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ORIGINAL PAPER

An Infinite Family of Planar Non-Hamiltonian Bihomogeneously Traceable Oriented Graphs

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Abstract We answer an open question on planar non-hamiltonian bihomogeneously traceable digraphs without opposite arcs by constructing an infinite family of such graphs.

Keywords Planar · Bihomogeneously traceable · Digraph

We shall consider in this paper only oriented graphs, i.e. digraphs without opposite arcs (cycles of length 2), calling them here just *graphs*. Z. Skupień introduced the notion of homogeneous traceability. A graph is said to be *homogeneously traceable* if every vertex is the initial point of a hamiltonian path. If, moreover, every vertex is also the endpoint of some hamiltonian path, the graph is called *bihomogeneously traceable*.

Obviously, each hamiltonian (or hypohamiltonian) graph is bihomogeneously traceable. But not every homogeneously traceable graph is hamiltonian [1]. Not even bihomogeneous traceability implies hamiltonicity. At a meeting in Kalamazoo (in 1980) Z. Skupień [5] proved that for each natural number $n \geq 7$ there exists a 2-diregular bihomogeneously traceable non-hamiltonian graph of order n ([5] appeared in 1981). Moreover, Skupień [6] later constructed exponentially many n-vertex bihomogeneously traceable graphs.

Independently, in another paper which also appeared in 1981, S. Hahn and T. Zamfirescu [4] also constructed an infinite sequence of bihomogeneously traceable non-hamiltonian graphs, and gave three special examples: the first is arc-minimal (has the smallest possible number of arcs) and has 7 vertices, the second is planar

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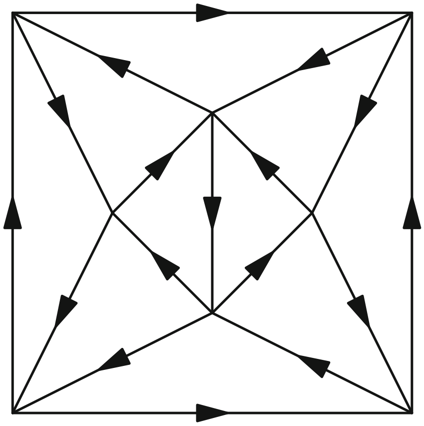

Fig. 1 The smallest planar non-hamiltonian bihomogeneously traceable graph

and has 8 vertices, and the third is both arc-minimal and planar, and has 9 vertices. Arc-minimality amounts in this context to 2-diregularity. The planar graph on 8 vertices is shown here in Fig. 1. It is proven in [4] that there are no smaller examples. Then, Hahn and Zamfirescu asked the natural question whether infinitely many planar bihomogeneously traceable non-hamiltonian graphs do exist.

In this paper we show that, indeed, there exist infinitely many such graphs.

If opposite arcs are permitted, infinite families of planar hypohamiltonian digraphs have been found long ago (see Fouquet and Jolivet [2], Thomassen [7], Grötschel and Wakabayashi [3]). Thomassen proved that a planar hypohamiltonian digraph with n vertices (and many opposite arcs, all but six) exists for each $n \geq 6$.

Theorem *There exists an infinite family of planar non-hamiltonian bihomogeneously traceable graphs.*

Proof The proof has three parts. First, we consider the infinite sequence $\{G_n\}_{n=1}^{\infty}$ of planar graphs depicted in Fig. 3. Secondly, we determine an infinite subsequence the graphs of which are non-hamiltonian. The last step will be proving that these graphs are all bihomogeneously traceable.

As seen in Figs. 2 and 3, which show G_1 and G_4, respectively, each graph G_n has the property that its vertex set is covered by two vertex-disjoint (directed) cycles c' and c''. The cycle c' has length $3n + 3$ (it is the boundary of the big central face) and c'' has length $3n + 12$ (it would become the boundary of the unbounded face, if the three edges ab, bc, ca were removed).

Obviously, G_n has $6n + 15$ vertices.

We will now focus on determining which graphs in this family are non-hamiltonian. Suppose G_n has a hamiltonian cycle H. We use for this proof the notation from Fig. 3. First, let us prove the following.

Claim *Neither ab, nor bc, nor ca lies in H.*

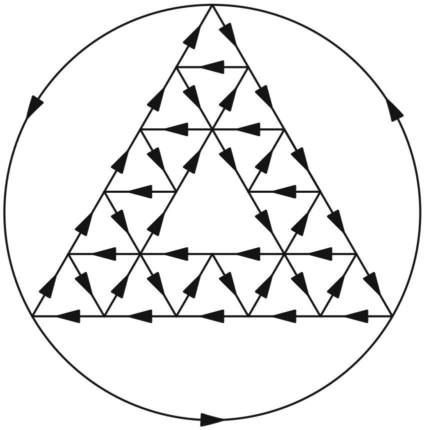

Fig. 2 The graph G_1

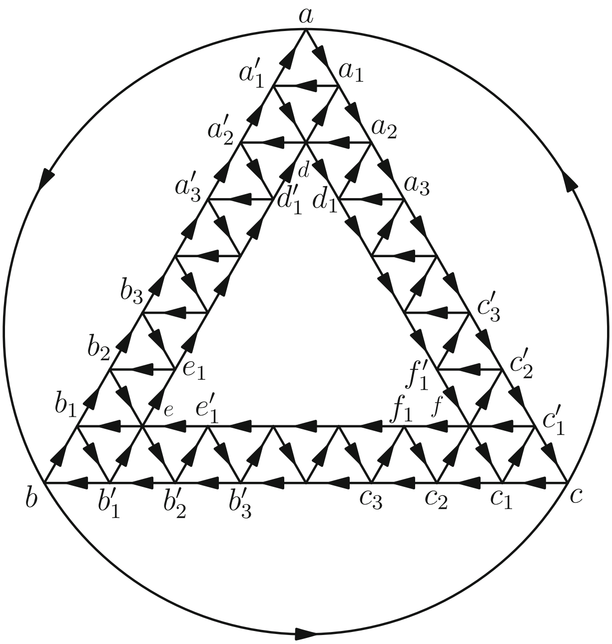

Fig. 3 The graph G_4

Indeed, since all three edges do not simultaneously belong to H, if we assume the claim not to be true, then we must have $bcc_1 \subset H$ (or $caa_1 \subset H$ or $abb_1 \subset H$). Continuing with the path bcc_1 in H, we distinguish between two cases:

Case A We go from c_1 to c'_1, a vertex adjacent to c. Then we must visit the 6-valent point f. Now, if we go to c'_2 then $f_1c_2c_3 \subset H$ and f_1 cannot be reached; if we go from f to f_1 then $c'_3c'_2f'_1 \subset H$ and we cannot leave from f'_1 anymore.

Case B From c_1 we continue to c_2. Thus $c'_2c'_1f \subset H$. Obviously the paths cc_1c_2 and $c'_2c'_1f$ must continue in a unique way ($cc_1c_2\ldots b'_2b'_1e$ and $c'_2c'_1ff_1f_2\ldots e'_2e'_1e$) and fatally meet at e.

Hence, the Claim is proved and it follows that $a'_2a'_1aa_1a_2 \cup b'_2b'_1bb_1b_2 \cup c'_2c'_1cc_1c_2 \subset H$. For H we have two possibilities to exit a_2:

Case 1 We go from a_2 to a_3. In this case, $a_3a_4 \not\subset H$. Indeed, if we go from a_3 to a_4, we cannot continue to d_2 because we would lose d_1. Similarly, we are forced to go via $a_4a_5a_6\ldots c'_1cc_1\ldots b'_1bb_1\ldots a'_1a$, but lose all vertices of c'.

Hence, from a_3 we must go to d_1 and continue (we are forced to) thus: $a_3d_1d_2d_3a_4a_5a_6d_4\ldots$ There are three subcases to be distinguished:

Subcase 1.1. $n = 1$ modulo 3.
Then the path continues as $\ldots c'_3c'_2f'_1$, which contradicts $c'_2c'_1 \subset H$.
Subcase 1.2. $n = 2$ modulo 3. Then the path continues as
$\ldots c'_4c'_3f'_2f'_1fc'_2c'_1cc_1c_2c_3f_1f_2f_3c_4\ldots bb_1b_2b_3e_1e_2e_3b_4\ldots d'_1da'_2a'_1a$,
thus concluding the hamiltonian cycle H.
Subcase 1.3. $n = 0$ modulo 3.
Then the path continues as
$\ldots c'_5c'_4f'_3f'_2f'_1c'_3c'_2c'_1cc_1c_2ff_1f_2c_3c_4\ldots b'_5e'_4e'_3e'_2b'_4b'_3b'_2e'_1$,
which contradicts $b'_2b'_1 \subset H$.

Case 2 We go from a_2 to d. From there, we cannot go to a'_2 and therefore $a_2dd_1 \subset H$. Hence $a'_3a'_2a'_1 \subset H$. A symmetry with respect to the vertical line through a and a reversal of edge orientations transforms G_n into itself and H into a hamiltonian cycle H', and places us in the situation that $a_1a_2a_3 \subset H'$, i.e. Case 1.

Consequently, G_n is hamiltonian precisely for $n = 2$ modulo 3. Let us consider $\mathcal{G} := \{G_n : n = 1 \text{ mod. } 3\}$. All graphs in $\mathcal{G}$ are non-hamiltonian and the second step of our proof is completed.

In the third step we will now show that each graph of $\mathcal{G}$ is bihomogeneously traceable. Consider an arbitrary graph $G_n \in \mathcal{G}$. The previously used symmetry implies that, if G is homogeneously traceable, then it is also bihomogeneously traceable. In fact, G_n is isomorphic to the graph obtained from G_n by reversing the direction of all of its arcs.

Let us now show that $G_n \in \mathcal{G}$ is homogeneously traceable. We do this by providing hamiltonian paths starting at each vertex. There are several different cases; by symmetry, it suffices to analyze the cases a, a_i, a'_i, d, d_i and d'_i. Let this starting vertex be v.

For all cases except $v = a_1$ and $v = a_2$, a hamiltonian path starting at v is easily constructed as follows. For $v \in c''$, go along c'' (never leaving it) up to the vertex, say u, that is the predecessor of v in c'', then move from u to a vertex of c' and continue along c' visiting all its vertices. For $v \in c'$ the proof is analogous. The Figs. 4 and 5 illustrate the remaining cases, i.e. show hamiltonian paths starting at a_1 or a_2 (with $d = d_0 = d'_0$, $k \geq 4$ and $m \geq 1$).

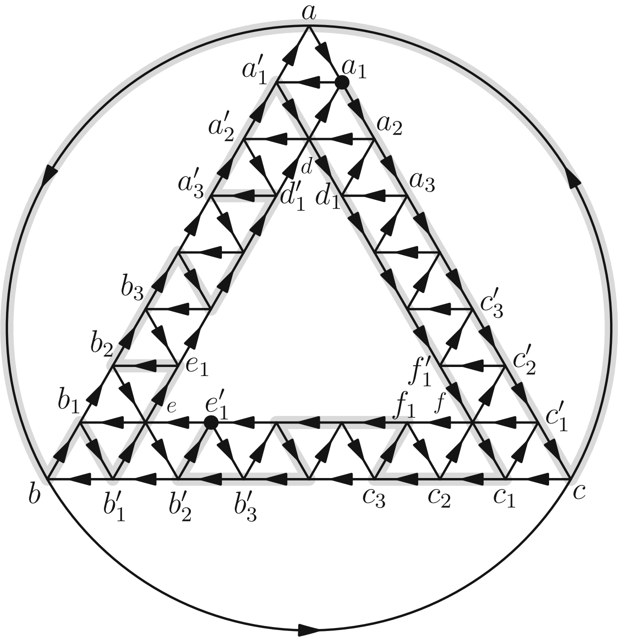

Fig. 4 The graph G_4 with a hamiltonian path starting in a_1

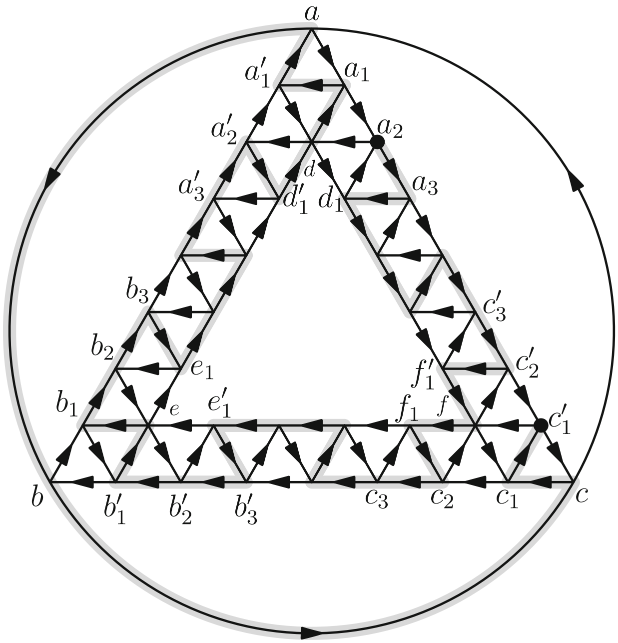

Fig. 5 The graph G_4 with a hamiltonian path starting in a_2

Case a_1: $a_1a_2 \ldots c'_1cabb_1b'_1ee_1b_2b_3b_4e_2e_3e_4b_5b_6 \ldots d'_2d'_1a'_3a'_2a'_1dd_1d_2 \ldots f'_1fc_1$
$c_2\ c_3 f_1 f_2 f_3c_4c_5 \ldots b'_3b'_2e'_1$

Case a_2: $a_2a_3d_1d_2d_3a_4a_5a_6d_4 \ldots c'_3c'_2 f'_1 ff_1c_2c_3c_4 f_2 f_3 f_4c_5 \ldots e'_2e'_1b'_3b'_2b'_1eb_1b_2$
$b_3e_1e_2e_3b_4 \ldots a'_3a'_2d'_1da_1a'_1abcc_1c'_1$

We have finished the proof by providing an infinite family $\mathcal{G}$ of graphs fulfilling all requirements in the statement. □

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