Improved bounds for acute triangulations of convex polygons

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Abstract. We present a novel method of constructing non-obtuse and acute triangulations of planar convex n-gons, improving existing bounds presented in [L. Yuan, Discrete Comput. Geom. 34, 697-706 (2005)] for $6 \leq n \leq 11$ and $6 \leq n \leq 56$, respectively.

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Introduction

A *triangulation* of a 2-dimensional space means a collection of (full) triangles covering the space such that the intersection of any two triangles is either empty, or a vertex or an edge. Furthermore, a triangulation is called *geodesic*, if all its triangles are geodesic, meaning that their edges are segments, i.e. shortest paths between the corresponding vertices. In this paper we shall always refer to geodesic triangulations. We have an *acute* triangulation if in each appearing geodesic triangle all angles are acute. The triangulation is a *right* one if each triangle is right, i.e. has two angles acute, and one right. Finally, the triangulation is called *non-obtuse* if each triangle in it is acute or right. For several results concerning the various kinds of triangulations mentioned above see the survey [15] by T. Zamfirescu.

In 1960, Y. D. Burago and V. A. Zalgaller [2] proved the existence of acute triangulations for polyhedral surfaces, without giving any specific upper bounds on the number of necessary triangles. Very recently Saraf gave in [10] a much simpler construction for proving the Burago-Zalgaller theorem.

Outside Russia, this result of Burago and Zalgaller remained – so it seems – unobserved, and the problem of finding the exact minimum size of the acute triangulation of a given convex polygon had one of its origins in a problem of Stover reported in 1960 by Gardner in his Mathematical Games section of the Scientific American (see [4], [5]). Stover asked whether a triangle with one obtuse angle can be cut into smaller triangles, all of them acute. By twofold coincidence, in the same year, Goldberg proposed the same problem in the American Mathematical Monthly (E1406, see [6]), and Burago and Zalgaller [2] gave an answer by proving that any obtuse triangle can be triangulated into seven triangles, which is best possible.

Twenty years later Cassidy and Lord [3] considered acute triangulations of the square, finding a construction with eight triangles, which is best possible. In 2000, T. Hangan, J. Itoh, and T. Zamfirescu proved in the paper [7], among other things, that this is true for any rectangle, and the number is best possible. In the same year, H. Maehara investigated acute triangulations of quadrilaterals [8] and showed that any convex quadrilateral can be triangulated with at most nine acute triangles; it is unknown whether this bound is optimal, as all known examples need at most eight triangles.

Concerning convex pentagons Yuan showed in [12] that to acutely triangulate a pentagon one requires at most 54 acute triangles.

For hexagons the best estimate so far is 9240, derived from the upper bound (greatly improved compared to Maehara's previous bound) contained in [11].

Hence, we ignore the best upper bound of the minimal size of an acute triangulation, for $n \geq 4$. While for $n = 4$, this is eight or nine (we recall that for convex quadrilaterals the precise bound is unknown), and for $n = 5$ we have the fairly small bound 54, there is a huge gap between the case of pentagons and that of hexagons. In [11] it is proved (by using methods from [1]) that an arbitrary n-gon P admits a non-obtuse triangulation of size $106n - 216$, which may be transformed into an acute triangulation of P of size $22(106n - 216)$.

The goal of this paper is to diminish the aforementioned gap. In fact we provide an algorithm which yields better upper bounds for all n from 6 to 56. For the hexagonal case we obtain an acute triangulation requiring at most 102 triangles.

Right Triangulations

We provide here a different approach, which improves the upper bound for the size of a non-obtuse triangulation of a convex n-gon for the cases where $6 \leq n \leq 11$.

Theorem 1. *Let P be a planar convex n-gon with $n \geq 6$. Then P admits a right triangulation of size at most*

$$r_n = \frac{2}{3}n^3 + n^2 - \frac{47}{3}n + 22.$$

Proof. Let P^1 be a convex n-gon, with vertices $\{\kappa_i^1\}_{i=1}^n$, and B_i be the bisector of the angle (which is less than π) in κ_i^1. Define (indices are taken modulo n)

$$\nu_i^2 = B_i \cap B_{i+1} \quad (1 \leq i \leq n) \quad \text{and} \quad \delta = \min_{\substack{1 \leq i \leq n, \\ x \in \kappa_i^1 \kappa_{i+1}^1}} \|\nu_i^2 - x\|.$$

Now, we consider the parallel body $P^1(\varepsilon)$ of all interior points of P^1 at distance at least ε from its boundary. This is obviously an n-gon with sides parallel to those of P^1 and vertices on the B_i's, as long as $\varepsilon < \delta$. For $\varepsilon = \delta$, $P^2 = P^1(\delta)$ is an m-gon with $m < n$, a line-segment or a point. We iterate this procedure until P^k (with $k \leq n-1$) is either a triangle, or a line-segment or a point.

For each vertex v of P^j let $V(v)$ be the set of points on $\mathrm{bd}P^{j-1}$ closest to v ($2 \leq j \leq k$). We consider all half-lines starting at v, and meeting $V(v)$. We do this for all vertices of P^j ($2 \leq j \leq k$). Moreover, if P^k is a triangle with largest angle at w, we also consider the half-line starting at w which is orthogonal to the opposite side and meets it.

Thus, we obtain a tiling of P^1 into right triangles and rectangles. Now the rectangles are easily cut into right triangles. We obtain

$$4n - 12 + \sum_{j=5}^{n} (2j-2) \cdot (n-j+1) = \frac{1}{3}n^3 - \frac{25}{3}n + 16$$

rectangles and

$$2 + \sum_{i=4}^{n} 2i = n^2 + n - 10$$

right triangles, whence altogether

$$\frac{2}{3}n^3 + n^2 - \frac{47}{3}n + 22$$

right triangles. □

Employing the technique used in the proof of Theorem 1 we shall improve existing bounds for the minimum size of acute triangulations of convex n-gons for $6 \leq n \leq 56$. To achieve this we shall also use the following remark and lemmas.

Remark. Let us call two rectangles with disjoint interiors but sharing an edge a *rectangle pair*. Say we have two rectangles $R_1 = \text{conv}\{a, x, y, d\}$ and $R_2 = \text{conv}\{b, x, y, c\}$ which share the edge xy, forming a rectangle pair. In [7] we learn that to acutely triangulate a rectangle, we need eight triangles. Without elaborating (for details, see [7]), we may triangulate $R = R_1 \cup R_2$ acutely into eight triangles, using x and y as vertices of the triangulation. We note that xy is not an edge of the triangulation of R. For an illustration, see the rectangle pair $\{R_j, Q_j\}$ from Fig. 1.

From [8] and several proofs in [12] we extract Lemmas 1 and 2, respectively.

Lemma 1. *Let $Q = \text{conv}\{a, b, c, d\}$ be a convex quadrilateral. Then Q admits an acute triangulation of size at most 9, featuring at most four vertices on $\text{bd}Q \setminus \{a, b, c, d\}$.*

Lemma 2. *Let $\Gamma = \text{conv}\{a, b, c, d, e\}$ be a convex pentagon. Then Γ admits an acute triangulation of size at most 54, featuring at most seven vertices on $\text{bd}\Gamma \setminus \{a, b, c, d, e\}$.*

Acute Triangulations

The following theorem is the main result of this paper.

Theorem 2. *Let P be a planar convex n-gon with $n \geq 6$. Then P admits an acute triangulation of size at most a_n, where*

$$a_n = \begin{cases} \frac{2}{3}n^3 + 2n^2 - \frac{71}{3}n + 28 & \text{for even } n \\ \\ \frac{2}{3}n^3 + 2n^2 - \frac{101}{3}n + 88 & \text{for odd } n. \end{cases}$$

Proof. Let P^1 be a planar convex n-gon with $n \geq 6$. In this first part we shall prove our result for even n. We proceed as described in the proof of Theorem 1, and obtain the finite sequence of polygons $P^1, ..., P^{k-1}$. We shall restrict ourselves to the generic case where $V(v)$ consists of at most 3 points (no three bisectors have a common point), and moreover $\mathrm{card}V(v) = 3$ for just one vertex v (which we call v') because otherwise k obviously becomes smaller and so becomes the size of the triangulation. (Notice, also, that $\{a_n\}_{n=6}^{\infty}$ is a monotone sequence.) Then $k = n - 2$.

Assume P^{n-3} is a quadrilateral. We triangulate P^{n-3} acutely with nine triangles using Lemma 1, generating at most four new vertices $v_1, ..., v_\ell \in \mathrm{bd}P^{n-3}$ $(\ell \leq 4)$. With the notation from the proof of Theorem 1, we consider for each vertex v of P^j two half-lines starting at v and meeting $V(v)$ $(2 \leq j \leq n-3)$; for $j = 2$ and $v = v'$, although $V(v')$ consists of three points on three consecutive sides of P^1, we do not join v' with the middle point v'' of $V(v')$. Moreover, we consider the half-lines starting at v_i $(i \leq \ell)$, orthogonal to $\mathrm{bd}P^{n-3}$ in a neighbourhood of v_i, and disjoint from the interior $\mathrm{int}P^{n-3}$ of P^{n-3}.

The next step is illustrated in Fig. 1. Let a *chain* be a maximal family of rectangles $C_1 = R_1, ..., R_q \subset P^m \setminus \mathrm{int}P^{m+1}$, $1 \leq m \leq n-4$, for which R_i has a common edge with R_{i-1} and R_{i+1}, $2 \leq i \leq q-1$. (The union of their boundaries includes some edge of P^{m+1}.) We now explain this step by illustrating it within $P^1 \setminus \mathrm{int}P^3$, from where we take two chains with non-empty intersection $C_1 = R_1, ..., R_q \subset P^2 \setminus \mathrm{int}P^3$ and $C_2 = Q_0, ..., Q_{q+1} \subset P^1 \setminus \mathrm{int}P^2$ (which shall be called a *double chain*), see Fig. 1. In this situation we triangulate acutely – following the Remark – precisely one rectangle pair in the double chain, consisting of one rectangle from C_1 and one from C_2, namely R_j and Q_j respectively, where j can be chosen arbitrarily in $\{1, ..., q\}$. Subsequently, we tile each of the remaining rectangles $R_1, ..., R_{j-1}, R_{j+1}, ..., R_q$ and $Q_0, ..., Q_{j-1}, Q_{j+1}, ..., Q_{q+1}$ into two right triangles in the way shown in Fig. 1. Now, apply the procedure de-

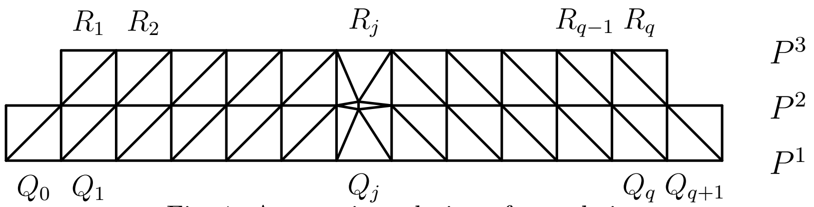

Fig. 1: Acute triangulation of two chains.

scribed above as follows. For $n = 6$ there are in total four double chains which are triangulated as shown above, and one chain consisting of two rectangles, which we triangulate as a rectangle pair (see the Remark); for $n = 8$ we first triangulate as before the chains of P^3. Then we are left with six double chains and one chain consisting of two rectangles, which we know how to triangulate. In this manner we triangulate non-obtusely any n-gon P^1, for even n.

Finally, we render the triangulation acute by applying gentle shifts to certain vertices. If the shifts are gentle enough, no acute angles which are affected by these shifts become right or obtuse. We shall explain the process on angles within $P^1 \setminus \text{int}P^2$, and then iterate. Take all right angles around vertices of P^1. Consider, for instance, $\{g\} = (\text{bd}P^1) \cap Q_{q+1} \cap Q_q$ (see Fig. 1). By shifting g in the direction of $(\text{bd}P^1) \cap Q_0 \cap Q_1$, we render all angles around g acute (note that by using this shift also all angles within $P^1 \setminus \text{int}P^2$ around $(\text{bd}P^2) \cap Q_{q+1} \cap Q_q$ become acute). Iterate this procedure for $P^t \setminus \text{int}P^{t+1}$ $(2 \le t \le n-4)$, and all angles within P^1 become acute.

One last thing remains to be proven: that by not joining v' to v'' no angles become non-acute. Let ab be the edge of the polygon P^1 containing v''. Now let us assume that $\angle av'b \ge \pi/2$, which implies $\angle v'ba + \angle bav' \le \pi/2$. Let c be adjacent to a and d adjacent to b (c and d exist as $n \ge 6$). Consider the half-lines L_a and L_b starting in a and b, respectively, which go through c and d, respectively. Note that by putting $\delta = \|v' - v''\|$ we have

$$\min_{x \in L_a} \|v' - x\| = \min_{y \in L_b} \|v' - y\| = \min_{z \in ab} \|v' - z\| = \delta.$$

Let a' (b') be the projection of v' onto the edge ac (bd), and put

$$\Omega = P^1 \setminus (\text{conv}\{a', a, v'', v'\} \cup \text{conv}\{b', b, v'', v'\}).$$

As at least one intersection point of consecutive bisectors lies in Ω, and as we have for any point $q \in \Omega$ the inequality $\min_{x \in P^1} \|q - x\| \le \delta$, a contradiction is obtained.

Let us now compute the number of triangles used – as this is an upper bound, the triangulation size at every step is maximal.

Firstly, nine triangles are required to triangulate the quadrilateral P^{n-3} acutely. Secondly, the total number of rectangles is

$$A_1 = \sum_{i=0}^{n-5}(2i+1) + \sum_{j=1}^{n-5} 2j(n-4-j) = \frac{1}{3}n^3 - \frac{49}{3}n + 44$$

plus

$$A_2 = 4n - 16,$$

where A_1 is the number of all rectangles before considering the four additional vertices on the boundary of the central quadrilateral P^{n-3} which appear when triangulating it, while A_2 is precisely the number of the additional rectangles. This yields

$$2(A_1 + A_2) = \frac{2}{3}n^3 - \frac{74}{3}n + 56$$

triangles. Thirdly, we now replace – as described in the third paragraph of this proof – the triangulations of rectangle pairs which have been triangulated into four triangles (by simply taking two diagonals) with triangulations as constructed in the Remark. This yields a total of

$$4\sum_{j=1}^{\frac{n}{2}-2}(2j+3) = n^2 - 16$$

additional triangles. Lastly, we count the triangles which occur directly as triangles:

$$2\left(\sum_{i=5}^{n} i\right) - 1 = n^2 + n - 21.$$

Hence, the acute triangulation of a planar convex n-gon with even n requires at most

$$a_n = 9 + \frac{2}{3}n^3 - \frac{74}{3}n + 56 + n^2 - 16 + n^2 + n - 21 = \frac{2}{3}n^3 + 2n^2 - \frac{71}{3}n + 28$$

triangles.

We now prove our result for the remaining, odd values of n. We proceed exactly as described in the proof of the even case, until we obtain the pentagon P^{n-4}. We triangulate P^{n-4} acutely with 54 triangles using Lemma 2, generating at most seven new vertices $v_1, ..., v_j \in \mathrm{bd}P^{n-4}$ $(j \leq 7)$. As in the proof of the even case, consider now $u_i \in \mathrm{bd}P^1$ such that the distance between v_i and u_i is minimal $(1 \leq i \leq j)$, and draw the line-segments $v_i u_i$.

The rest of the second part of the proof is perfectly analogous to the even case. Still, as this construction is based on triangulating the pentagon, a different formula is obtained, namely

$$a_n = \frac{2}{3}n^3 + 2n^2 - \frac{101}{3}n + 88,$$

and this completes the proof. □

Let us now shift our focus from polygons to double polygons. Defining a more general term, we start with a planar convex set C and call the *double convex set* $2C$ the (degenerate convex) surface homeomorphic to the sphere, which is the union of two planar convex sets isometric to C, the boundaries of which are identified in accordance with the isometry.

There are several papers providing triangulations of certain double polygons, the first of which contains a theorem on acute triangulations of the double triangle, offering for the size the optimal upper bound 12 [14]. A further result treats the case of the double convex quadrilateral, where at most 20 acute triangles are needed [13]. For the case of double convex pentagons, the best upper bound so far is 76 [12].

Using Theorems 1 and 2, we obtain further upper bounds for non-obtuse and for acute triangulations of double convex n-gons.

Theorem 3. *Let P be a convex n-gon with $n \geq 6$. Then $2P$ admits a right triangulation of size at most $2r_n$, and an acute triangulation of size at most $2a_n$.*

Proof. The needed triangulation is obtained directly from that of P. What is essential to remark is the following. In order to obtain a true triangulation, no triangle of the triangulation of P may have three vertices on bd$P$. This is dealt with by our construction, because from each vertex of P a bisector departs, and thus, avoiding a situation like the one just described. □

Thus, any double hexagon admits a triangulation using at most 204 acute triangles.

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