Acute triangulations of the double triangle

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Abstract. We prove that every doubly covered triangle can be triangulated with 12 acute triangles, and this number is best possible.

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MSC: 52B12, 52C20.

Introduction

A *triangulation* of a 2-dimensional space means a collection of (full) triangles covering the space, such that the intersection of any two triangles is either empty or consists of a vertex or of an edge. A triangle is called *geodesic* if all its edges are *segments*, i.e., shortest paths between the corresponding vertices. The triangles of our triangulations will always be geodesic, and we are especially interested in *acute triangulations* which, by definition, contain only triangles all angles of which are acute ($< \pi/2$).

In [7] T. Hangan, J. Itoh and T. Zamfirescu considered the following problem.

Problem 1. Does there exist a number N such that every compact convex surface in $\mathbb{R}^3$ admits an acute triangulation with at most N triangles?

Of course, one should estimate N, if it exists.

Indeed, the first compact surfaces to be investigated should be the convex ones, as formulated in Problem 1, and among these the polyhedral surfaces play a central role. The acute triangulation of the Platonic solids was treated in [7], [8], [10], and [9].

The case of the arbitrary polyhedral surfaces is much more difficult, even for a small number of vertices. So, for example, even the family of all tetrahedral surfaces is not easy to treat.

We shall study here the acute triangulation of all doubly covered triangles, for short *double triangles,* which can be regarded as the simplest polyhedral case, although these surfaces cannot be isometrically embedded in $\mathbb{R}^3$ as the boundary of a 3-dimensional polytope. Even this case is less trivial than it might appear at the first look.

The problem

Consider the congruent copies Δ, Δ' of an equilateral triangle, as faces of our double triangle T. Trisect the angles of Δ. The pairs of trisectors closer to each side of Δ meet at the vertices of an equilateral triangle ABC. The bisectors of Δ' meet at I. Then the trisectors of Δ, the bisectors of Δ', the sides of ABC and the segments AI, BI, CI determine an acute triangulation of T with 10 triangles.

Many double triangles not too different from T can be acutely triangulated in the same combinatorial manner.

However, the problem we want to solve is this: Find the minimal integer N such that *every* double triangle can be triangulated with at most N acute triangles. We shall see that N is not 10.

The result

Theorem. *Every double triangle can be triangulated with at most* 12 *acute triangles. There exist double triangles for which no smaller acute triangulation is possible.*

Proof. Clearly, each triangulation of any 2-dimensional manifold has an even number of triangles.

First we show how to construct a triangulation with 12 acute triangles.

Let a, b, c be the vertices of an arbitrary double triangle, of faces Δ, Δ', and assume that no side is longer than bc. Consider the orthogonal projection a^* of a onto bc and the point $d \in \Delta \setminus bc$ close to a^*, but outside the circle of diameter ab. Let d^* be the orthogonal projection of d onto ac and e the orthogonal projection of d^* onto bc. Choose a point $f \in \Delta \setminus ac$ on the bisector of $\angle ced$, close to ac (and outside the circle of diameter ad). Finally, let d' and f' be the images of d and f through the isometry from Δ to Δ' which identifies their boundaries. The geodesic segments ab, ad, ad', af, af', bd, bd', ce, cf, cf', dd', de, $d'e$, df, $d'f'$, ef, ef', ff' determine a triangulation with 12 triangles, which are easily seen to be acute.

Now we show that for a certain double triangle any acute triangulation T has at least 12 triangles.

Consider a triangle with an angle of $3\pi/4$. Then the curvature of the double triangle at the corresponding vertex v is $\pi/2$. This means that any geodesic triangle containing v in its interior has excess $\pi/2$ and therefore at least one angle larger or equal to $\pi/2$. So, our acute triangulation T must have v as a vertex. On the other hand, the degree at v cannot be less than 4 since the total angle there is $3\pi/2$. Hence T has at least 10 as sum of the degrees at the vertices of the triangle.

Let n, m, f be the number of vertices, edges and faces of T, respectively. For any triangulation, $2m = 3f$, and Euler's formula gives $m = 3n - 6$. The sum $2m$ of all degrees is at least $10 + (n-3)5$. Hence

$$2(3n-6) \geq 5n-5,$$

i.e., $n \geq 7$. But there are no triangulations of the sphere with 7 vertices satisfying our degree conditions. Indeed, the degree sequence must be precisely $(3, 3, 4, 5, 5, 5, 5)$, and it happens that this sequence is not realizable; by the way, there is no triangulation with 7 vertices out of which exactly one has degree 4 (see [13], page 246).

Thus, $n \geq 8$. This implies

$$f = 2m/3 = 2n - 4 \geq 12.$$

The theorem is proven.

Remark. Maehara [12] and Yuan [14] gave estimates for the minimal number of triangles in an acute triangulation of an arbitrary n-gon, depending of course on n. These imply estimates for the analogous number of the doubly covered arbitrary n-gon, where, however, the dependence on n is not any more natural.

So, the problem for doubly covered polygons, as well as for doubly covered 2-dimensional convex bodies, is still widely open.

Historical notes

The investigation of acute triangulations has one of its origins in a problem of Stover reported in 1960 by Gardner in his Mathematical Games section of the Scientific American (see [4], [5]). There the question was raised whether a triangle with one obtuse angle can be cut into smaller triangles, all of them acute. In the same year, independently, Burago and Zalgaller [1] investigated in considerable depth acute triangulations of polygonal complexes, beeing led to them by the problem of their isometric embedding into $\mathbb{R}^3$. (Accidentally, their paper also includes a solution to Stover's problem!)

In 1980, Cassidy and Lord [2] considered acute triangulations of the square. Recently, Maehara investigated acute triangulations of quadrilaterals [11] and other polygons [12].

Triangulations with triangles which are close to equilateral were considered by Gerver [6] and, on Riemannian surfaces, by Colin de Verdière and Marin [3].

For a short survey of existing results about acute triangulations, see [15].

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