph of order n. To G and each of these graphs we apply Lemma 3. □

2.2 Beyond almost hypohamiltonicity

A 2-connected graph G is *k-hypohamiltonian* if G is non-hamiltonian, there exists a set $W \subset V(G)$ of cardinality $k \leq |V(G)| - 1$, so that for every $w \in W$ the graph G_w

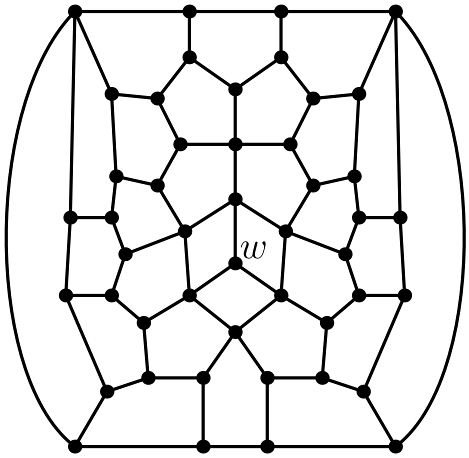


Fig. 7: A planar almost hypohamiltonian graph with cubic exceptional vertex w.

is non-hamiltonian, and for every $v \in V(G) \setminus W$ the graph G_v is hamiltonian. The vertices in W are *exceptional*. Denote the family of all k-hypohamiltonian graphs by $\mathcal{H}_k$. $\mathcal{H}_0 = \mathcal{H}$ is the family of all hypohamiltonian graphs, and $\mathcal{H}_1$ the family of all almost hypohamiltonian graphs.

Somewhat surprisingly, it turns out that for $k \geq 2$ it is easy to construct very small k-hypohamiltonian graphs, even if one adds the condition of planarity: consider a 4-cycle $v_1v_2v_3v_4v_1 = C$. For $k \geq 4$, add to C the path $v_2w_1w_2...w_{k-3}w_{k-2}v_4$. The graph one obtains is k-hypohamiltonian with $v_2, v_4, w_1, ..., w_{k-2}$ as exceptional vertices. For $k = 2$ take $K_{2,3}$, and for $k = 3$ consider the construction for $k = 5$ to which the edge v_1w_2 is added. Summarizing, if we define α_k ($\overline{\alpha}_k$) as the order of the smallest (smallest planar) k-hypohamiltonian graph, then

$$\alpha_0 = 10, \quad \alpha_1 \leq 17, \quad \alpha_2 = \overline{\alpha}_2 = 5, \quad \overline{\alpha}_3 \leq 7, \quad \alpha_4 = \overline{\alpha}_4 = 6, \quad \overline{\alpha}_5 \leq 7,$$

$$\text{and} \quad \overline{\alpha}_\ell = \ell + 1 \quad \text{for all} \quad \ell \geq 6,$$

where for α_2 and $\overline{\alpha}_2$ the equalities follow from the fact that all three 2-connected graphs on fewer than 5 vertices are hamiltonian. Concerning α_4 and $\overline{\alpha}_4$, the four 2-connected non-isomorphic spanning subgraphs of K_5 with at least eight edges are hamiltonian. Among the three with seven edges, two are hamiltonian, while the third one is 2-hypohamiltonian. Among the two with six edges, one is hamiltonian, the other one – which is $K_{2,3}$ – is 2-hypohamiltonian. The only one with five edges is the 5-cycle. No other spanning subgraphs are 2-connected. To justify the last equality, let $\ell \geq 6$. Between two fixed vertices take three paths, one of length 2, one of length 3, and one of length $\ell - 3$. This graph is ℓ-hypohamiltonian and has order $\ell + 1$. By [17] and Lemma 2, we have

$$\overline{\alpha}_0 \leq 40 \quad \text{and} \quad \overline{\alpha}_1 \leq 39,$$

noticing a striking discrepancy between the cases $k = 1$ and $k = 2$.

Let H be a 2-connected graph containing three vertices v_1, v_2, v_3, and write $\{v_1, v_2, v_3\} = \partial H$. Additionally, for any i, j with $i \neq j$, there exists a hamiltonian path between v_i and v_j. We call such a graph H *nice*.

Theorem 5. *Let $G \in \mathcal{H}_j$, $j \geq 0$, v be a cubic vertex in G with no exceptional vertex in $N(v)$, and let H be a nice graph. Join by three edges the vertices of ∂H to those of $N(v)$, according to a bijection, and delete v. Then for the resulting graph Γ we have*

$$\Gamma \in \begin{cases} \mathcal{H}_{j+|V(H)|-1} & \text{if } v \text{ is exceptional in } G \\ \mathcal{H}_{j+|V(H)|} & \text{otherwise.} \end{cases}$$

If G and H are planar and ∂H lies in the boundary of a face, then Γ is planar. If all vertices in $V(G)$ and $V(H) \setminus \partial H$ are cubic, and all vertices in ∂H have degree 2, then Γ is cubic.

Proof. Let $\partial H = \{v_1, v_2, v_3\}$ and $N(v) = \{v'_1, v'_2, v'_3\}$ with $v_i v'_i \in E(\Gamma)$ for all $i \in \{1, 2, 3\}$. We consider G_v and H to be subgraphs of Γ.

First we show that Γ is non-hamiltonian and that Γ_x is non-hamiltonian for all $x \in V(H)$. Assume the contrary. A hamiltonian cycle of Γ or Γ_x intersects G_v (which we here consider as a subgraph of Γ or Γ_x, respectively) along a hamiltonian path $\mathfrak{p}$. W.l.o.g. suppose that v'_2 and v'_3 are the end-vertices of $\mathfrak{p}$. In G, $\mathfrak{p} \cup v'_3 v v'_2$ is a hamiltonian cycle, a contradiction.

Consider the set W of exceptional vertices in G and $w \in W$. Assume there exists a hamiltonian cycle $\mathfrak{h}$ in Γ_w. W.l.o.g. $v'_1 v_1 \notin E(\mathfrak{h})$. Now $(\mathfrak{h} \cap G_v) \cup v'_2 v v'_3$ is a hamiltonian cycle in G_w, a contradiction, as w is exceptional in G. By construction, $v \notin V(\Gamma)$, so $\Gamma \in \mathcal{H}_{j+|V(H)|-1}$ if v is exceptional and $\Gamma \in \mathcal{H}_{j+|V(H)|}$ otherwise.

Finally, we show that Γ_z is hamiltonian if $z \in V(G) \setminus (\{v\} \cup W)$. Let $\mathfrak{h}'$ be a hamiltonian cycle in G_z, which exists, as z is non-exceptional. W.l.o.g. $v'_1 v v'_2 \subset \mathfrak{h}'$. Put $\mathfrak{p}' = \mathfrak{h}' \cap G_v$. There exists a hamiltonian path $\mathfrak{p}''$ between v_1 and v_2 in H since H is nice. Now $\mathfrak{p}' \cup v'_1 v_1 \cup v'_2 v_2 \cup \mathfrak{p}''$ is the desired hamiltonian cycle in Γ_z. $\square$

Actually, the above operation can be applied simultaneously to k cubic vertices – recall that there exist infinitely many cubic hypohamiltonian graphs [3], even if one adds planarity as condition [26].

A strengthening of Lemma 3 follows.

Theorem 6. *Let $i, j \geq 0$, $G \in \mathcal{H}_i$ have the set of exceptional vertices W, and $H \in \mathcal{H}_j$ have the set of exceptional vertices W'. Let $x \in V(G)$ and $y \in V(H)$ be cubic vertices with the property that $N[x] \cap W = \emptyset$ and $N[y] \cap W' = \emptyset$. Then $G_x H_y \in \mathcal{H}_{i+j}$ with $W \cup W'$ as set of exceptional vertices. If G and H are planar, then so is $G_x H_y$.*

Proof. Let $i \leq j$. The case $i = j = 0$ coincides with a result of Thomassen [22], while $i = 0$ and $j = 1$ is Lemma 3. When $i = 0$ and $j \geq 2$ the proof is very similar to the proof of Lemma 3, so we skip it and assume in the following $i \geq 1$. We denote by z_1, z_2, z_3 the vertices in $G_x H_y$ obtained when identifying $N(x)$ with $N(y)$. Abusing notation, we also write $N(x) = \{z_1, z_2, z_3\}$ in G and $N(y) = \{z_1, z_2, z_3\}$ in H, where z_k in $G_x H_y$ is the vertex obtained when identifying z_k in G with z_k in H, for all $k \in \{1, 2, 3\}$. We treat G_x and H_y as subgraphs of $G_x H_y$.

We first show that $G_x H_y$ is non-hamiltonian. Assume $G_x H_y$ contains a hamiltonian cycle $\mathfrak{h}$. W.l.o.g. both edges in $\mathfrak{h}$ incident with z_1 lie in $E(G_x)$. But then $(\mathfrak{h} \cap G_x) \cup z_2 x z_3$ yields a hamiltonian cycle in G, a contradiction.

Now we show that for all $v \in W \cup W'$, the graph $G_x H_y - v$ is non-hamiltonian. W.l.o.g. let $v \in V(G) \setminus \{x\}$. Assume there exists a hamiltonian cycle $\mathfrak{h}$ in $G_x H_y - v$. Among the vertices in $\{z_1, z_2, z_3\}$ there exists exactly one, say z_1, for which either both edges in $\mathfrak{h}$ incident with z_1 lie in (a) $E(G_x)$ or (b) $E(H_y)$. If (a) holds, then $(\mathfrak{h} \cap G_x) \cup z_2 x z_3$ is an $(n-1)$-cycle in G avoiding v, a contradiction, as v is an exceptional vertex of G. If (b) holds, then $(\mathfrak{h} \cap H_y) \cup z_2 y z_3$ yields a hamiltonian cycle in H, once more a contradiction.

Next we prove that $G_x H_y - z_1$ is hamiltonian. Let $\mathfrak{h}_G$ be a hamiltonian cycle in $G - z_1$, and $\mathfrak{h}_H$ a hamiltonian cycle in $H - z_1$; these exist as z_1 is non-exceptional in both G and H. Put $\mathfrak{p}_G = \mathfrak{h}_G - x$. $\mathfrak{p}_G$ avoids z_1 and has end-vertices z_2 and z_3. Similarly we obtain $\mathfrak{p}_H$, which avoids z_1 and has end-vertices z_2 and z_3. Now $\mathfrak{p}_G \cup \mathfrak{p}_H$ is a cycle of length $|V(G_x H_y)| - 1$ avoiding z_1. Analogously, $G_x H_y - z_2$ and $G_x H_y - z_3$ are hamiltonian.

Finally we show that $G_x H_y - u$ is hamiltonian, for all $u \in V(G_x H_y) \setminus (W \cup W' \cup \{z_1, z_2, z_3\})$; w.l.o.g. $u \in V(G)$. Consider a hamiltonian cycle $\mathfrak{h}_G$ in $G - u$. Assume w.l.o.g. that $z_2 x, x z_3 \in E(\mathfrak{h}_G)$. Now consider a hamiltonian cycle $\mathfrak{h}_H$ in $H - z_1$. Delete from $\mathfrak{h}_G$ the vertex x (and edges incident to x), thus obtaining a path $\mathfrak{p}_G$, and delete from $\mathfrak{h}_H$ the vertex y (and edges incident to y), thereby obtaining a path $\mathfrak{p}_H$. Now $\mathfrak{p}_G \cup \mathfrak{p}_H$ is the desired cycle. □

Consider $k \geq 0$. Let n_k be the smallest integer such that for every $n \geq n_k$ there exists a planar k-hypohamiltonian graph of order n.

Corollary 2. *For every $k \geq 0$ we have $n_k < \infty$.*

Proof. Jooyandeh, McKay, Östergård, Pettersson, and the author [17] showed that $n_0 \leq 42$, and in Theorem 1 we proved $n_1 \leq 76$. For every $n \geq 76$, let G_n denote the graph of order n constructed in the proof of Theorem 1, and put $\{G_n\}_{n \geq 76} = \mathcal{G}_1$. Due to the nature of Lemma 5, it is clear that each G_n contains many cubic vertices. By applying Theorem 6 to G_{76} and every graph $G \in \mathcal{G}_1$, we obtain an infinite family $\mathcal{G}_2$ of graphs proving $n_2 \leq 147$. (Note that in Theorem 6, $|V(G_x H_y)| = |V(G)| + |V(H)| - 5$.) Now apply Theorem 6 to G_{76} and every $G \in \mathcal{G}_2$, whence, $n_3 \leq 218$. This can be continued ad infinitum. We obtain $n_p \leq n_{p-1} + 71$, for every $p \geq 2$. □

Finally, Theorem 7 is a natural strengthening of Lemma 6. Its proof is analogous to the proof of Lemma 6, so we skip it.

Theorem 7. *Let $G \in \mathcal{H}_k$ with the set W of exceptional vertices contain a cubic 4-cycle C with $W \cap V(C) = \emptyset$. Then $G_C^\star \in \mathcal{H}_k$. If G is cubic, then so is $G_C^\star$.*

2.3 A planar counter-example to a conjecture of Chvátal

Chvátal [4] conjectured that if the deletion of an edge e from a hypohamiltonian graph G does not create a vertex of degree two, then $G - e$ is hypohamiltonian. Thomassen [23] gave numerous counter-examples to aforementioned conjecture, yet none of them is planar. We now provide a planar counter-example.

Consider the graph from Fig. 1 which has 48 vertices (i.e. the left-most one), the edge denoted by e_1, and the vertex denoted by v. This graph is hypohamiltonian [32]. Using Grinberg's Criterion, it is clear that it remains hypohamiltonian if we add an edge such that the octagon becomes two pentagons. Call this graph G. Notice that $G - e_1 = G'$ has no vertices of degree two. $G' - v$ contains exactly one heptagon (the unbounded face) and exactly one dodecagon. Assume that $G' - v$ contains a hamiltonian cycle $\mathfrak{h}$. Then Grinberg's Criterion yields

$$3(f_5 - f'_5) - 5 \pm 10 = 0,$$

where as before f_5 (f'_5) is the number of pentagons inside (outside) of $\mathfrak{h}$. This can only hold if the ambiguous sign is "−", which implies that the dodecagon, like the heptagon, lies on the outside of $\mathfrak{h}$. But as e_2 and e_3 (see Fig. 1) lie in $\mathfrak{h}$, we have a contradiction. So $G' - v$ is not hamiltonian, whence, G' is not hypohamiltonian. As both G and G' are obviously planar, we are done.

Inspired by Chvátal's conjecture, we note here the following. Consider $G \in \mathcal{H}$. If there exists an edge $e \in E(G)$ such that there is exactly one vertex $w \in V(G)$ with the property that for every hamiltonian cycle $\mathfrak{h}$ in $G - w$ we have $e \in E(\mathfrak{h})$, then $G - e$ is almost hypohamiltonian with exceptional vertex w.

2.4 Strengthening a theorem of Araya and Wiener

We now turn our attention to the family of planar cubic hypohamiltonian graphs. A brief motivation follows. Hamiltonian paths and cycles in planar cubic graphs have been investigated extensively since Tait tried to prove the four colour conjecture based on the conjecture that every 3-connected planar cubic graph is hamiltonian. This conjecture was disproved by Tutte [27] in 1946. Before 1968, when Grinberg proved his hamiltonicity criterion [9], such graphs were quite difficult to find. Since then, several non-hamiltonian planar cubic 3-connected graphs have been constructed. However, for the smallest example, the Lederberg-Bosák-Barnette graph on 38 vertices, the proof does not use Grinberg's Criterion. In 1986, Holton and McKay [14] (finalizing the efforts of several authors) showed that all planar cubic 3-connected graphs on fewer than 38 vertices are hamiltonian.

We require the following two results of Thomassen.

Lemma 8 [26]. *Let $G \in \overline{\mathcal{H}}$ contain a cubic quadrilateral face bounded by the cycle $v_1v_2v_3v_4v_1 = C$. Then* $\mathrm{Th}(G_C) \in \overline{\mathcal{H}}$.

Lemma 9 [26]. *Let G be a planar cubic graph containing a quadrilateral adjacent to four heptagons, and suppose furthermore that any other face is a k-gon, where $k = 2 \bmod 3$. Then G is non-hamiltonian.*

We will show in Lemma 11 the existence of a 76-vertex planar cubic hypohamiltonian graph, which we call Λ (see Fig. 8), with which we strengthen the main result of Araya and Wiener in [2]; they showed the following.

Lemma 10 [2]. *There exist planar cubic hypohamiltonian graphs on $70+4k$ vertices for every $k \geq 0$, and on n vertices for every even $n \geq 86$.*

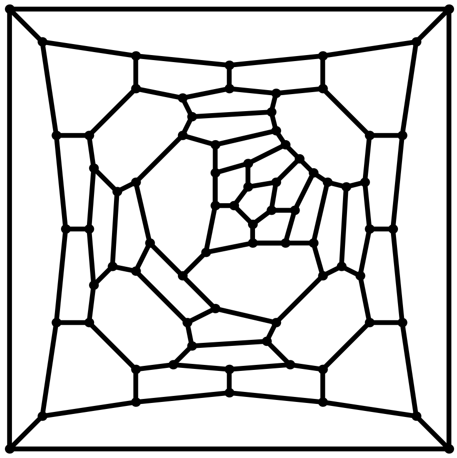


Fig. 8: A planar cubic hypohamiltonian graph of order 76.

Lemma 11. *There exists a planar cubic hypohamiltonian graph on* $76 + 4k$ *vertices for every* $k \geq 0$.

Proof. Fig. 8 shows the graph Λ, which is obviously planar and cubic. Λ contains precisely one quadrilateral surrounded by four heptagons, while all other faces are pentagons, octagons or hendecagons. By Lemma 9, Λ is non-hamiltonian. In the Appendix one can find, for each vertex of Λ, a cycle of length 75 avoiding it. By applying successively Lemma 8, the proof is complete.

Theorem 8. *There exist planar cubic hypohamiltonian graphs on* 70 *vertices and on* n *vertices for every even* $n \geq 74$.

Proof. Combining Lemmas 10 and 11, the statement is verified. □

3 Open questions

We mention here a few open questions. The first five questions are new and involve almost hypohamiltonian graphs, the latter three are interesting unsolved problems concerning hypohamiltonicity.

Problem 1.
Do (planar) cubic almost hypohamiltonian graphs exist?

Problem 2.
Do almost hypohamiltonian graphs of order less than 17 exist, or of order $n \in \{18, 19, 21, 24\}$?

Problem 3.
What is the smallest order of a (planar) almost hypohamiltonian graph?

Problem 4.
Do 5-connected almost hypohamiltonian graphs exist?

Problem 5.
Thomassen [25] showed that every planar hypohamiltonian graph contains a cubic vertex. Taking a 4-cycle $v_1v_2v_3v_4v_1$, adding the vertex v_5, and the edges v_1v_3, v_1v_5

and v_3v_5, we obtain a planar 2-hypohamiltonian graph with no cubic vertex. Does Thomassen's result extend to almost hypohamiltonian graphs?

Problem 6.
Thomassen [25] asks whether there exists a hypohamiltonian graph with (a) minimum degree 4, or even (b) connectivity 4.

Problem 7.
Máčajová and Škoviera [19] ask whether there exist infinitely many hypohamiltonian cubic graphs with both cyclic connectivity and girth 7.

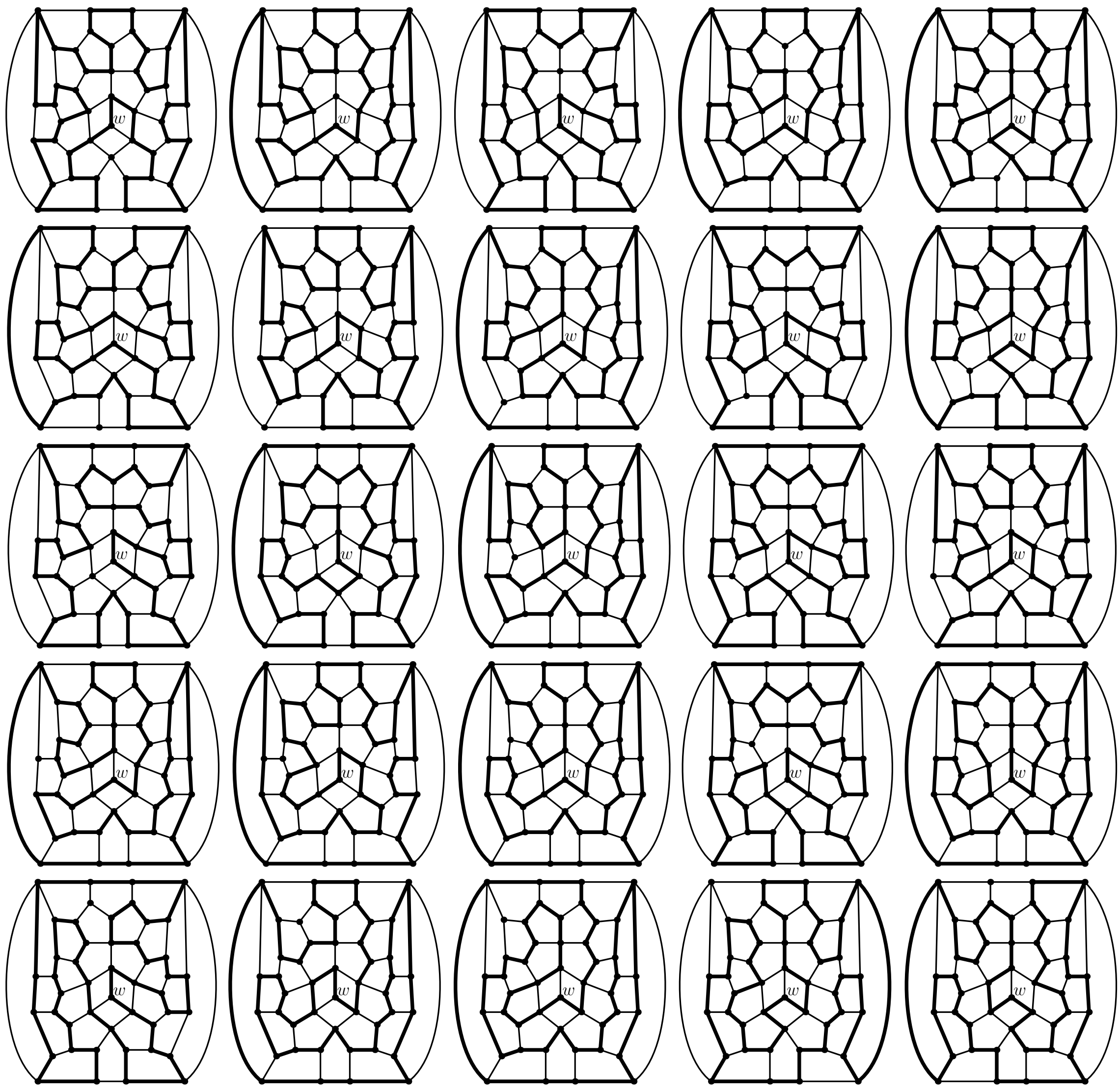
Problem 8.
Häggkvist [13] conjectures that every cubic hypohamiltonian graph has six perfect matchings which together cover every edge exactly twice.

Notes. Very recently, B. D. McKay succeeded to construct several planar cubic almost hypohamiltonian graphs: three on 68 vertices and twenty-five on 74 (private communication). This solves Problem 1.

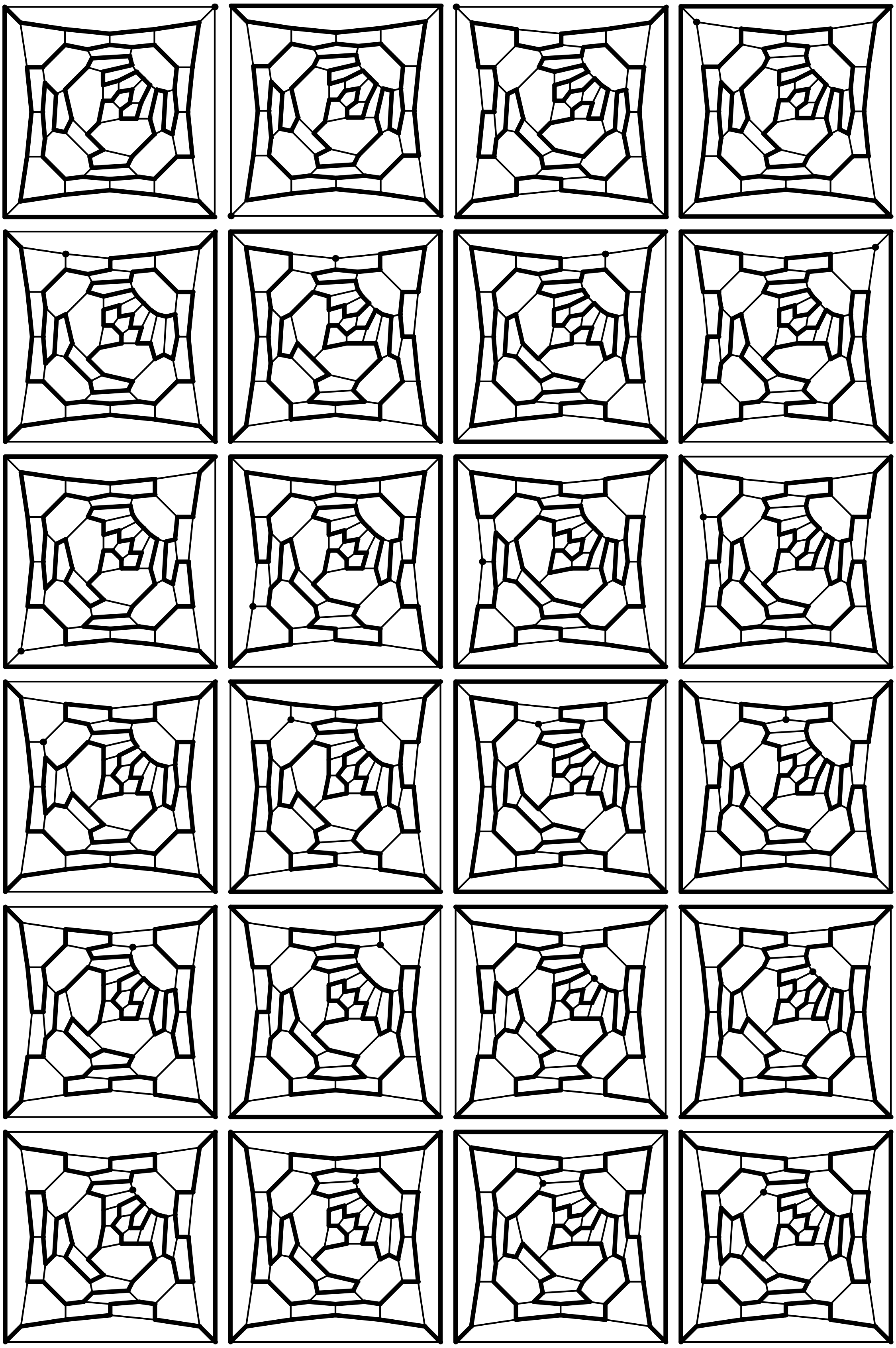
Máčajová and Škoviera use Coxeter's graph to construct an infinite family of cubic hypohamiltonian graphs of girth 7 and cyclic connectivity 6, whereas no hypohamiltonian graph of girth greater than 7 is known.

Note that in [13], there is a minor yet confusing error in the definition of hypohamiltonicity: it lacks the demand that a hypohamiltonian graph must be nonhamiltonian!

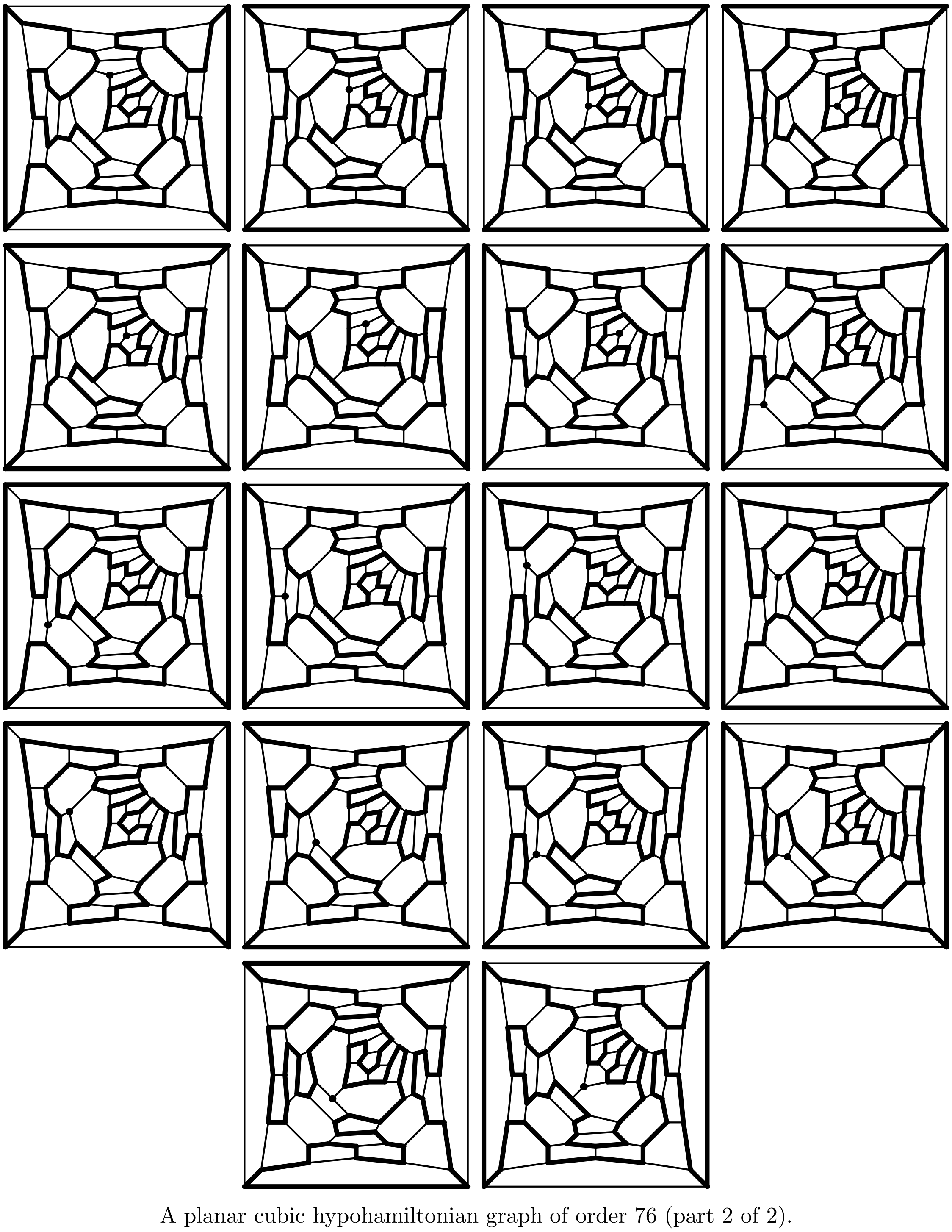
4 Appendix



A planar almost hypohamiltonian graph of order 47, the exceptional vertex of which is cubic.



A planar cubic hypohamiltonian graph of order 76 (part 1 of 2).



A planar cubic hypohamiltonian graph of order 76 (part 2 of 2).

References

[1] R. E. L. Aldred, B. D. McKay, and N. C. Wormald. Small Hypohamiltonian Graphs. *J. Combin. Math. Combin. Comput.* **23** (1997) 143–52.

[2] M. Araya and G. Wiener. On cubic planar hypohamiltonian and hypotraceable graphs. *Electron. J. Combin.* **18** (2011) #P85.

[3] J. A. Bondy. Variations on the Hamiltonian Theme. *Canad. Math. Bull.* **15** (1972) 57–62.

[4] V. Chvátal. Flip-flops in hypohamiltonian graphs. *Canad. Math. Bull.* **16** (1973) 33–41.

[5] V. Chvátal, D. A. Klarner, and D. E. Knuth. Selected Combinatorial Research Problems. STAN-CS-72-292, Stanford Univ., CA, USA, 1972.

[6] J. B. Collier and E. F. Schmeichel. New flip-flop constructions for hypohamiltonian graphs. *Discrete Math.* **18** (1977) 265–71.

[7] J. B. Collier and E. F. Schmeichel. Systematic searches for hypohamiltonian graphs. *Networks* **8** (1978) 193–200.

[8] T. Gaudin, J.-C. Herz, and P. Rossi. Solution de problème no. 29. *Rev. Franç. Rech. Opérat.* **8** (1964) 214–8.

[9] E. J. Grinberg. Plane homogeneous graphs of degree three without Hamiltonian circuits. *Latvian Math. Yearbook* **4** (1968) 51–8 (in Russian).

[10] M. Grötschel. On the Monotone Symmetric Travelling Salesman Problem: Hypohamiltonian/Hypotraceable Graphs and Facets. *Math. Oper. Res.* **5** (1980) 285–92.

[11] B. Grünbaum. Vertices Missed by Longest Paths or Circuits. *J. Combin. Theory, Ser. A* **17** (1974) 31–8.

[12] W. Hatzel. Ein planarer hypohamiltonscher Graph mit 57 Knoten. *Math. Ann.* **243** (1979) 213–6.

[13] R. Häggkvist. Problem 443. Special case of the Fulkerson Conjecture. In: "Research problems from the 5th Slovenian Conference", Bled, 2003 (eds.: B. Mohar, R. J. Nowakowski, and D. B. West). *Discrete Math.* **307** (2007) 650–8.

[14] D. A. Holton and B. D. McKay. The smallest non-hamiltonian 3-connected cubic planar graphs have 38 vertices. *J. Combin. Theory, Ser. B* **45** (1988) 315–9.

[15] D. A. Holton and J. Sheehan. The Petersen Graph, Chapter 7: Hypohamiltonian graphs, Cambridge University Press, New York, 1993.

[16] J. D. Horton. A Hypotraceable Graph. Research Report CORR 73–4, Dept. Combin. and Optim., Univ. Waterloo, 1973.

[17] M. Jooyandeh, B. D. McKay, P. R. J. Östergård, V. H. Pettersson, and C. T. Zamfirescu. Planar hypohamiltonian graphs on 40 vertices. Submitted.

[18] S. F. Kapoor, H. V. Kronk, and D. R. Lick. On detours in graphs. *Canad. Math. Bull.* **11** (1968) 195–201.

[19] E. Máčajová and M. Škoviera. Infinitely many hypohamiltonian cubic graphs of girth 7. *Graphs Combin.* **27** (2011) 231–41.

[20] J. Park, H. Lim, and H. Kim. Panconnectivity and pancyclicity of hypercube-like interconnection networks with faulty elements. *Theoret. Comput. Sci.* **377** (2007) 170–80.

[21] R. Sousselier. Problème no. 29: Le cercle des irascibles. *Rev. Franç. Rech. Opérat.* **7** (1963), 405–6.

[22] C. Thomassen. Hypohamiltonian and hypotraceable graphs. *Discrete Math.* **9** (1974) 91–6.

[23] C. Thomassen. On hypohamiltonian graphs. *Discrete Math.* **10** (1974) 383–90.

[24] C. Thomassen. Planar and infinite hypohamiltonian and hypotraceable graphs. *Discrete Math.* **14** (1976) 377–89.

[25] C. Thomassen. Hypohamiltonian graphs and digraphs. Theory and Applications of Graphs, Lecture Notes in Mathematics **642**, Springer, Berlin (1978) 557–71.

[26] C. Thomassen. Planar cubic hypohamiltonian and hypotraceable graphs. *J. Combin. Theory, Ser. B* **30** (1981) 36–44.

[27] W. T. Tutte. On hamiltonian circuits. *J. London Math. Soc.* **21** (1946) 98–101.

[28] W. T. Tutte. A theorem on planar graphs. *Trans. Amer. Math. Soc.* **82** (1956) 99–116.

[29] D. B. West. Introduction to Graph Theory (Second Edition). Pearson Education, 2001.

[30] G. Wiener and M. Araya. On planar hypohamiltonian graphs. *J. Graph Theory* **67** (2011) 55–68.

[31] C. T. Zamfirescu. Hypohamiltonian graphs and their crossing number. *Electron. J. Combin.* **19** (2012) #P12.

[32] C. T. Zamfirescu and T. I. Zamfirescu. A planar hypohamiltonian graph with 48 vertices. *J. Graph Theory* **55** (2007) 338–42.

Carol T. ZAMFIRESCU
Fakultät für Mathematik,
Technische Universität Dortmund,
Germany
E-mail address: *czamfirescu@gmail.com*

Errata

In the following, we list all substantial changes made with respect to the published version, i.e. [*J. Graph Theory* **79**, Iss. 1 (2015) 63–81].
I thank Prof. Gunnar Brinkmann for pointing out several of these omissions.

A recurring mistake has been corrected in which the neighbours of a vertex on a fixed hamiltonian cycle were not denoted properly. Furthermore, ambiguous notation when identifying vertices has been replaced.

In Section 2.3, we are not taking into consideration the octagon present in the left-most graph from Fig. 1. This has been fixed by adding an edge such that the octagon is subdivided into two pentagons.

In Theorem 1, the proof of the first part of the statement was incorrect, since we cannot apply Lemma 5 directly to the graph from Fig. 3 (Lemma 5 requires a cubic quadrilateral, which the graph from Fig. 3 does not possess). The proof has been corrected. In the second part of the statement, it should be 76, not 74. This has been corrected in the abstract and Corollary 2, as well.