(2)-pancyclic graphs

CAROL T. ZAMFIRESCU

Abstract. We introduce the class of (2)-*pancyclic* graphs, which are simple undirected finite connected graphs of order n having exactly two cycles of length p for each p satisfying $3 \leq p \leq n$, analyze their properties, and give several examples of such graphs, among which are the smallest.

Key Words. Pancyclic Graphs.

MSC 2010. 05C38.

Introduction

Throughout this paper all graphs will be simple (i.e. without multiple edges or loops), undirected, finite, and connected. In the classic book on graph theory by J. A. Bondy and U. S. R. Murty [1] we find a series of 50 open problems, among which problem number 10 shall be the starting point of our investigations. It is as follows. Determine all connected graphs G having exactly one cycle of each length p, $3 \leq p \leq n$, where n is the order of the graph G – such graphs are called *uniquely pancyclic*. According to [1], this problem was posed in 1973 by R. C. Entringer. Finding the smallest four such graphs is an easy task, and proving that they are the only uniquely pancyclic graphs on less than nine vertices is left to the reader. In 1986, Y. Shi [8] constructed three new uniquely pancyclic graphs, conjecturing that there are no other uniquely pancyclic graphs than these seven (see Fig. 1). Recently, K. Markström [7] confirmed Shi's conjecture for graphs of order 59 or less by computer search. We study a closely related problem. We consider in the following (2)-*pancyclic* graphs, which are defined as graphs having exactly two cycles of every possible length t, $3 \leq t \leq n$. Let an *n-cycle* denote a subgraph isomorphic to C_n. Further definitions follow where they find their first use.

It seems almost nothing is known on (2)-pancyclic graphs. All we could find were four such graphs on a website maintained by G. Exoo [4], which we reproduce in Fig. 4.

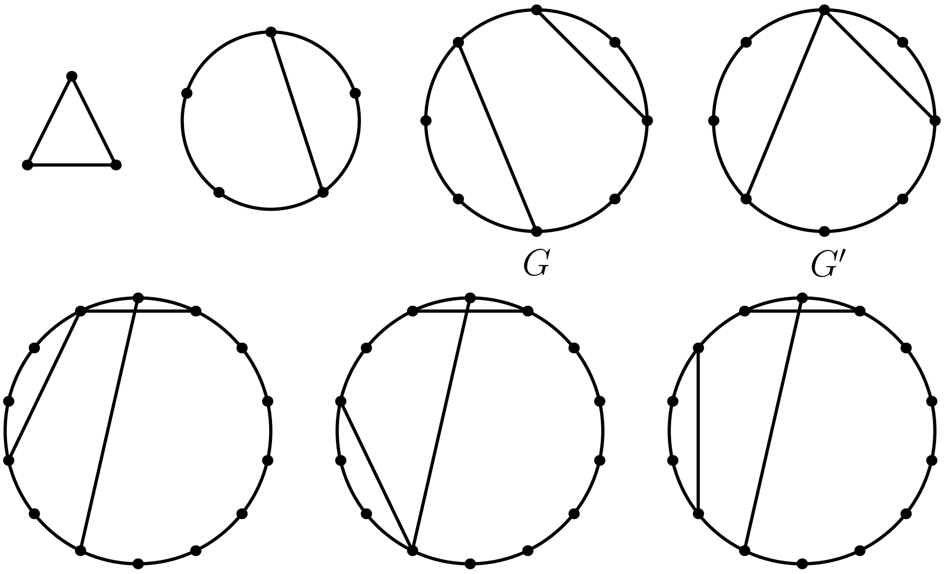

Fig. 1: All seven known uniquely pancyclic graphs;
they are of order 3, 5, 8, 8, 14, 14, and 14, respectively

We remark that we do not use the already existing terms *bipancyclic* or *uniquely r-pancyclic*; these terms describe graphs containing cycles of all even lengths and graphs containing exactly one cycle of length t for all t fulfilling $r \leq t \leq n$, respectively. On the latter subject several results have appeared, see [15], [12], [13], and [10].

Related investigations study *uniquely bipancyclic* graphs (these are graphs containing exactly one cycle of every even length), see e.g. [14], and *uniquely r-bipancyclic* graphs, the definition of which is obvious at this point, see e.g. [11].

A related topic concerns graphs of order n in which no two cycles have the same length. Let $f(n)$ be the maximum number of edges in such a graph. P. Erdős posed in 1975 the problem (see problem 11 in [1]) of determining $f(n)$. For results, see [9], [2], [6], and [5]. The same problem can be posed for 2-connected graphs, and results can be found in [9], [3], and [2].

Preparation

We start with several observations which help us shorten proofs considerably. Let G be a connected graph on n vertices.

If G is (2)-pancyclic, then it is Hamiltonian, i.e. it has a subgraph isomorphic to C_n. Thus, we can construct G by considering in the plane a circular n-cycle C_n called in the following the *exterior* cycle, and add to it chords lying in the disc D determined by C_n.

If we construct a graph G by adding k chords to an exterior cycle, we call the *chordal crossing number* $\mu(G)$ the total number of interior points of D obtained as intersections of chords.

We now formulate the following simple necessary condition for a graph G to be (2)-pancyclic.

Lemma 1. *Label the vertices of $C_n \subset G$ by $v_1, ..., v_n$ in cyclic order. For G to be (2)-pancyclic, it must contain at most one pair of chords $v_i v_j$ and $v_{i+1} v_{j+1}$ (indices mod. n). Additionally, there must exist chords c_1 and c_2 such that $\mu(C_n \cup \{c_1, c_2\}) = 1$.*

Proof. Let two chords realizing the first (second) condition from Lemma 1 be called *intertwined* (*intersecting*).

For G to be (2)-pancyclic, there must exist exactly two n-cycles in G. One of them is automatically given by the Hamiltonian cycle $v_1 ... v_n v_1$. Adding one pair of intertwined chords as prescribed in Lemma 1 yields precisely one new n-cycle, namely (w.l.o.g. put $i < j$) $v_1 v_2 ... v_{i-1} v_i v_j v_{j-1} ... v_{i+2} v_{i+1} v_{j+1} v_{j+2} ... v_n v_1$.

If there is more than one pair of intertwined chords in G, then at least three n-cycles occur, whence, G cannot be (2)-pancyclic.

Let us now show that in a (2)-pancyclic graph there always exist intersecting chords. Assume the contrary to be true. Then there would be only one n-cycle, leading directly to a contradiction. □

Now three Lemmas follow. Lemmas 2 and 3 are separated for improved referencing; they give lower and upper bounds for the number of chords occurring in a (2)-pancyclic graph of fixed order n. Lemma 4 is a simple remark on the existence of intertwined chords when the chordal crossing number is 1.

Lemma 2. *A (2)-pancyclic graph of order n has at most $\left\lfloor \frac{\sqrt{16n+1}-5}{2} \right\rfloor$ chords.*

Proof. Let G be a (2)-pancyclic graph of order n, isomorphic to an n-cycle to which k chords have been added. By Lemma 1 we know that G contains two intersecting chords, say c_1 and c_2.

Claim. The $(k+1)$-st chord added to G creates at least $k+3$ new cycles.

Let this $(k+1)$-st chord be e. e forms 2 cycles by halving G, one cycle in conjunction with every chord, yielding k additional cycles, and one cycle by using both c_1 and c_2. Summa summarum this gives $k+3$ new cycles, as stated.

From the claim we can deduce that $k \geq 3$ chords, added to a cycle C_n (the resulting graph shall be called G) yield at least $-4 + \sum_{i=1}^{k+2} i = \sigma$ cycles. For G to be (2)-pancyclic, σ must be less than or equal to $2n-4$, the number of cycles in a (2)-pancyclic graph on n vertices. This leads directly to the inequality $k^2 + 5k + 6 - 4n \leq 0$, which gives $k \leq (\sqrt{16n+1} - 5)/2$. □

Lemma 3. *A (2)-pancyclic graph of order n has at least $\lceil \log_2(2n-3) \rceil - 1$ chords.*

Proof. We employ precisely the same notation as in Lemma 2.

Claim. The k-th chord added to G creates at most 2^k new cycles.

Let us denote the added chord by e. There exist exactly two cycles using as chords only e. With every other chord, e may form at most two new cycles, i.e. we have at most $2 + 2k - 2$ new cycles. The same holds for pairs of chords, triples of chords, etc. Therefore, we may state that e forms no more than $2 + 2\sum_{i=1}^{k-1} \binom{k-1}{i} = 2^k$ new cycles.

We proceed as in the proof of Lemma 2. The claim proved above implies that adding k chords to an n-cycle yields a graph with at most $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$ cycles. For G to be (2)-pancyclic, we must have $2^{k+1} - 1 \geq 2n - 4$, the right side being, by definition, the number of cycles in a (2)-pancyclic graph, which finally yields $k \geq \lceil \log_2(2n-3) \rceil - 1$. □

Lemma 4. *If there exists exactly one pair of intersecting chords in a* (2)*-pancyclic graph G, i.e. $\mu(G) = 1$, then the chords of the pair must be intertwined.*

Proof. There must be two Hamiltonian cycles C' and C'' in the (2)-pancyclic graph G, one of which is $C' = C_n$. Clearly, C'' must use at least one pair of intersecting chords – as there is but one such pair, C'' must use it. As C'' has length n, the chords of the pair must be intertwined. □

Results

Theorem 1. *Every* (2)*-pancyclic graph has minimal degree* 2 *and maximal degree at most* $\left\lfloor \frac{\sqrt{16n+1}-3}{2} \right\rfloor$.

Proof. First, let us prove the statement concerning the minimal degree of a (2)-pancyclic graph G of order n. Let G have minimal degree greater than or equal to 3. Then G has at least $\left\lceil \frac{n}{2} \right\rceil$ chords. Using Lemma 2, we obtain a contradiction if

$$\left\lfloor \frac{\sqrt{16n+1}-5}{2} \right\rfloor < \left\lceil \frac{n}{2} \right\rceil .$$

We are led to the inequality $n^2 - 6n + 26 > 0$, valid for all n, whence, the first part of Theorem 1 holds.

The maximal degree (minus two) is trivially bounded by the maximal number of chords that can occur, $\lfloor (\sqrt{16n+1} - 5)/2 \rfloor$ (see Lemma 2), except for one chord, as at least one pair of chords intersects in D (see Lemma 1). □

Corollary. *Each* (2)*-pancyclic graph has connectivity* 2.

We now investigate the order of the smallest (2)-pancyclic graphs – they are presented in Fig. 2. Astonishingly, both descend from the uniquely pancyclic graph (in contrast to all other known (2)-pancyclic graphs, see Fig. 4, which do not have a uniquely pancyclic subgraph), in the sense that graphs (a) and (b) from Fig. 2 can be obtained from the graph G in Fig. 1 by adding a single chord (in Fig. 2, chord v_1v_3). Also, by deleting the chord v_2v_4 from the graph from Fig. 1 (a), we obtain the uniquely pancyclic graph G' from Fig. 1. Furthermore, we checked that one cannot construct a (3)-pancyclic graph from the graphs shown in Fig. 2 by adding one or more chords.

Theorem 2. *There exist exactly two (up to isomorphism)* (2)*-pancyclic graphs of order* 8*, and there are none on fewer vertices. They are the* 8*-cycle* $v_1 \ldots v_8 v_1$, *to which we add the chords v_1v_3 and v_2v_4, plus the chord (a) v_1v_6 or (b) v_5v_8, respectively.*

Proof. We shall examine all relevant graphs on seven or less vertices, and then determine all (2)-pancyclic graphs of order 8. Let k denote the number of chords. Obviously, there is no (2)-pancyclic graph on 3 vertices.

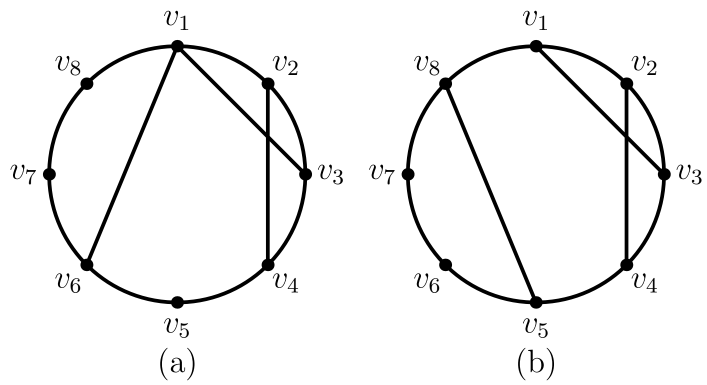

Fig. 2: The two smallest (2)-pancyclic graphs, of order 8

For $n = 4$, using Lemmas 2 and 3, we obtain $k \leq 1$ and $k \geq 2$, which is absurd.

In the case $n = 5$, Lemmas 2 and 3 yield $k = 2$. W.l.o.g. we may consider the 5-cycle $v_1v_2v_3v_4v_5v_1$ and add to it two chords – these must intersect due to Lemma 1, which makes the graph unique (up to isomorphism); w.l.o.g. let the chords be v_1v_3 and v_2v_4. As three 4-cycles occur, it is not (2)-pancyclic.

For graphs on 6 or 7 vertices, Lemmas 2 and 3 yield $k \leq 2$ and $k \geq 3$, a contradiction.

For a (2)-pancyclic graph G of order 8, Lemmas 2 and 3 imply that G must have exactly three chords, two of which intersect (by Lemma 1). As there are but three chords, we have $1 \leq \mu(G) \leq 3$. We may dismiss $\mu(G) = 3$, as 15 cycles would occur, and there should be only 12. Let $C = v_1v_2...v_7v_8v_1$ be the exterior cycle of G. Two situations remain to be discussed: $\mu(G) = 1$ and $\mu(G) = 2$. Note that certain cases are equivalent, in the sense that the treated graphs are isomorphic.

Case 1: $\mu(G) = 1$. By Lemma 4, the chords which meet must be intertwined. We have three cases for C plus these chords, namely $C \cup \{v_1v_3, v_2v_4\}$, $C \cup \{v_1v_4, v_2v_5\}$, and $C \cup \{v_1v_5, v_2v_6\}$. The second and third case immediately lead to contradiction, as three 4-cycles and four 5-cycles, respectively, occur. It remains $G = C \cup \{v_1v_3, v_2v_4\}$.

To simplify the proof, we consider the various possibilities for the third chord, and then either show the contradiction reached, or specify whether the obtained graph is isomorphic to (a) or (b) from Fig. 2.

v_1v_4: four 3-cycles, v_1v_5: three 4-cycles, v_1v_6: (a), v_1v_7: three 3-cycles, v_4v_i: equivalent to v_1v_{13-i}, v_5v_7: three 3-cycles, v_5v_8: (b), and v_6v_8: equivalent to v_5v_7.

Case 2: $\mu(G) = 2$. If no pair of chords has a common end-vertex, 14 cycles arise (too many), whence, some pair of chords must share an end-vertex, say v_1. We will use the following.

($\star$) *In order for G to be (2)-pancyclic, it must contain two n-cycles. In our case, this occurs only if either there exists a pair of intertwined chords, or if the three chords are v_1v_j, v_1v_{j+1} ($3 \leq j \leq 6$) and v_2v_8.*

When using ($\star$), we simply write $\star$.

Subcase 2.1 $C \cup \{v_1v_3, v_1v_4\}$.

v_2v_5: three 4-cycles, v_2v_6 and v_2v_7: $\star$, and v_2v_8: three 3-cycles.

Subcase 2.2 $C \cup \{v_1v_3, v_1v_5\}$.

v_2v_6 and v_2v_7: one 3-cycle, and v_2v_8: three 7-cycles.

Subcase 2.3 $C \cup \{v_1v_3, v_1v_6\}$.

v_2v_7: one 3-cycle, and v_2v_8: three 7-cycles.

Subcase 2.4 $C \cup \{v_1v_3, v_1v_7\}$.

v_2v_8: three 3-cycles.

Subcase 2.5 $C \cup \{v_1v_4, v_1v_5\}$.

v_2v_6: one 3-cycle, v_2v_7: $\star$, v_2v_8: one 4-cycle, v_3v_6 and v_3v_7: $\star$, and v_3v_8: one 3-cycle.

Subcase 2.6 $C \cup \{v_1v_4, v_1v_6\}$.

Three 4-cycles (for any third chord).

Thus, we have (i) proved that there are no (2)-pancyclic graphs on less than 8 vertices, (ii) found two non-isomorphic such graphs on 8 vertices (see Fig. 2), and (iii) showed that these two are indeed all (2)-pancyclic graphs on 8 vertices. □

Clearly, we may deduce that a (2)-pancyclic graph features at least three chords, two of which must intersect.

Theorem 3. *There exist no* (2)*-pancyclic graphs on* 9 *or* 10 *vertices*

Proof. Let G be a (2)-pancyclic graph of order 9. Lemmas 2 and 3 imply that G has exactly 3 chords, c_1, c_2 and c_3. Evidently, a (2)-pancyclic graph of order 9 has 14 cycles. By Lemma 1, two of the chords, say c_1 and c_2, must intersect. $C_9 \cup \{c_1, c_2\}$ has exactly seven cycles.

If we add c_3 such that $\mu(G) = 1$, only five new cycles form, which is too few. If c_3 intersects both c_1 and c_2 in D, eight new cycles are created, which is too many.

The case remains where c_3 intersects precisely one of c_1, c_2, say c_1. We note that if $c_3 \cap c_2 \neq \emptyset$, i.e. c_2 and c_3 have a common end-vertex, only six cycles form. When $c_3 \cap c_2 = \emptyset$, we have indeed 14 cycles: see Fig. 3. The following is an exhaustive list of cycle lengths present in G (x_i represents the number of vertices of degree 2 on various subpaths determined on the exterior cycle of G, see Fig. 3).

$x_1+x_2+x_3+4$, x_1+x_6+3, $x_1+x_2+x_5+x_6+5$, $x_1+x_4+x_5+5$, $x_1+x_2+x_4+5$, $x_1+x_3+x_5+6$, x_2+x_5+4, $x_2+x_4+x_6+6$, $\left(\sum_{i=2}^{5} x_i\right)+5$, $x_2+x_3+x_6+5$, $x_3+x_5+x_6+5$, x_3+x_4+3, $x_4+x_5+x_6+4$, $\left(\sum_{i=1}^{6} x_i\right)+6$.

As there must be two 3-cycles present in the graph (and we ruled out any adjacency between chords), we may immediately deduce that $x_1 = x_3 = x_4 = x_6 = 0$. Similarly, as there must be two 4-cycles, $x_2 = x_5 = 0$ and this yields the existence of a third 4-cycle.

For (2)-pancyclic graphs of order 10, we summon Lemmas 2 and 3, and obtain $k \leq 3$ and $k \geq 4$, absurd. □

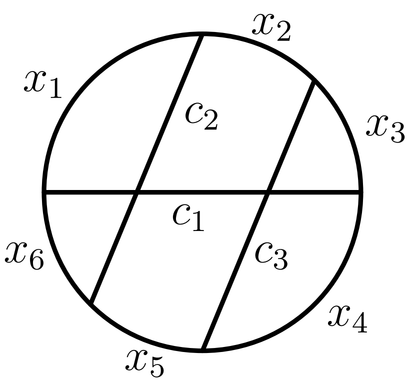

Fig. 3: Is there a (2)-pancyclic graph on 9 vertices?

Theorem 4. *An edge in a* (2)*-pancyclic graph lies on at least* $\lceil \log_2(2n-3) \rceil + 1$ *and at most* $2n - \lceil \log_2(2n-3) \rceil - 4$ *cycles.*

Proof. Let G be a (2)-pancyclic graph on n vertices and k chords. We know from Theorem 2 that $k \geq 3$. Let C be the exterior cycle of G, and let $c(x)$ be the number of cycles using a given edge $x \in E(G)$.

We will now prove the lower bound. Consider $e \in E(C)$. Due to the fact that for every chord of the graph there is a unique cycle which contains e and uses only that chord, we have $c(e) \geq k + 1$. In addition to this, due to Lemma 1, we know that there exist chords c_1 and c_2, which intersect. There must exist a cycle using precisely c_1, c_2, and e, whence $c(e) \geq k+2$. For any other pairs of chords, or triples of chords (and obviously also quadruples, and so forth), there is no guarantee that there exist cycles using e and exactly two or three chords.

Now, let p, q be two intersecting chords. q lies on two cycles using as chords only q itself. There exist precisely two cycles which use the chords p and q, and no other chords.

For all other $k - 2$ chords we can only be sure of at least one cycle using one of them and q, and no other chord. As in the first part of this proof, we know that there exist triples of chords which lie on no cycle. But if we choose p, q, and a further chord, these three must lie on a cycle which uses no further chord. Thus, $c(q) \geq 2 + 2 + (k - 2) + (k - 2) = 2k$. This bound is sharp in the sense that there exists a (2)-pancyclic graph with k chords which has a chord q with $c(q) = 2k$: see Fig. 2(b), and choose q to be the chord which does not lie on a 3-cycle.

As $k \geq 3$, we have $2k \geq k + 2$, and thus, through any given edge at least $k + 2$ cycles pass.

Let us now prove the upper bound. Consider $e \in E(C)$. Every chord halves G, and a given edge lies in precisely one of these halves. Thus, we have $c(e) \leq 2n-k-4$ (we start with the trivial upper bound $2n - 4$ and exclude cycles).

Consider now two chords in G. There exists at least one and at most two cycles which use only these two chords. By Lemma 1, there exists at least one pair of intersecting chords, say p and q. Let A and B be the cycles using as chords only p and q. Clearly, $e \in E(C)$ cannot lie on both A and B, as $E(A \cap B) = \{p, q\}$, but surely lies on one of them, as $C \subset A \cup B$. We obtain the slight improvement $c(e) \leq 2n - k - 5$.

What is the situation for $q \in E(G \setminus C)$? First of all, q halves G, and each half has as boundary a cycle containing q; hence, $c(q) \leq 2n - 6$. Additionally, for each chord excluding q, there exist exactly two cycles which use only this chord. Trivially, q does not lie on these cycles. We obtain $c(q) \leq 2n - 2k - 4$. We may stop here, as $c(q) \leq c(e)$.

In summary, we have for $s \in E(G)$

$$k + 2 \leq c(s) \leq 2n - k - 5.$$

In order to render the inequalities above independent of k, we use (twice) Lemma 2.
□

The (2)-pancyclic graphs in Fig. 4 on 11, 13, 17 and 19 vertices are due to G. Exoo, and can be found on his website [4].

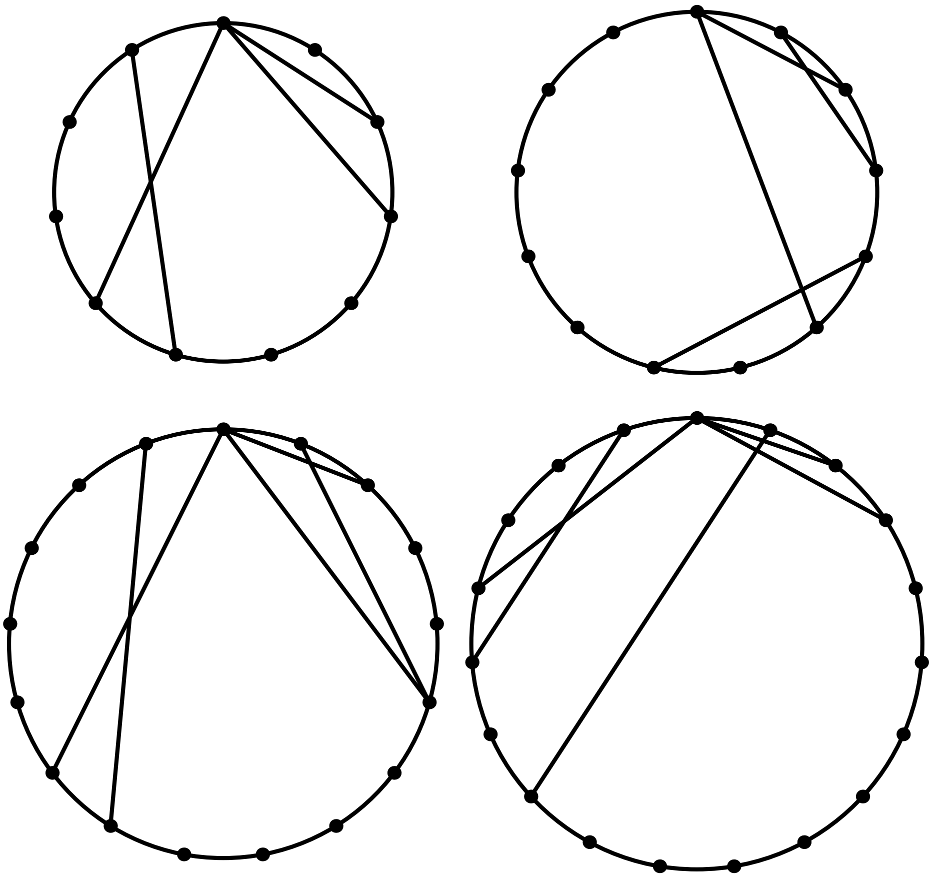

Fig. 4: (2)-pancyclic graphs on 11, 13, 17, and 19 vertices

Theorem 5. *All* (2)*-pancyclic graphs on* 17 *or fewer vertices are non-Eulerian.*

Proof. Let G be an Eulerian (2)-pancyclic graph on n vertices, constructed by adding k chords to an n-cycle C. It is well known that a graph is Eulerian iff the degree of each vertex is even. By Lemma 1, there exist intersecting chords in G and thus, $k \geq 4$. Now, let a *chordal cycle* be a cycle in G all edges of which are chords.

Obviously, G is Eulerian iff there exists a set $\{C_1, ..., C_s\}$ of pairwise edge-disjoint chordal cycles in G such that $G = C \cup \bigcup_{i=1}^{s} C_i$.

Consider $\mu(G) = 1$, whence, by Lemma 4, G has two intertwined chords (and no other chords intersect). If we use but four chords this implies (as all cycles must be either C itself, or chordal) a graph featuring K_4 as subgraph; this trivially leads to contradiction, as at least four 3-cycles arise. If we put $\mu(G) \geq 2$, at least five chords must occur in G and hence, $k \geq 5$.

We prove that for $k = 5$, we have – with respect to the chordal crossing number – three distinct situations: $\mu(G) = 1$, $\mu(G) = 2$, and $\mu(G) = 5$. We have $\mu(G) \geq 1$ due to Lemma 1, and Fig. 5 shows realizations of graphs with $\mu(G) = 2$ and $\mu(G) = 5$. It remains to be shown that the cases $\mu(G) = 3$ and $\mu(G) = 4$ are impossible.

As $k = 5$ and $\mu(G) \geq 1$, we can only have exactly one chordal cycle, featuring five edges and five vertices, four of which, say $v_1, ..., v_4$, are prescribed by the two intersecting chords. It is immediately seen that there are only three combinatorially distinct possibilities, shown in Fig. 5. Thus, $\mu(G) = 1$, $\mu(G) = 2$ or $\mu(G) = 5$.

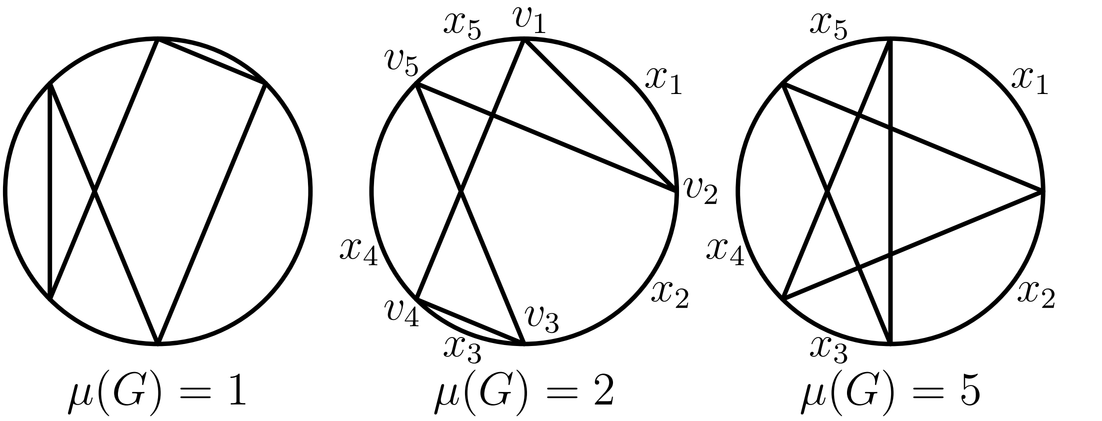

Fig. 5: (2)-pancyclic graphs on five chords featuring all possible chordal crossing numbers, under the assumption that all chords lie within one chordal cycle

In the case $\mu(G) = 1$, Lemma 4 yields that the chords must be intertwined. This implies that (at least) three 4-cycles occur, a contradiction.

For $\mu(G) = 2$, we argue as follows using the notation from Fig. 5, where x_i denotes the number of vertices of degree 2 on the respective path. We know that there must be two n-cycles in G, one of which is given by the exterior cycle C. Clearly, $x_1 \geq 1$ and $x_3 \geq 1$. This implies that the second n-cycle must contain the paths on C between v_1 and v_2, and between v_3 and v_4. Two cases arise.

If the second n-cycle goes from v_4 to v_5 along C, it is forced to use the chord v_5v_2 (as it otherwise becomes C), leading us to v_1. As there is no unused path from v_1 to v_3, we are led to a contradiction.

In the second case, the second n-cycles uses the chord v_4v_1. Then it must contain the path from v_1 to v_2 on C, as $x_1 \geq 1$. Then, in order not to miss v_5, it must contain $v_2v_5v_3$, and return to v_4 on C. This implies $x_2 = x_4 = x_5 = 0$, as otherwise it would not span all vertices. Now at least three 3-cycles occur; a contradiction.

Lastly, we treat the situation $\mu(G) = 5$. As there exist precisely two 3-cycles in G, precisely two numbers in $\{x_1, ..., x_5\}$ must vanish. If they are consecutive (mod. 5), we obtain three 3-cycles; if not there is a single 4-cycle or another number in $\{x_1, ...x_5\}$ is 1, and in this case there are at least three 4-cycles.

Hence, there exist no Eulerian (2)-pancyclic graphs on 5 chords, i.e. $k \geq 6$, which yields by Lemma 2 the desired bound $n \geq 18$. □

In the following, we introduce (analogous to uniquely r-pancyclic graphs, see e.g. [15]) *r-(2)-pancyclic* graphs, which are connected graphs of order n featuring exactly two cycles of each length t with $r \leq t \leq n$.

Theorem 6. *Let $n \geq 6$. If $2^{k-1} + k - 1 \leq n < 2^k + k$, then there exists a p_n-(2)-pancyclic graph, where $p_n = 2^{k-2} + 3$.*

Proof. Consider an n-cycle $C = v_1 v_2 ... v_n v_1$, and add to it the following chords. $v_1 v_3 = c_1$, $v_2 v_4 = c_2$, $v_4 v_7 = c_3$, $v_7 v_{12} = c_4$, $v_{12} v_{21} = c_5$, $v_{21} v_{38} = c_6$, and so forth. We continue this way as long as possible, allowing also that the second endpoint of the last chord coincides with v_1. The construction is shown in Fig. 6.

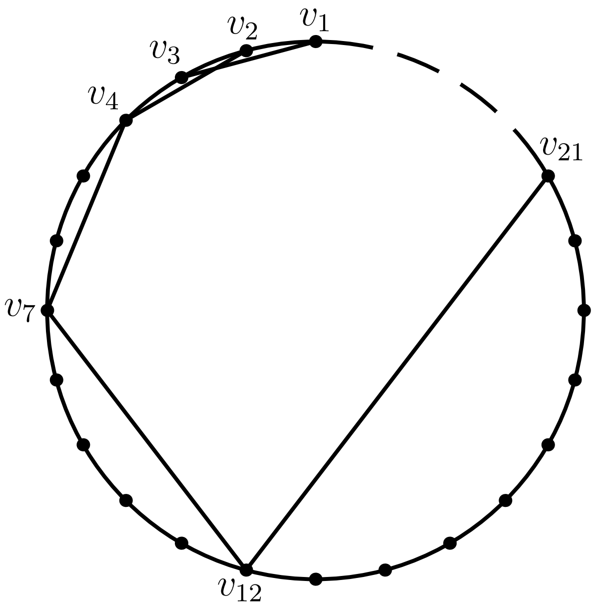

Fig. 6: Constructing p_n-(2)-pancyclic graphs

We obtain in the described way, say, k chords. Let C_k be the cycle containing c_k and not meeting any other chord (except at the endpoints of c_k). A simple inspection of this graph shows that for all large m, from n downwards, we find precisely two cycles of length m. The largest such m for which a third cycle of length m appears equals the length of C_k. Thus, for example, if $6 \leq n < 11$, we get $p_n = 5$; if $11 \leq n < 20$, we have $p_n = 7$; if $20 \leq n < 37$, then $p_n = 11$; and if $37 \leq n < 70$, then $p_n = 19$; in general, if $2^{k-1} + k - 1 \leq n < 2^k + k$, then $p_n = 2^{k-2} + 3$. □

Corollary. *There exists a q_n-(2)-pancyclic graph of order n, where*

$$q_n = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1 & \text{for } n \geq 7, \\ \left\lceil \frac{n}{2} \right\rceil & \text{for } n \geq 21 \text{ and} \\ \left\lfloor \frac{n}{2} \right\rfloor & \text{for } n \geq 38. \end{cases}$$

Proof. This also includes a more detailed proof of Theorem 6. Let

$$A_j = [2 + j + 2^{j+1}, 2 + j + 2^{j+2}], \quad j \geq 0.$$

We have

$$k = \sum_{j=0}^{\infty} (j+2) \cdot \chi_{A_j}(n)$$

chords, where χ denotes the indicator function. For the k-th chord ($k \geq 3$), we skip 2^{k-2} vertices. Let $\mathcal{C}$ be the family of all cycles in our graph using at least one chord but neither c_1, nor c_2. For a given chord, there exist precisely two cycles of unequal lengths using that chord, and only that chord; we choose the shorter one. Let $\mathcal{C}' \subset \mathcal{C}$ be the subfamily obtained this way.

We can now prove the assertion. Let us write $c_j^S = 1$ if the cycle S uses the chord c_j, and $c_j^S = 0$ if it does not, where $3 \leq j \leq k$. The set of all possible lengths of cycles using any (ranging from one to all) of the $k-2$ chords $c_3, c_4, ..., c_k$ (observe that c_1 and c_2 are not included, as well as the case of not using any chords – we deal with them in an instant) is $N \cup M$, where

$$N = \{2 + 2^{j-2} : 3 \leq j \leq k\}$$

and

$$M = \left\{n - \left(2 \cdot c_3^S + 4 \cdot c_4^S + 8 \cdot c_5^S + ... + 2^{k-2} \cdot c_k^S\right) : S \in \mathcal{C} \setminus \mathcal{C}'\right\}.$$

Here, N is given by the cycles in $\mathcal{C}'$ and M by all remaining cycles. Note that every number in M has the same parity as n. More exactly, we have $M = \{n - \sum_{j=3}^{k} 2^{j-2}, ..., n-4, n-2\}$ (it does not reach n, as we required there to be at least one chord in use).

In addition to this, we have two 3-cycles and two $n-1$ cycles (by using either c_1 or c_2 as only chord), one 4-cycle (by using c_1 and c_2), and two n-cycles which form by using all vertices and either no chord, or both c_1 and c_2. In further investigations we may ignore the two 3-cycles and the 4-cycle, as $q_n \geq 6$ for all $n \geq 7$.

We have listed all cycle lengths occurring in the construction from Fig. 6. Three situations occur, where a cycle S uses at least one chord c_j, $3 \leq j \leq n$. Let us mention here that lengths appearing in N do not interest us, as $q_n > \max N$, which we discuss at the end of this proof.

(i) Among the cycles in $\mathcal{C}$, for each length $\ell \in M$ we have exactly one cycle of length ℓ.

(ii) To each cycle from *(i)* there exist two cycles which are almost identical, but one uses the chord c_1 (and not c_2), while the other uses c_2 (but not c_1). Both have length $\ell - 1$, with $\ell \in M$.

(iii) To each cycle from *(i)* there exists one cycle which is almost identical to the one from *(i)*, but uses the chords c_1 and c_2. It has length ℓ, where $\ell \in M$.

If we add to these cycles the pairs of cycles of length $n-1$ and n mentioned above, we may conclude that every cycle of length $\ell \in M \setminus N$ appears exactly twice.

We now show that $((M \setminus N) \cup \{n-1, n\}) \supset [q_n, n] \cap \mathbb{N}$, i.e. $q_n > \min M$ and $q_n > \max N$. We have

$$\min M = n - \sum_{j=3}^{k} 2^{j-2} = n + 2 - 2^{k-1} \geq k + 2^{k-1} + 2 - 2^{k-1} = k + 2$$

and

$$q_n \geq \frac{n}{2} + 1 \geq \frac{k + 2^{k-1}}{2} + 1 > 2 + 2^{k-2} = \max N. \qquad (\dagger)$$

For $k \geq 4$, clearly $\max N \geq \min M$, so we are done, and for $k = 3$ we have $q_n \geq 6$ and $\min M = 5 > 4 = \max N$, which completes the proof of the assertion that for all $n \geq 7$ there exists an $\left(\lceil \frac{n}{2} \rceil + 1\right)$-(2)-pancyclic graph of order n. Strengthening the bound is done by using inequalities similar to ($\dagger$), where for $n \geq 21$ we have $k \geq 5$, and for $n \geq 38$ we have $k \geq 6$. $\square$

Discussion

We recall that r-(2)-pancyclic graphs are graphs of order n having exactly two cycles of each length t, where $r \leq t \leq n$. Let us denote by r_n the minimal value of r for which there exists an r-(2)-pancyclic graph on n vertices. Please find below a table listing the sets, depending on n, in which r must reside (optimal results are denoted in bold).

By Theorem 2 and Exoo's graphs we know that $r_8 = r_{11} = r_{13} = r_{17} = r_{19} = 3$, and Theorem 3 yields $r_9 \neq 3$ and $r_{10} \neq 3$. The remaining upper bounds for r_n are given by Theorem 6.

n	3	4	5	6	7	8	9	10	11	12	13	14
r_n	-	-	**5**	**5**	**5**	**3**	$\{4,5\}$	$\{4,5\}$	**3**	$\{3,...,7\}$	**3**	$\{3,...,7\}$

n	15	16	17	18	19	20	21	...
r_n	$\{3,...,7\}$	$\{3,...,7\}$	**3**	$\{3,...,7\}$	**3**	$\{3,...,11\}$	$\{3,...,11\}$	...

Let us comment briefly on the values of r_n for $5 \leq n \leq 7$. For $n = 5$, adding one chord to a 5-cycle yields a uniquely pancyclic graph. Adding a second chord either yields only one 5-cycle or two 5-cycles and three 4-cycles, so $r_5 = 5$. On six and seven vertices, by Theorem 6 we have $r_6 \leq 5$ and $r_7 \leq 5$. Theorem 2 implies that $r_6 \geq 4$ and $r_7 \geq 4$. Finally, a case by case analysis gives the optimality of $r_6 = 5$ (add to a 6-cycle $v_1v_2...v_6v_1$ the chords v_1v_3 and v_2v_4) and $r_7 = 5$ (add to a 7-cycle $v_1v_2...v_7v_1$ the chords v_1v_3, v_1v_5 and v_2v_4).

During our investigation, the following conjecture and questions arose, the formulation of which requires a natural extension of (2)-pancyclicity: we call a graph (oriented or not) (k)*-pancyclic*, if it contains precisely k cycles of length p, where $3 \leq p \leq n$.

Conjecture. *There are finitely many* (2)*-pancyclic graphs.*

Problem 1. Is every (2)-pancyclic graph planar?

Problem 2. Do (k)-pancyclic graphs exist for $k \geq 3$?

Consider now an oriented graph, i.e. a digraph with exactly one orientation on every edge. We show in Fig. 7 a simple construction yielding uniquely pancyclic oriented graphs of any order $n \geq 3$.

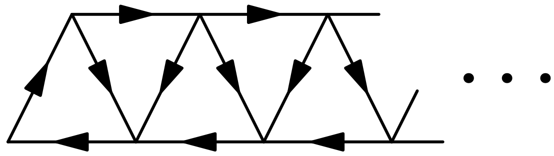

Fig. 7: Uniquely pancyclic oriented graphs

We also have examples of (2)-pancyclic oriented graphs. They will be presented in a subsequent paper.

Problem 3. For which $k \geq 2$ do infinite families of (k)-pancyclic oriented graphs (or digraphs) exist?

References

[1] J.A. Bondy and U.S.R. Murty. Graph Theory with Applications. North-Holland (1976).

[2] E. Boros, Y. Caro, Z. Füredi, and R. Yuster. Covering non-uniform hypergraphs. *J. Comb. Th. B* **82** (2001) 270–284.

[3] G. Chen, J. Lehel, M.S. Jacobson, and W.E. Shreve. Note on graphs without repeated cycle lengths. *J. Graph Th.* **29** No. 1 (1998) 11–15.

[4] G. Exoo. http://ginger.indstate.edu/ge/Graphs/PANCYCLIC/index.html

[5] C. Lai. On the size of graphs with all cycles having distinct lengths. *Discrete Math.* **122** (1993) 363–364.

[6] C. Lai. Graphs without repeated cycle lengths. *Australas. J. Comb.* **27** (2003) 101–105.

[7] K. Markström. A note on uniquely pancyclic graphs. *Australas. J. Comb.* **44** (2009) 105–110.

[8] Y. Shi. Some theorems of uniquely pancyclic graphs. *Discrete Math.* **59** (1986) 167–180.

[9] Y. Shi. On maximum cycle-distributed graphs. *Discrete Math.* **71** (1988) 57–71.

[10] Y. Shi and J. Sun. Two results on uniquely r-pancyclic graphs. *Chin. Q. J. Math.* **7** No. 2 (1992) 56–60 (in Chinese).

[11] Y. Shi and J. Sun. On uniquely r-bipancyclic graphs. *J. Shanghai Teachers Univ.* **26** No. 4 (1997) 1–10.

[12] Y. Shi, H.P. Yap, and S.K. Teo. On uniquely r-pancyclic graphs. *Ann. N. Y. Acad. Sci.* **576** (1989) 487–499.

[13] J. Sun. Some results on uniquely r-pancyclic graphs. *J. Nanjing Univ., Nat. Sci. Ed.* **27** Spec. Issue (1991) 165–166 (in Chinese).

[14] H. Sun, Y. Shi, and J. Sun. A Theorem of Uniquely Bipancyclic Graphs. *J. Shanghai Teachers Univ.* **29** No. 4 (2000) 24–34.

[15] H.P. Yap and S.K. Teo. On uniquely r-pancyclic graphs. *Lect. Notes Math.* **1073** (1984) 334–335.

Carol T. ZAMFIRESCU
Fakultät für Mathematik,
Technische Universität Dortmund,
Germany
E-mail address: *czamfirescu@gmail.com*